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BARGMANN-TYPE KERNELS AND UNBOUNDED SUBNORMALS

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ABSTRACT. A class of positive definite kernels $k_p!$ on the complex plane is introduced such that the multiplication operator $M(k_p!)$ in the Hilbert space $\mathcal{H}(k_p!)$ associated with $k_p!$ is an unbounded subnormal, $M(k_1!)$ in particular being the multiplication operator in the classical Bargmann space. Multivariable unbounded subnormal weighted shifts, analogous to the tuple of creation operators corresponding to the multidimensional Schrödinger representation, are also discussed and the subnormality of such shifts is related to the multidimensional Stieltjes Moment problem.

1. Preliminaries. Let $\mathcal{H}$ be a separable Hilbert space, and let $S = (S_1, \dots, S_m)$ be a tuple of (possibly unbounded) linear operators in $\mathcal{H}$ such that the operators $S_1, \dots, S_m$ have a common invariant dense domain $\mathcal{D}(S)$ in $\mathcal{H}$ and commute with each other on $\mathcal{D}(S)$. The tuple S is said to be subnormal if there exist a Hilbert space $\mathcal{K}$, a tuple $N = (N_1, \dots, N_m)$ of normal operators $N_1, \dots, N_m$ in $\mathcal{K}$ and a dense subspace $\mathcal{D}(N)$ of $\mathcal{K}$ such that $\mathcal{D}(N)$ is invariant for all N_i and N_i^*, N_i are commuting on $\mathcal{D}(N)$, $\mathcal{D}(S) \subset \mathcal{D}(N)$ and $N_i/\mathcal{D}(S) = S_i$ for all i. The study of unbounded subnormal tuples S has received some attention in recent years with the special emphasis on the case $m = 1$ [**13, 16, 18, 21, 22, 23**]. In what follows, if $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_m)$ are multiindices of nonnegative integers and $z = (z_1, \dots, z_m)$ is a tuple of complex numbers, then z^α will denote $z_1^{\alpha_1} z_2^{\alpha_2} \dots z_m^{\alpha_m}$ and $\alpha + \beta$ will denote the tuple $(\alpha_1 + \beta_1, \dots, \alpha_m + \beta_m)$. The symbol $\epsilon(j)$ will stand for the m-tuple $(0, \dots, 1, \dots, 0)$ which has 1 in the jth coordinate place and zeros elsewhere. For a subset A of the real line $\mathbf{R}$ or the complex plane $\mathbf{C}$, A^n will denote the Cartesian product of A with itself n times. Occasionally, we might forego the multiindex notation in the interest of clarity. Finally, $\mathbf{N}$ will denote the set of nonnegative integers.

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If $\{a_\alpha\}_{\alpha\in\mathbf{N}^m}$ is a multisequence of real numbers such that $a_\alpha = \int_k x^\alpha\, d\mu(x)$ for some subset K of $\mathbf{R}^m$ and some positive measure μ satisfying $\int_k |x^\alpha|\, d\mu(x) < \infty$ for all α in $\mathbf{N}^m$, then $\{a_\alpha\}$ is said to be a K-moment sequence. According to whether $K = [0,1]^m$, $(-\infty,\infty)^m$ or $[0,\infty)^m$, $\{a_\alpha\}$ will be referred to as a Hausdorff, Hamburger or Stieltjes moment sequence, respectively. The problem of determining whether a given sequence $\{a_\alpha\}$ is a Hausdorff, Hamburger or Stieltjes moment sequence is the Hausdorff, Hamburger or Stieltjes Moment Problem, respectively, and the reader is referred to [**5, 6, 10, 20** and **24**] for extensive treatments of such problems. The connection of the Hausdorff Moment Problem with the bounded subnormal operator theory is well known [**3, 9, 12, 14, 19**], and that of the Stieltjes Moment Problem with the unbounded subnormal operator theory of a single operator has been established and explored in [**21, 22** and **23**]. The m-tuple $a^+ = (a_1^+, \dots, a_m^+)$ of creation operators corresponding to the m-dimensional harmonic oscillator or the $2m$-dimensional Schrödinger representation of quantum mechanics is the most famous example of an unbounded subnormal tuple (see [**7, 11, 13, 21** and **22**]). Now a^+ is really the tensor product of the one-dimensional creation operators a_i^+, the domain for a^+ being the m-fold tensor product of the Schwarz space with itself. Also, a^+, as restricted to the linear span of the products of one-dimensional Hermite functions, is seen to be a multivariable weighted shift. Recall that if $\{e_\alpha\}_{\alpha\in\mathbf{N}^m}$ is an orthonormal basis of a Hilbert space $\mathcal{H}$ and $\{w_\alpha^{(j)}\}$, $j = 1, \dots, m$, is a set of positive numbers, then $T = (T_1, \dots, T_m)$, defined on the linear span of $\{e_\alpha\}$ by $T_j e_\alpha = w_\alpha^{(j)} e_{\alpha+\varepsilon(j)}$, $\alpha \in \mathbf{N}^m, j = 1, \dots, m$, is called a multivariable weighted shift with weights $\{w_\alpha^{(j)}\}$. We always assume that the coordinates T_i of such a multivariable weighted shift are commuting so that $w_\alpha^{(j)} w_{\alpha+\varepsilon(j)}^{(k)} = w_\alpha^{(k)} w_{\alpha+\varepsilon(k)}^{(j)}$ for all α in $\mathbf{N}^m$, and for all $j, k \geq 1$.

A positive definite kernel k on a subset A of $\mathbf{C}^m$ is a function $k(z,w)$ from $A^2 = A \times A$ into $\mathbf{C}$ satisfying $\sum_{\alpha,\beta} k(z_\alpha, z_\beta) c_\alpha \bar{c}_\beta \geq 0$ for all possible choices of a finite sequence of points $\{z_\alpha\}$ from A, and complex numbers c_α. We will be exclusively concerned with positive definite kernels $k(z,w) = \sum_\alpha a_\alpha z^\alpha \bar{w}^\alpha$ ($a_\alpha > 0$ and $^-$ denotes the conjugate), defined on either $\mathbf{C}^m$ or the unit polydisk $\mathbf{D}^m$ of $\mathbf{C}^m$. It is well-known (see [**2**] and [**8**]) that a positive definite kernel k gives rise to a functional Hilbert space $\mathcal{H}(k)$ such that the tuple $M(k) = (M_{z_1}, \dots, M_{z_m})$ of multiplications by coordinate functions in $\mathcal{H}(k)$, acting on the linear span of $\{z^\alpha\sqrt{a_\alpha}\}$, can be identified with the multivariable weighted shift $T=(T_1,\dots,T_m)$ with weights $\{w_\alpha^{(j)}=\sqrt{a_\alpha/a_{\alpha+\varepsilon(j)}}\}$. The tuple a^+ referred to earlier can be looked upon as $M(k)$ in $\mathcal{H}(k)$, where $k(z,w)$ is $\exp(z_1\bar{w}_1+\cdots+z_m\bar{w}_m)$ [**7, 11, 13**] and $\mathcal{H}(k)$ is the famous Bargmann space. The Bargmann space has already received some attention from the viewpoint of operator theory (see [**7, 13** and the references therein]).

In this paper we introduce a class of multiplication tuples analogous to a^+ in the Bargmann space by considering positive definite kernels on $\mathbf{C}^m$ that are analogs of the Fischer or Bargmann kernel $\exp(z_1\bar{w}_1+\cdots+z_m\bar{w}_m)$ and relate the question of the subnormality of a class of multivariable weighted shifts, of which a^+ is a prototype, to the multidimensional Stieltjes moment problem.

2. Subnormality of a single operator. Let $k(z,w)=\sum_{n=0}^\infty a_n z^n\bar{w}^n$ be a positive definite kernel on $\mathbf{C}$ or the unit disk $\mathbf{D}$ in $\mathbf{C}$. We associate with k a positive definite kernel $k!$ on $\mathbf{C}$ defined by $k!(z,w)=\sum_{n=0}^\infty (a_n/n!)z^n\bar{w}^n$. For example, the Szegö kernel $1/(1-z\bar{w})$ on $\mathbf{D}$ gets associated with the Bargmann kernel $\exp(z\bar{w})$ on $\mathbf{C}$. Later, we will provide a stronger motivation for considering the association $k\to k!$ by referring to the solutions of the confluent hypergeometric equation of the theory of special functions. First we record some elementary observations related to the association $k\to k!$. Recall that a densely defined operator T in $\mathcal{H}$ is said to be hyponormal if $\mathcal{D}(T)\subset\mathcal{D}(T^*)$ and $\|Tx\|\geq\|T^*x\|$ for all x in $\mathcal{D}(T)$ (see [**18**]), where $\|\cdot\|$ denotes the norm of $\mathcal{H}$; and, for a weighted shift operator T, this amounts to the corresponding weight sequence being nondecreasing [**18**]. The proof of Proposition 1 below is straightforward.

Proposition 1. *If $M(k)$ is hyponormal, then so are $M(k!)$ and $M(k-k!)$.*

Proposition 2. *Let $k=\sum_{n=0}^\infty a_n z^n\bar{w}^n$ be a positive definite kernel on $\mathbf{C}$ such that $M(k)$ is an unbounded subnormal in $\mathcal{H}(k)$, and let $l=\sum_{n=0}^\infty b_n z^n\bar{w}^n$ be a positive definite kernel on $\mathbf{D}$ such that $M(l)$ is a bounded subnormal on $\mathcal{H}(l)$. Then both $M(k!)$ and $M(l!)$ are subnormal.*

Proof. Note that $\{n! = \int_0^\infty x^n \exp(-x)\,dx\}$ is a Stieltjes moment sequence. If $M(k)$ is an unbounded subnormal in $\mathcal{H}(k)$, then $\{1/a_n\}$ is a Stieltjes moment sequence [**22**] and hence so is $\{n!/a_n\}$ (see [**20** or **24**]); so that $M(k!)$ is subnormal [**22**]. If $M(l)$ is a bounded subnormal on $\mathcal{H}(l)$, then $\{1/a_n\}$ is a Hausdorff (and hence a Stieltjes) moment sequence [**19**]; so that $\{n!/a_n\}$ is again a Stieltjes moment sequence, leading to $M(l!)$ being subnormal. □

We plan to make the content of Proposition 2 more explicit for the special class $\{k_n\}$ of positive definite kernels on $\mathbf{D}$ defined by $k_n(z,w) = (1 - z\bar{w})^{-n}$. (The kernel k_1 is the Szegö kernel and k_2 is, modulo the factor $1/\pi$, the Bergman kernel. The importance of the class $\{k_n\}$ for the bounded subnormal operator theory was highlighted in [**1**] and the appropriate generalizations of k_n have been explored in [**2**] and [**3**]). First, however, we address the question of computing the measure $d\mu(x)$ associated with $\{n!/a_n\}$, where a_n is as in Proposition 2. For this, we need the theory of Stieltjes-Mellin transforms as expounded in [**24**, Chapter VI]. If $d\alpha(x)$ is the measure associated with $\{a_n\}$ for some nondecreasing function α on $[0,\infty)$ and $\beta(y) = -\exp(-y)$ on $[0,\infty)$, then $n!/a_n = \int_0^\infty x^n\,d\alpha(x) \int_0^\infty y^n\,d\beta(y)$. If

$$\alpha'(t) = -\alpha(\exp(-t)) \quad \text{and} \quad \beta'(u) = -\beta(\exp(-u)),$$

then

$$n!/a_n = \int_{-\infty}^{\infty} \exp(-nt)\,d\gamma'(t),$$

where

$$\gamma'(t) = \int_{-\infty}^{\infty} \alpha'(t-u)\beta'(u)\,du$$

is the Stieltjes resultant of α' and β'. ($\int_{-\infty}^{\infty} |d\alpha'(t)| < \infty$ forces $\alpha'(-\infty)$ to be finite; hence we may assume $\alpha'(-\infty) = 0$ and use [**24**, Chapter VI, Theorem 10]). Thus, we get $n!/a_n = \int_{0,\infty} x^n\,d\gamma(x)$ by using the substitution $x = \exp(-t)$; and in fact $\int_{[0,\infty)} x^n\,d\gamma(x)$ can be interpreted as the Cauchy limit $\int_{0+}^{\infty} x^n\,d\gamma(x)$, where $\gamma(x) = -\gamma'(\log x^{-1})$.

We now find out the measures $d\mu_p(x)$ associated with $M(k_p!)$, $p \geq 2$. Note that, modulo constant factors, $k_p(z,w) = \sum_{n=0}^{\infty} a_n^{(p)} z^n \bar{w}^n$, where $1/a_n^{(p)} = \int_0^1 x^n (1-x)^{p-2}\,dx$. Thus $n!/a_n^{(p)} = \int_0^\infty y^n \exp(-y)\,dy \int_0^1 x^n (1-$

$x)^{p-2}\,dx$, and using the substitution $(x,y) = (\exp(-s), \exp(-t))$, one has

$$\begin{aligned} n!/a_n^{(p)} &= \int_{-\infty}^{\infty} \exp(-\exp(-t))\,dt \int_0^{\infty} \exp(-n(s+t)) \\ &\quad (1-\exp(-s))^{p-2}\exp(-s-t)\,ds \\ &= \int_{-\infty}^{\infty} \exp(-\exp(-t))\,dt \int_t^{\infty} \exp(-nu) \\ &\quad (1-\exp(t-u))^{p-2}\exp(-u)\,du \\ &= \int_{-\infty}^{\infty} \exp(-nu)\biggl(\int_{-\infty}^{u} \exp(-\exp(-t)) \\ &\quad (1-\exp(t-u))^{p-2}\,dt\biggr)\exp(-u)\,du \\ &= \int_{(0,\infty)} x^n \biggl(\int_{-\infty}^{-\log x} \exp(-\exp(-t)) \\ &\quad (1-x\exp(t))^{p-2}\,dt\biggr)\,dx; \end{aligned}$$

so that

$$d\mu_p(x) = \biggl(\int_{-\infty}^{-\log x} \exp(-\exp(-t))(1-x\exp(t))^{p-2}\,dt\biggr)\,dx.$$

It is easy to see that the weighted shift operator with weights

$$\left\{\sqrt{a_n^{(p)}/a_{n+1}^{(p)}}\right\}$$

can be identified with $M(k_p!)$ in $\mathcal{H}(k_p!)$, where $\mathcal{H}(k_p!)$ is the Hilbert space obtained by completing the set of analytic polynomials in $L^2(\mathbf{C}, \nu_p)$ with $d\nu_p(r\exp(i\theta)) = d\mu_p(r^2)\exp(i\theta)/2\pi$. □

The study of Toeplitz operators on the spaces $\mathcal{H}(k_p!)$, $p \geq 2$, along the lines of [**7**] seems desirable. Here we only note that the analog of Theorem 5 in [**7**] is true for the spaces $\mathcal{H}(k_p!)$ and illustrate the required argument for the case $p = 2$. If φ is in $L^\infty(\mathbf{C}, \nu_2)$, then the Toeplitz operator T_φ on $\mathcal{H}(k_2!)$ is defined by $T_\varphi f = P(\varphi f)$, $f \in \mathcal{H}(k_2!)$, where P denotes the projection of $L^2(\mathbf{C}, \nu_2)$ onto $\mathcal{H}(k_2!)$.

Theorem 1. *If φ is in $L^\infty(\mathbf{C}, \nu_2)$ has compact support, then T_φ is a compact operator.*

Proof. Note that $\{e_n = (n!/(n+1))^{-1/2} z^n\}$ is an orthonormal sequence in $\mathcal{H}(k_2!)$. If φ in $L^\infty(\mathbf{C}, \nu_2)$ is such that $|\varphi| \le K$ a.e. $[\nu_2]$ and $\varphi(z) = 0$ for $|z| > a$; and if $\langle .,. \rangle$ denotes the inner product of $\mathcal{H}(k_2!)$, then

$$\begin{aligned}\langle T_\varphi e_k, e_j\rangle &= (k!j!/((k+1)(j+1)))^{-1/2} < P(\varphi z^k), z^j\rangle \\ &= (k!j!/((k+1)(j+1)))^{-1/2} \int_{\mathbf{C}} \varphi(z) z^k \bar{z}^j \, d\nu_2(z),\end{aligned}$$

so that

$$\begin{aligned}|\langle T_\varphi e_k, \bar{e}_j\rangle| \le{}& (k!j!/((k+1)(j+1)))^{-1/2} K a^{k+j} \\ & \int_{(0,\infty)} \left(\int_{-\infty}^{-\log r^2} \exp(-\exp(-t))\, dt \right) d(r^2) \\ ={}& (k!j!/((k+1)(j+1)))^{-1/2} K a^{k+j},\end{aligned}$$

since $\int_{(0,\infty)} (\int_{-\infty}^{-\log r^2} \exp(-\exp(-t))\, dt)\, d(r^2) = (n!/n+1)_{n=0} = 1$. But then $\sum_{k,j} |\langle T_\varphi e_k, e_j\rangle|^2$ is easily seen to be finite, leading to T_φ being compact. □

Since the multiplication operator $M(k_1!)$ in the Bargmann space has a natural connection with the differential operator $x - d/dx$ acting on the linear span of Hermite functions (see [**7, 11** and **22**]), it is relevant to ask whether $M(k_p!)$, $p \ge 2$, is related in a natural way to some differential operator acting on an appropriate family of orthogonal functions in $L^2(\mathbf{R}, \mu_p)$. The answer to this question might also illustrate the physical significance, if any, of the operators $M(k_p!)$. To provide a stronger motivation for the above comments and the need to consider the association $k \to k!$ in general, the reader is referred to the confluent hypergeometric equation (see [**15**])

$$\text{(A)} \qquad z d^2u/dz^2 + (c - z)\, du/dz - au = 0,$$

where we assume that a and c are positive integers, u being a function of z. One of the linearly independent solutions of (A) is given by

$$\Phi(a, c; z) = (\Gamma(c)/\Gamma(a)) \sum_{n=0}^{\infty} (\Gamma(a+n)/(\Gamma(c+n)\Gamma(n+1))) z^n,$$

where Γ denotes the Gamma function.

Note that the positive definite kernels $k_p!$, $p \geq 1$, are precisely the Kummer functions $\Phi(p,1,z)$, $p \geq 1$, with z replaced by $z\bar{w}$! The author admittedly lacks in his knowledge of the theory of special functions to pursue this topic any more at this juncture.

The reader is reminded that two densely defined operators S and T in $\mathcal{H}$ and $\mathcal{K}$, respectively, are said to be quasisimilar if there exist bounded linear operators $A : \mathcal{H} \to \mathcal{K}$ and $B : \mathcal{K} \to \mathcal{H}$ such that A and B are injective and have dense ranges, and such that $AS \subset TA$ and $BT \subset SB$ [**18**]. Note that quasisimilarity is an equivalence relation on the class of densely defined operators. The operators $M(k_p!)$ discussed above are seen to determine disjoint quasisimilarity orbits. For example, consider the operators $M(k_1!)$ and $M(k_2!)$. If B is a bounded linear operator from $\mathcal{H}(k_2!)$ to $\mathcal{H}(k_1!)$ and $\{e_n^{(1)}\}$, $\{e_n^{(2)}\}$ are orthonormal bases in $\mathcal{H}(k_1!)$, $\mathcal{H}(k_2!)$, respectively, then we may write $Be_n^{(2)} = \sum_{j=0}^{\infty} \alpha_{n,j} e_j^{(1)}$, $n \geq 0$. The relationship $BM(k_2!) \subset M(k_1!)B$ would force

$$((n+1)/\sqrt{n+2}) \sum \alpha_{n+1,j} e_j^{(1)} = \sum \sqrt{j+1}\alpha_{n,j} e_{j+1}^{(1)},$$

which in turn would imply that $\alpha_{n,0} = 0$ for all $n \geq 1$ and $\alpha_{n+1,n+1} = \sqrt{n+2}\alpha_{0,0}$. If B is to be bounded, one must have $\alpha_{0,0} = 0$ as well and this shows that B cannot be one-to-one. The reader may similarly consider any $M(k_p!)$ and $M(k_q!)$, $p \neq q$. The gist of the arguments here seems to lie in the asymptotic behavior of the solutions of Equation (A) (see [**15**]) and suggests that the questions related to the intertwining of unbounded multiplication operators in functional Hilbert spaces associated with positive definite kernels on $\mathbf{C}^m$ may be answered in a meaningful way by some analogs of Theorem 4.6 in [**8**].

3. Subnormality of a tuple of operators. The positive definite kernel $\exp(z_1\bar{w}_1 + \cdots + z_m\bar{w}_m)$ on $\mathbf{C}^m$ corresponding to a^+ has the coefficients $\{1/(n_1!n_2!\cdots n_m!)\}$ in its series expansion, the reciprocals whereof form a multidimensional Stieltjes moment sequence in an obvious way. More interestingly, one may consider $\{1/(n_1+\cdots+n_m)!\}$ and the associated kernel $\sum_{n_i\geq 0, 1\leq i\leq m} z^n\bar{w}^n/(n_1+\cdots n_m)!$ on $\mathbf{C}^m$. Note that $\{(n_1+\cdots+n_m)!\}$ is a Stieltjes moment sequence; for example, $(n_1+n_2)! = \int_{[0,\infty)^2} x_1^{n_1}x_2^{n_2} \exp(-(x_1+x_2)/2)\, d\mu(x_1,x_2)$, where $d\mu(x_1,x_2)$ is the linear measure on the diagonal $x_1 = x_2$ in the positive quadrant of $\mathbf{R}^2$. In view of our discussion in the preceding section about the significance of the kernels $k_p!$, one would like to form interesting subnormal tuples out of $M(k_p!)$ by means other than that of simply tensoring $M(k_p!)$. In this context, Theorem 2 and Corollary 1 below are relevant. The proof of Theorem 2 combines a result of Nussbaum [**17**] with the arguments in [**22**] and makes the content of Theorem 10 in [**21**] more transparent for the case of a multivariable weighted shift; it also relates the notion of joint subnormality for a class of multivariable weighted shifts, of which a^+ is prototypical, to the (multidimensional) Stieltjes moment problem.

Lemma 1. *For $m > 1$, let $\{\alpha(n_1, \dots, n_m)\}$ be a multisequence of real numbers satisfying*

$$\text{(B)} \qquad \sum_{n_i=1}^{\infty} 1/\alpha(0, \dots, 0, n_i, 0, \dots, 0)^{1/2n_i} = \infty \quad \textit{for every } i.$$

Then $\{\alpha(n_1, \dots, n_m)\}$ is a Stieltjes moment sequence if and only if

$$\text{(C)} \qquad \sum_{0 \le i_k, j_k \le p_k, 1 \le k \le m} \alpha(i_1 + j_1 + \varepsilon_1, \dots, i_m + j_m + \varepsilon_m) c_{i_1, i_2, \dots, i_m} \bar{c}_{j_1, j_2, \dots, j_m} \ge 0$$

for all possible choices of a finite sequence of complex numbers $c_{i_1, i_2, \dots, i_m}$ $(0 \le i_k \le p_k,\ 1 \le k \le m)$ and for any choice $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$ in $\{0, 1\}^m$.

Remark. Condition (C) may be described by saying that $\{\alpha(n_1, \dots, n_m)\}$ is a completely positive definite sequence (see [**6**]).

Proof of Lemma 1. If $\{\alpha(n_1, \dots, n_m)\}$ is a Stieltjes moment sequence with a corresponding measure $d\mu(x)$, then the left-hand side of (C) is equal to

$$\int_{[0,\infty)^m} \Big| \sum_{0 \le i_k \le p_k, 1 \le k \le m} c_{i_1, i_2, \dots, i_m} x_1^{i_1 + \eta_1} x_2^{i_2 + \eta_2} \cdots x_m^{i_m + \eta_m} \Big|^2 d\mu(x),$$

where η_i are some nonnegative integers (taking values 1/2 or 0), and is clearly nonnegative.

Conversely, suppose (C) is satisfied. We illustrate the proof for the case $m = 2$. Define a new sequence $\{\beta(n_1, n_2)\}$ as follows: $\beta(2n_1, 2n_2) = \alpha(n_1, n_2)$ for all (n_1, n_2) in $\mathbf{N}^2$ and $\beta(m_1, m_2) = 0$ if (m_1, m_2) is not equal to $(2n_1, 2n_2)$ for any (n_1, n_2) in $\mathbf{N}^2$. We now establish that $\{\beta(n_1, n_2)\}$ is positive definite; that is, sums such as

$$\sum_{0\le i_k, j_k\le m_k, 1\le k\le 2} \beta(i_1+j_1, i_2+j_2) c_{i_1,i_2}\bar{c}_{j_1,j_2}$$

are nonnegative (cf. [**24**, Theorem 13a, Chapter III]).

We treat here the case of m_1 being odd and m_2 being even. The other three cases can be handled similarly. Thus, one has,

$$\begin{aligned}
&\sum_{0\le i_k, j_k\le m_k, 1\le k\le 2} \beta(i_1+j_1, i_2+j_2) c_{i_1,i_2}\bar{c}_{j_1,j_2} \\
&= \sum_{i_1,j_1=0}^{(m_1-1)/2} \sum_{i_2,j_2=0}^{m_2/2} \beta(2i_1+2j_1, 2i_2+2j_2) c_{2i_1,2i_2}\bar{c}_{2j_1,2j_2} \\
&\quad + \sum_{i_1,j_1=0}^{(m_1-1)/2} \sum_{i_2,j_2=0}^{(m_2/2)-1} \beta(2i_1+2j_1, 2i_2+2j_2+2) c_{2i_1,2i_2+1}\bar{c}_{2j_1,2j_2+1} \\
&\quad + \sum_{i_1,j_1=0}^{(m_1-1)/2} \sum_{i_2,j_2=0}^{m_2/2} \beta(2i_1+2j_1+2, 2i_2+2j_2) \\
&\quad c_{2i_1+1,2i_2+1}\bar{c}_{2j_1+1,2j_2+1} \\
&= \sum_{i_1,j_1=0}^{(m_1-1)/2} \sum_{i_2,j_2=0}^{m_2/2} \alpha(i_1+j_1, i_2+j_2) c_{2i_1,2i_2}\bar{c}_{2j_1,2j_2} \\
&\quad + \sum_{i_1,j_1=0}^{(m_1-1)/2} \sum_{i_2,j_2=0}^{(m_2/2)-1} \alpha(i_1+j_1, i_2+j_2+1) c_{2i_1,2i_2+1}\bar{c}_{2j_1,2j_2+1} \\
&\quad + \sum_{i_1,j_1=0}^{(m_1-1)/2} \sum_{i_2,j_2=0}^{m_2/2} \alpha(i_1+j_1+1, i_2+j_2) c_{2i_1+1,2i_2}\bar{c}_{2j_1+1,2j_2} \\
&\quad + \sum_{i_1,j_1=0}^{(m_1-1)/2} \sum_{i_2,j_2=0}^{(m_2/2)-1} \alpha(i_1+j_1+1, i_2+j_2+1) c_{2i_1+1,2i_2+1}\bar{c}_{2j_1+1,2j_2+1},
\end{aligned}$$

and the last expression is seen to be nonnegative in view of (C). The generalization to the case $m > 2$ should now be obvious. Also, in view of (B), we have

$$\sum_{n_i=1}^{\infty} 1/\beta(0,\dots,0,2n_i,0,\dots,0)^{1/2n_i} = \sum_{n_i=1}^{\infty} 1/\alpha(0,\dots,0,n_i,0,\dots,0)^{1/2n_i} = \infty.$$

But then it follows from a result of Nussbaum (see the remark preceding Theorem 10 of [**17**]) that $\{\beta(n_1,\dots,n_m)\}$ is a Hamburger moment sequence. Hence there exists a positive measure $d\mu(x)$ on $\mathbf{R}^m$ such that

$$\beta(n_1,\dots,n_m) = \int_{\mathbf{R}^m} x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}\,d\mu(x_1,\dots,x_m)$$

for all $(n_1,\dots,n_m)$ in $\mathbf{N}^m$. Hence we have $\alpha(n_1,\dots,n_m) = \beta(2n_1,\dots,2n_m) = \int_{\mathbf{R}^m} x_1^{2n_1}\cdots x_m^{2n_m}\,d\mu(x_1,\dots,x_m)$ and, using the transformation $(y_1,\dots,y_m) = (x_1^2,\dots,x_m^2)$, $\alpha(n_1,\dots,n_m) = \int_{[0,\infty)^m} y_1^{n_1}\cdots y_m^{n_m}\,d\nu(y_1,\dots,y_m)$ for some positive measure $d\nu(y_1,\dots,y_m)$. □

Corollary 1. *If for $m > 1$, $\{\alpha(n_1,\dots,n_m)\}$ is a multisequence satisfying* (B) *and* (C)*, then*

$$\{\beta(n_1,\dots,n_m) = \alpha(0,\dots,0,k(L_1),0,\dots,0,k(L_2),0,\dots,0,k(L_p),0,\dots,0)\}$$

is a Stieltjes moment sequence, where $k(L_j)$, occurring in the L_j'th coordinate place, is the sum $\sum_{i\in A_j} n_i$, $\{A_1,\dots,A_p\}$ being a fixed partition of $\{1,2,\dots,m\}$.

Proof. Let $\{\alpha(n_1,\dots,n_m)\}$ satisfy (B) and (C). Then clearly $\{\beta(n_1,\dots,n_m)\}$ satisfies (B). By Lemma 1, $\{\alpha(n_1,\dots,n_m)\}$ is a Stieltjes moment sequence so that

$$\beta(n_1,\dots,n_m) = \int_{[0,\infty)^m} x_{L_1}^{k(L_1)}x_{L_2}^{k(L_2)}\cdots x_{L_p}^{k(L_p)}\,d\mu(x_1,\dots,x_m)$$

for some positive measure $d\mu(x_1, \dots, x_m)$, and it easily follows from this that $\{\beta(n_1, \dots, n_m)\}$ satisfies condition (C) as well. $\square$

Theorem 2. *Let $T = (T_1, \dots, T_m)$, $m > 1$, be a multivariable weighted shift on the linear span L of an orthonormal basis $\{e_\alpha\}_{\alpha \in \mathbf{N}^m}$ of $\mathcal{H}$, such that $e_{0,\dots,0}$ is a quasi-analytic vector for T; that is,*

$$\text{(D)} \qquad \sum_{n=1}^{\infty} 1/||T_i^n e_{0,\dots,0}||^{1/n} = \infty$$

for each i, where $||\cdot||$ denotes the norm of $\mathcal{H}$. Then the following are equivalent.

(a) *T is subnormal.*

(b) *T satisfies the multidimensional Halmos-Bram conditions; that is,*

$$\text{(E)} \qquad \sum_{0 \le i_k, j_k \le p_k, 1 \le k \le m} \langle T^i f_j, T^j f_i \rangle \ge 0$$

for all possible choices of a finite sequence $\{f_j = f_{j_1,\dots,j_m}\}$ $(0 \le j_k \le p_k,\ 1 \le k \le m)$ of vectors in L, and where $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathcal{H}$.

(c) *$\{||T^n e_{0,\dots,0}||^2\}_{n \in \mathbf{N}^m}$ is a (multidimensional) Stieltjes moment sequence.*

Proof. That (a) implies (b) is easy to verify. Suppose now that (b) holds. Letting

$$f_{i_1,\dots,i_m} = c_{i_1,\dots,i_m} T_1^{i_1+\varepsilon_1} \cdots T_m^{i_m+\varepsilon_m} e_{0,\dots,0},$$

where $(\varepsilon_1, \dots, \varepsilon_m) \subset \{0,1\}^m$, and substituting in (E) we see that the sequence $\{||T^n e_{0,\dots,0}||^2\}_{n \in \mathbf{N}^m}$ satisfies condition (C) of Lemma 1. In view of hypothesis (D), condition (B) in Theorem 1 is also satisfied by $\{||T^n e_{0,\dots,0}||^2\}$. Hence (c) holds.

Now suppose (c) is true. Then, as in the proof of Theorem 4 in [**22**], we can consider

$$\begin{aligned}\langle T^p e_{0,\dots,0}, T^q e_{0,\dots,0} \rangle \\ &= \delta(p_1, q_1)\delta(p_2, q_2) \cdots \delta(p_m, q_m) ||T^p e_{0,\dots,0}||^2 \\ &= \delta(p_1, q_1) \cdots \delta(p_m, q_m) \int_{[0,\infty)^m} x^p \, d\mu(x),\end{aligned}$$

where $\delta(k,l)=1$ if $k=l$ and $\delta(k,l)=0$ if $k\neq l$. Clearly, we have

$$\langle T^p e_{0,\dots,0}, T^q e_{0,\dots,0}\rangle = \delta(p_1,q_1)\cdots\delta(p_m,q_m)\int_{[0,\infty)^m} x_1^{(p_1+q_1)/2}\cdots x_m^{(p_m+q_m)/2}\,d\mu(x).$$

Set

$$\alpha(n_1,\dots,n_m)=\int_{[0,\infty)^m} x_1^{n_1/2}\cdots x_m^{n_m/2}\,d\mu(x).$$

Then it is easy to see that $\{\alpha(n_1,\dots,n_m)\}$ satisfies condition (C) of Lemma 1. Note also that $\sum 1/\alpha(0,\dots,2n_i,\dots,0)^{1/2n_i} = \sum 1/ ||T_i^{n_i}e_{0,\dots,0}||^{1/n_i} = \infty$. Hence it follows from the remark preceding Theorem 10 of [**17**] that $\{\alpha(n_1,\dots,n_m)\}$ is a Hamburger moment sequence. If $d\nu(x)$ is the corresponding positive measure, then

$$\begin{aligned}\langle T^p e_{0,\dots,0}, T^q e_{0,\dots,0}\rangle &= \delta(p_1,q_1)\dots\delta(p_m,q_m)\int_{\mathbf{R}^m} x^{(p+q)/2}\,d\mu(x)\\ &= \int_{\mathbf{R}^m} x^{p+q}\,d\nu(x)\int_{\mathbf{T}^m} z^{p-q}\,d\varphi,\end{aligned}$$

where $\mathbf{T}^m$ is the unit polycircle in $\mathbf{C}^m$, $d\varphi$ is the normalized product arc-length measure $(d\theta_1\otimes\cdots\otimes d\theta_m)/(2\pi)^m$ on $\mathbf{T}^m$ and $p-q$ denotes $(p_1-q_1,\dots,p_m-q_m)$. Thus, one has

$$\langle T^p e_{0,\dots,0}, T^q e_{0,\dots,0}\rangle = \langle M_x^{p+q}1_\mu, 1_\mu\rangle_\mu \langle M_z^{p-q}1_\varphi, 1_\varphi\rangle_\varphi,$$

where M_x denotes the multiplication tuple in $L^2(\mathbf{R}^m,\mu), \langle .,.\rangle_\mu$ the inner product of $L^2(\mathbf{R}^2,\mu)$ and 1_μ the constant function 1 in $L^2(\mathbf{R}^2,\mu)$; similar interpretations holding for M_z, $\langle .,.\rangle_\varphi$ and 1_φ with reference to $L^2(\mathbf{T}^m,\varphi)$. This leads to

$$\begin{aligned}\langle T^p e_{0,\dots,0}, T^q e_{0,\dots,0}\rangle &= \langle \prod_{i=1}^m (M_{x_i}\otimes M_{z_i})^{p_i}(1_\mu\otimes 1_\varphi),\\ &\quad \prod_{i=1}^m (M_{x_i}\otimes M_{z_i})^{q_i}(1\mu\otimes 1_\varphi)\rangle_{\mu\otimes\varphi},\end{aligned}$$

where $\otimes$ denotes the tensor product and allows us to identify each T_i with $M_{x_i} \otimes M_{z_i}/S$, where S is the linear space generated by $1_\mu \otimes 1_\varphi$ in $L^2(\mathbf{R}^m \otimes \mathbf{T}^m, \mu \otimes \varphi)$. Since $(M_{x_1} \otimes M_{z_1}, \dots, M_{x_m} \otimes M_{z_m})$ is a normal tuple of operators in $L^2(\mathbf{R}^m \otimes \mathbf{T}^m, \mu \otimes \varphi)$, the desired conclusion follows. $\square$

Using Corollary 1 and Theorem 2 above, one can easily build new Stieltjes moment sequences out of the ones associated with $M(k_p!)$ of Section 1. Thus, $\{(n_1 + \cdots + n_m)!\}$, $\{(n_1 + \cdots + n_m)!/(n_1 + \cdots + n_m + 1)\}$, $\{(n_1 + n_2)!(n_3 + n_4)!/(n_3 + n_4 + 1)\}$ are examples of Stieltjes moment sequences associated with $M(k_1!)$ and $M(k_2!)$. The study of multivariable weighted shifts associated with such multisequences will hopefully shed more light on the structure theory of unbounded subnormals.

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