ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 1, Winter 1995

STABILITY PROPERTY AND PHASE SPACE

JUNJI KATO

1. Introduction. Recently Murakami and Yoshizawa [**16**] have discussed the relationship between the BC-stability and the ρ-stability in a class of functions bounded by a priori bound for a functional differential equation defined on a phase space X with a seminorm $||\cdot||_X$. The BC-stability means that the solution remains small if the initial function is small with respect to the BC-norm, $|\cdot|_{(-\infty,0]}$, while the ρ-stability corresponds to the ρ-metric:

$$\rho(\varphi) := \sum_{k=1}^{\infty} 2^{-k} \frac{|\varphi|_{[-k,0]}}{1+|\varphi|_{[-k,0]}},$$

where $|\varphi|_I := \sup_{s\in I} |\varphi(s)|$ for an interval I.

The situation above is rather complex; there appear three metrics, and the restriction to the class of functions bounded by a bound will be observed to effect on these metrics.

The purpose of this paper is to clarify the relationships between these metrics and to give a unified aspect on the concepts of the stability by allowing more flexibility in the choice of the phase space. Haddock and Hornor [**7**] have introduced the concept of the H-stability related with a fading memory subspace H of X, see the latter, Example 3, for the definition. Our idea will show that this turns out to be a problem of the choice of the suitable phase space.

Consider the equation

$$\text{(E)} \qquad \dot{x}(t) = f(t, x_t),$$

where $f(t,\varphi)$ is defined and continuous on $[0,\infty)\times X$ for a phase space X. Then it will be easier to see the existence of a solution in a space with a weaker topology if $f(t,\varphi)$ endows an adequate regularity there. However, the weaker the topology of the space is, the more meager

Received by the editors on September 24, 1992, and in revised form on January 4, 1993.

the set of continuous functions on the space becomes. For example, any linear function $f(\varphi)$ on CC, that is, the space of continuous functions $C((-\infty,0],R^n)$ with the ρ-metric is not continuous unless $f(\varphi)=f(\psi)$ if $\varphi(s)\equiv\psi(s)$ on $[-h,0]$ for an $h>0$, which asserts that the phase space can be reduced to $C([-h,0],R^n)$, the space of continuous functions defined on $[-h,0]$ with the uniform norm $|\cdot|_{[-h,0]}$, while

$$f(t,\varphi)=\int_{-\infty}^{0} k(t-s)\varphi(s)\,ds$$

is continuous on $[0,\infty)\times BC$ if $k\in L^1[0,\infty)\cap C[0,\infty)$.

The space CC is one of the examples with a weak topology, while the space of bounded continuous functions, BC with the BC-norm is a typical example with a rather strong topology.

There is no doubt that if $f(t,\varphi)$ in (E) is completely continuous on $[0,\infty)\times CC$, then a solution exists for given initial condition in CC and remains in CC as long as it exists. On the other hand, Seifert [**18**] has presented an example which shows that even if $f(t,\varphi)$ is completely continuous on $[0,\infty)\times BC$, no solution exists for some initial function in BC. However, in this case, if the initial function belongs to $BUC\subset BC$, the space of uniformly continuous functions, then the solution exists and it stays in BUC as long as it exists (cf. [**5, 6, 15**]). These facts show us that the choice of the phase space is very important.

The concept of the stability related with two metrics is also considered. Recently, similar problems have been discussed by several authors, cf. [**13, 14**] and their references. However, our notation is deeply related with the choice of the phase space and different from theirs. One of our motivations is to give a unified explanation to the fact: why the boundedness condition on the righthand side of the equation could be dropped in the result of Burton [**1**] based on a Liapunov functional.

2. Phase space. Axiomatic approaches for the phase space have been considered by several authors [**4, 5, 6, 8, 12, 17**], etc., and summarized in [**9**]. However, mainly they have restricted the phase space to be a (semi-) normed space. This feature prevents CC from being a phase space (cf. [**10**]) or requires special considerations when supplemental conditions are posed on the initial functions (cf. [**6**]).

Extending the axioms set on the normed phase space, we shall give the following definitions. First of all, the following notations will be utilized throughout the paper:

$x_t(s)$ $:= x(t+s)$ for $s \le 0$;

$|\cdot|$: is a norm in R^n;

$\mathcal{U}_+^m$: is the set of neighborhoods of 0 in the first quadrant $R_+^m := \{(u^1, u^2, \dots, u^m) : u^1, u^2, \dots, u^m \ge 0\}$;

$X_\tau C$: is the set of functions x defined on $(-\infty, \infty)$ such that $x_\tau \in X$ and $x|_{[\tau,\infty)}$ is continuous;

$X_\tau C^L$, $X_\tau C_M, X_\tau C_M^L \subset X_\tau C$: the lower suffix M indicates that $|x|_{[\tau,\infty)} \le M$, while the upper suffix L means that $|x(t) - x(s)| \le L|t-s|$ for all $t, s \ge \tau$;

$\mathcal{K}$: is the class of strictly increasing, continuous functions $p(u)$ defined on a $U \in \mathcal{U}_+^1$ and satisfying $p(0) = 0$;

$\mathcal{LK}^m$: is the class of continuous functions $P(t,u)$ defined on $[0,\infty) \times U$ for a $U \in \mathcal{U}_+^m$, nonincreasing in t, strictly increasing in each of $u^k, k = 1, 2, \dots, m$ and $P(t, 0) = 0$ for all $t \ge 0$;

and we shall say that the sequence φ^k is *compactly convergent* to a φ on I if $\varphi^k(s)$ is convergent to $\varphi(s)$ uniformly on any compact subset of I.

Let (X, μ) be a subset of R^n-valued functions defined on $(-\infty, 0]$ with an invariant (pseudo) metric μ; $\mu(0) = 0$, $0 \le \mu(\varphi)$ and

$$\mu(\varphi + \psi) \le \mu(\varphi) + \mu(\psi) \quad \text{if } \varphi, \psi, \varphi + \psi \in X.$$

Then (X, μ) is said to be a *phase space* if the following basic axioms are satisfied:

(A) $0 \in X$, and for any $x \in X_\tau C_M$, with some M, we have

(i) $x_t \in X$ for all $t \ge \tau$;

(ii) $p(|x(t)|) \le \mu(x_t) \le P(t - \tau, |x|_{[\tau,t]}, \mu(x_\tau))$ for some $p \in \mathcal{K}$, $P \in \mathcal{LK}^2$ and all $t \ge \tau$;

(iii) x_t is continuous in $t \ge \tau$;

(B) If φ^k is a Cauchy sequence in (X, μ) and compactly convergent to a φ, then $\varphi \in X$ and $\mu(\varphi^k - \varphi) \to 0$ as $k \to \infty$.

The following proposition says that this provides a natural extension of the concept of the usual phase space with the norm-setting

Proposition 1 [**12**]. *If the phase space* (X, μ) *is a norm space, then* $p(u)$ *and* $P(t, u, v)$ *in* (A) (ii) *can be linear in* u *and in* (u, v)*, respectively.*

A phase space is said to admit a *uniform fading memory*, or simply a *UFM*, if the function $P(t, u, v)$ in (A) (ii) satisfies

$$P(t, u, v) \to 0 \quad \text{as } t \to \infty \quad \text{and } u \to 0,$$

which is equivalent to assuming

$$\text{(C)} \qquad P(t, 0, v) \to 0 \quad \text{as } t \to \infty,$$

and we can state the following proposition

Proposition 2. *The phase space* (X, μ) *admits a* UFM *if and only if, for any* $v > 0$ *and* $\varepsilon > 0$, $\varepsilon \leq \varepsilon_0$*, there exists a* $T(\varepsilon, v) \geq 0$ *such that*

$$\mu(x_t) < \varepsilon \quad \text{if } t \geq \tau + T(\varepsilon, \mu(x_\tau))$$

when $x \in X_\tau C$ *with* $x|_{[\tau,\infty)} = 0$.

Proof. The "only if" part is obvious. Now we shall prove the "if" part. First of all, it will be noted that we may assume that $T(\varepsilon, v)$ is decreasing in ε, nondecreasing in v and tends to ∞ as $\varepsilon \to +0$.

Let $x \in X_\tau C$ be given, and define

$$y(t) := \begin{cases} x(t) & t \leq \tau \\ x(\tau)(1 + \tau - t) & \tau \leq t \leq \tau + 1 \\ 0 & \tau + 1 \leq t, \end{cases}$$
$$z(t) := x(t) - y(t).$$

Then $y \in X_{\tau+1}C$, $y|_{[\tau+1,\infty)} = 0$ and

$$\begin{aligned}\mu(y_t) &\le P(0, |x(\tau)|, \mu(x_\tau))\\ &\le P(0, p^{-1}(\mu(x_\tau)), \mu(x_\tau))\\ &=: q(\mu(x_\tau))\end{aligned}$$

for all $t \ge \tau$, and hence $q \in \mathcal{K}$ and we have

$$\begin{aligned}\mu(y_t) &\le T_\varepsilon^{-1}(t-\tau-1, \mu(y_{\tau+1}))\\ &\le T_\varepsilon^{-1}(t-\tau-1, q(\mu(x_\tau)))\end{aligned}$$

for $t \ge \tau+1$, where $T_\varepsilon^{-1}(t,v)$, defined for $t > T(\varepsilon_0, v)$ is the inverse function of $T(\varepsilon, v)$ with respect to ε, that is, the function such that $T(T_\varepsilon^{-1}(t,v), v) = t$, and obviously we may assume that $T_\varepsilon^{-1}(t,v) \in \mathcal{LK}$ and $T_\varepsilon^{-1}(t,v) \to 0$ as $t \to \infty$. On the other hand, $z \in X_\tau C$ and $\mu(z_t) \le P(0, |x|_{[\tau,t]}, 0)$ for all $t \ge \tau$.

Therefore, we have

$$\begin{aligned}\mu(x_t) &\le \mu(y_t) + \mu(z_t)\\ &\le P(0, |x|_{[\tau,t]}, 0) + T_\varepsilon^{-1}(t-\tau-1, q(\mu(x_\tau)))\end{aligned}$$

for all $t \ge \tau + 1 + T(\varepsilon_0, q(\mu(x_\tau)))$, while

$$\mu(x_t) \le P(0, |x|_{[\tau,t]}, 0) + q(\mu(x_\tau))$$

for all $t \ge \tau$, which imply the existence of $P(t,u,v)$ satisfying the axioms (A) (ii) and (C). □

For the phase space with UFM we can state the following proposition, which is well known in the case of norm-setting.

Proposition 3. *Suppose that* (X, μ) *is a phase space with* UFM. *Then*:

(i) *the set* $\Gamma(t, Y, M, L) := \{x_s; x \in Y_0 C_M^L, s \ge t\}$ *is relatively compact for any compact set* $Y \subset X$*, any* $t \ge 0$ *and any positive constants* M *and* L;

(ii) *the set* $\Gamma^\infty(Y, M, L) := \cap_{t\ge 0}\overline{\Gamma(t, Y, M, L)}$ *is compact for any bounded set* $Y \subset X$ *and any positive constants* M *and* L.

3. Examples. The above definition is no more than a simple modification of the usual definition, and it will not be difficult to see that the fundamental theorems on the functional differential equations can be verified even for the equations defined on a phase space presented here by the standard arguments. However, by extending the norm-setting to the metric-setting, we can obtain a systematical view in the theory of functional differential equations and unify the various observations on stability.

Example 1. Consider the space (CC, ρ) of continuous functions on $(-\infty, 0]$ with the compact open topology, that is, metrized by the metric:

$$\rho(\varphi) := \sum_{k=1}^{\infty} 2^{-k} \frac{|\varphi|_{[-k,0]}}{1 + |\varphi|_{[-k,0]}}.$$

First of all, for any $\varepsilon > 0$ choose $m = m(\varepsilon)$ so that $\sum_{k=m}^{\infty} 2^{-k} < \varepsilon/2$, while for given $x \in CC_\tau C$ and $t \geq \tau$ choose $\delta = \delta(\varepsilon, x, t) > 0$, $\delta < 1$, so that $|x(r) - x(s)| < \varepsilon/2$ if $|r - s| < \delta$ and $r, s \in [t - m - 1, t + 1]$. Therefore, (A) (iii) will be verified by

$$\rho(x_{t*} - x_t) < \sum_{k=m}^{\infty} 2^{-k} + \max_{-m \leq s \leq 0} |x(t^* + s) - x(t + s)| < \varepsilon$$

if $|t^* - t| < \delta$.

On the (A) (ii) we may set

$$p(u) := \frac{u}{1 + u}, \qquad P(t, u, v) := u + 2^{1-t} v.$$

In fact, we have

$$\sum_{k=1}^{\infty} 2^{-k} \frac{|x|_{[t-k,t]}}{1 + |x|_{[t-k,t]}} \leq \sum_{k > t - \tau} 2^{-k} \frac{|x|_{[t-k,\tau]}}{1 + |x|_{[t-k,\tau]}} + |x|_{[\tau,t]}.$$

The first term of the righthand side is equivalent to or less than

$$\sum_{k=1}^{\infty} 2^{1-(t-\tau+k)} \frac{|x|_{[\tau-k,\tau]}}{1 + |x|_{[\tau-k,\tau]}} = 2^{1-(t-\tau)} \rho(x_\tau).$$

Thus, the space (CC, ρ) is a phase space and admits a UFM.

Example 2. In [**15**] the property

(∗) if $\varphi^k \in X \cap BC, \sup_k |\varphi^k|_{(-\infty,0]} < \infty$, compactly converges to a φ on $(-\infty, 0]$, then $\varphi \in X$ and $\mu(\varphi^k - \varphi) \to 0$ as $k \to \infty$.

is assumed instead of (B) as a basic axiom for a phase space (X, μ), and it is thought of as an independent axiom from the UFM. However, we have the following

Proposition 4. *Let (X, μ) be a phase space. Then, for any $M > 0$ the space (X^*, μ), $X^* := \{\varphi \in X \cap BC; |\varphi|_{(-\infty,0]} \le M\}$, admits a* UFM *if and only if the axiom* (∗) *holds.*

Proof. Suppose that the axiom (∗) does not imply a UFM for (X^*, μ). Then there are sequences $t_k \ge \tau_k \ge 0$ and $x^k \in X^*_{\tau_k}C$ such that $t_k - \tau_k \ge k$, $x|_{[\tau_k,\infty)} = 0$, $\mu(x^k_{\tau_k}) \le v$ and $\mu(x^k_{t_k}) \ge \varepsilon$ for positive constants v and ε. Set $\varphi^k := x^k_{t_k}$. Then obviously φ^k converges compactly to 0 on $(-\infty, 0]$, and hence $\mu(\varphi^k) < \varepsilon$ for large k by (∗), which yields a contradiction.

Conversely, let (X^*, μ) admit a UFM. Then, by the standard argument we can know that $X^* \supset BC$ and $\mu(\varphi) \le P(0, |\varphi|_{(-\infty,0]}, 0)$ for a $\varphi \in BC$. Therefore, we can choose $T(\varepsilon) \ge 0$ and $\eta(\varepsilon) > 0$ for $\varepsilon > 0$ so that $P(t, u, P(0, 2M, 0)) < \varepsilon$ if $t \ge T(\varepsilon)$ and $u < \eta(\varepsilon)$, and hence $\varphi, \psi \in X^*$ satisfies $\mu(\varphi - \psi) < \varepsilon$ if $|\varphi - \psi|_{[-T(\varepsilon),0]} < \eta(\varepsilon)$. Now let $\varphi^k \in X^*$ be a sequence which compactly converges to a φ on $(-\infty, 0]$. Obviously, $\varphi \in X^*$, and $|\varphi^k - \varphi|_{[-T(\varepsilon),0]} < \eta(\varepsilon)$ for large k implies that $\mu(\varphi^k - \varphi) \to 0$. □

Example 3 (cf. [7]). Let (X, μ) be a phase space. The metric space (Y, ν), $Y \subset X$, is said to be an admissible subspace if for any $x \in Y_\tau C$ we have

(i) $x_t \in Y$ for all $t \ge \tau$;

(ii) $\mu(x_t) \le Q(t - \tau, |x|_{[\tau,t]}, \mu(x_\tau), \nu(x_\tau))$ for a $Q \in \mathcal{LK}^3$,

and it is said to be a uniform fading memory subspace if, in addition, we have

$$Q(t,u,v,w) \to 0 \quad \text{as } t \to \infty \quad \text{and } u \to 0 \text{ in (ii)}.$$

Suppose that (Y,ν) is a uniform fading memory subspace of a phase space (X,μ). Then we can see that the space (X^*,μ), $X^* := \{\varphi \in Y; \nu(\varphi) \le M\}$ admits a UFM for an M. In fact, we have

$$\begin{aligned}\mu(x_t)^2 &\le P(t-\tau, |x|_{[\tau,t]}, \mu(x_\tau))Q(t-\tau, |x|_{[\tau,t]}, \mu(x_\tau), \nu(x_\tau)) \\ &\le P(0, |x|_{[\tau,t]}, \mu(x_\tau))Q(t-\tau, |x|_{[\tau,t]}, \mu(x_\tau), M)\end{aligned}$$

for any $x \in X^*_\tau C$, which shows that (X^*,μ) admits a UFM since $P^*(t,u,v) \to 0$ as $t \to \infty$ and $u \to 0$, where

$$P^*(t,u,v) := \{P(0,u,v)Q(t,u,v,M)\}^{1/2}.$$

Example 4. Let $g, h : (-\infty, 0] \to [1,\infty)$ be continuous, nonincreasing functions, and set

$$X := \{\varphi : \mu(\varphi) < \infty, \nu(\varphi) \le M\}$$

for a fixed M, where

(i) $\mu(\varphi) := \sup_{s\le 0}(|\varphi(s)|/g(s))$, $\nu(\varphi) := \sup_{s\le 0}(|\varphi(s)|/h(s))$; or

(ii) $\mu(\varphi) := |\varphi(0)| + \int_{-\infty}^0 (|\varphi(s)|/g(s))\, ds$, $\nu(\varphi) := \int_{-\infty}^0 (|\varphi(s)|/h(s))\, ds$

with the condition

$$\text{(g)} \qquad \int_{-\infty}^0 \frac{1}{g(s)}\, ds < \infty;$$

or

(iii) $\mu(\varphi) := |\varphi(0)| + \int_{-\infty}^0 (|\varphi(s)|/g(s))\, ds$, $\nu(\varphi) := \int_{-\infty}^0 (|\varphi(s)|/h(s))^p\, ds$ for a $p > 1$ with the condition (g).

First of all, it is not difficult to see that

$$\mu(x_t) \le K|x|_{[\tau,t]} + \mu(x_\tau)$$

for $x \in X^*_\tau C$, $X^* := \{\varphi : \mu(\varphi) < \infty\}$, under one of the conditions (i), (ii) and (iii), where $K := 1$ for (i) and $K := 1 + \int_{-\infty}^0 (1/g(s))\, ds$ for (ii) and (iii).

(1^0) If $M(t) := \sup_{s \le -t}(h(s+t)/g(s)) \to 0$ as $t \to \infty$, then (X, μ) admits a UFM when μ and ν are given by (i) or (ii). In fact, for $x \in Y_\tau C$, $Y := \{\varphi : \nu(\varphi) < \infty\}$, we have

$$\mu(x_t) \le K|x|_{[\tau,t]} + M(t-\tau)\nu(x_\tau).$$

(2^0) If $M(t) := \int_{-\infty}^0 (g(s)h(s)/g(s-t)^2)^{p/(p-1)}\, ds \to 0$ as $t \to \infty$, then (X, μ) admits a UFM when μ and ν are given by (iii), because

$$\mu(x_t) \le K|x|_{[\tau,t]} + M(t-\tau)^{(p-1)/2p}\mu(x_\tau)^{1/2}\nu(x_\tau)^{1/2p}.$$

4. Stability. Consider the equation

$$\text{(E)} \qquad \dot{x}(t) = f(t, x_t)$$

with a completely continuous function $f(t, \varphi)$ defined on a phase space (X, μ), and let

$$d : X \to [0, \infty)$$

with the domain

$$X^d := \{\varphi \in X : d(\varphi) < \infty\}$$

an invariant metric, that is,

$$d(0) = 0, \qquad 0 \le d(\varphi), \qquad d(\varphi + \psi) \le d(\varphi) + d(\psi)$$

for $\varphi, \psi, \varphi + \psi \in X^d$.

Then the zero solution of (E) is said to be (d, R^n)-*stable*, if for any $\varepsilon > 0$ and $\tau \ge 0$ there exists a $\delta(\varepsilon, \tau) > 0$ such that

$$\text{(S)} \qquad d(x_\tau) < \delta(\varepsilon, \tau) \quad \text{implies} \quad |x(t)| < \varepsilon \text{ for all } t \ge \tau,$$

and (d, R^n)-(*equi*) *asymptotically stable*, if in addition to (S) there exist $\gamma(\tau) > 0$ and $T(\varepsilon, \tau) \ge 0$ for given $\varepsilon > 0$ and $\tau \ge 0$, such that

$$\text{(AS)} \qquad d(x_\tau) < \gamma(\tau), \quad t \ge \tau + T(\varepsilon, \tau) \quad \text{imply} \quad |x(t)| < \varepsilon$$

when x is a solution of (E). In a usual manner, we can also introduce the concepts of (d, R^n)-*uniform* (or *total*, or *uniformly asymptotic*) *stability*.

The assertions (S) and (AS) can be rewritten in

$$\text{(S)} \qquad d(x_\tau) < \delta(\varepsilon, \tau) \quad \text{implies} \quad |x|_{[\tau,\infty)} < \varepsilon,$$

$$\text{(AS)} \qquad d(x_\tau) < \gamma(\tau), t \geq \tau + T(\varepsilon, \tau) \quad \text{imply} \quad |x|_{[t,\infty)} < \varepsilon,$$

respectively.

Generally, let

$$\lambda(\cdot; \tau, t) : X_\tau^d C|_{(-\infty,t]} \to [0, \infty)$$

be a functional for any $\tau \geq 0$ and $t \geq \tau$ such that $\lambda(x; \tau, t)$ is continuous, nondecreasing in $t \geq \tau$ for each $x \in X_\tau^d C$ and $\lambda(\cdot; \tau, \infty) := \lim_{t\to\infty} \lambda(\cdot; \tau, t)$. Then we can introduce the concept of the (d, λ)-*stabilities* by replacing $|x|_{[t,\infty)}$ in (S) and (AS) with $\lambda(x; t, \infty)$, that is,

$$(S_\lambda) \qquad d(x_\tau) < \delta(\varepsilon, \tau) \quad \text{implies} \quad \lambda(x; \tau, \infty) < \varepsilon;$$

and

$$(AS_\lambda) \qquad d(x_\tau) < \gamma(\tau), \quad t \geq \tau + T(\varepsilon, \tau) \quad \text{imply} \quad \lambda(x; t, \infty) < \varepsilon.$$

Put $\lambda^R(x; \tau, t) := |x|_{[\tau,t]}$. Then the (d, λ^R)-stability is not different from the (d, R^n)-stability. The (d, λ^d)-stability, $\lambda^d(x; \tau, t) := \sup_{\tau \leq s \leq t} d(x_s)$, may be called as (d, d)-*stability*.

Remark. In [**13, 14**] and their references, the concepts of the stability in two measures have been discussed, which corresponds to the notion

$$d(\tau, x_\tau) < \delta(\varepsilon, \tau) \quad \text{implies} \quad d^*(t, x_t) < \varepsilon \quad \forall\, t \geq \tau$$

and so on, where $d(t, \cdot)$ and $d^*(t, \cdot)$ are metrics on the phase space. Our concepts are a different concept in general.

The pair (d,λ) is said to admit a *UFM* (*on* $X^d_\tau C$) if

(D) $x_t \in X^d$ and $d(x_t) \le P^*(t-\tau, \lambda(x;\tau,t), d(x_\tau))$ for $x \in X^d_\tau C$, all $t \ge \tau$ and for a $P^* \in \mathcal{LK}^2$ with $P^*(t,u,v) \to 0$ as $t \to \infty$ and $u \to 0$

is satisfied.

Then we have the following

Proposition 5. *If the pair* (d,λ) *admits a UFM, then* (d,λ)*-stabilities imply* (d,d)*-stabilities.*

Proof. For example, let the zero solution of (E) be (d,λ)-asymptotically stable, namely, there are $\delta(\varepsilon,\tau) > 0$, $\gamma(\tau) > 0$ and $T(\varepsilon,\tau) \ge 0$, for given $\varepsilon > 0$ and $\tau \ge 0$, satisfying (S_λ) and (AS_λ) when x is a solution of (E).

Let $\varepsilon > 0$ be given. Choose $\eta(\varepsilon) > 0$ so that

$$u < \eta(\varepsilon) \quad \text{and} \quad v < \eta(\varepsilon) \quad \text{imply} \quad P^*(0,u,v) < \varepsilon.$$

Then we have

$$d(x_\tau) < \min\{\eta(\varepsilon), \delta(\eta(\varepsilon),\tau)\} \quad \text{implies} \quad d(x_t) < \varepsilon \text{ for } t \ge \tau.$$

Now choose $T^*(\varepsilon) \ge 0$ and $\eta^*(\varepsilon) > 0$ so that

$$P^*(t,u,\alpha) < \varepsilon \quad \text{if } t \ge T^*(\varepsilon) \quad \text{and } u < \eta^*(\varepsilon)$$

for a fixed $\alpha > 0$. Hence we can see that $d(x_t) < \varepsilon$ if $d(x_\tau) < \min\{\eta(\alpha), \delta(\eta(\alpha),\tau), \gamma(\tau)\}$ and $t \ge \tau + T^*(\varepsilon) + T(\eta^*(\varepsilon),\tau)$ because

$$d(x_t) \le P^*(t-\sigma, \lambda(x;\sigma,t), P^*(0, \lambda(x;\tau,\sigma), d(x_\tau)))$$

for the $\sigma = \tau + T(\eta^*(\varepsilon),\tau)$ and $P^*(0,\lambda(x;\tau,\sigma), d(x_\tau)) < \alpha$. $\square$

Obviously, if $d(\varphi) := \mu(\varphi)$ for $\varphi \in X$ and $\lambda(x;\tau,t) := |x|_{[\tau,t]}$, then the property (D) is equivalent to requiring that (X,μ) admits a UFM. In order to introduce various concepts of the stability, it will be interesting to consider various functionals λ. For example, we have the following

Example 5. The pair (d,λ) admits a UFM on $X_\tau C_M$ if

(i) $d(\varphi) := p(|\varphi(0)|) + W(\int_{-\infty}^{0} e^{\gamma s}\alpha(|\varphi(s)|)\,ds)$ and $\lambda(x;\tau,t) := q(|x(t)|) + \int_{\tau}^{t}\beta(|x(s)|)\,ds$; or

(ii) $d(\varphi) := \inf_{-\xi\le u\le 0}\{p(|\varphi(u)|)+\int_{-\infty}^{u} e^{\gamma s}\alpha(|\varphi(s)|)\,ds\}$ and $\lambda(x;\tau,t) := \int_{\tau}^{t}\beta(|x(s)|)\,ds$.

The pair (d,λ) admits a UFM on $X_\tau C^L$ if

(iii) $d(\varphi) := \sup_{s\le 0} e^{\gamma s}|\varphi(s)|$ and $\lambda(x;\tau,t) := \int_{\tau}^{t}\beta(|x(s)|)\,ds$.

Here and hereafter $\gamma,\xi > 0$ are constants and $p,q,W,\alpha,\beta\in\mathcal{K}$.

Consider the case (i). First of all, we shall show that for an $x\in X_\tau C_M$, we have

$$\int_{-h}^{0} e^{\gamma s}\alpha(|\varphi(s)|)\,ds \le \theta\biggl(\int_{-h}^{0}\beta(|\varphi(s)|)\,ds\biggr) \tag{1}$$

for a $\theta\in\mathcal{K}$ and all h. Let $J := \{s\in[-k,0] : \alpha(|\varphi(s)|)\ge\varepsilon\}$ for a k and an ε, which will be suitably chosen. Then,

$$\begin{aligned} A &:= \int_{-h}^{0} e^{\gamma s}\alpha(|\varphi(s)|)\,ds \\ &\le \alpha(M)m(J) + e^{-k\gamma}\alpha(M)/\gamma + \varepsilon(k - m(J)), \end{aligned}$$

where the second term in the righthand side will not appear when $k\ge h$ and $m(J)$ denotes the Lebesgue measure of the set J, which implies that $m(J)\ge\{A - e^{-k\gamma}\alpha(M)/\gamma - \varepsilon k\}/(\alpha(M)-\varepsilon)$, while we have

$$\int_{-h}^{0}\beta(|\varphi(s)|)\,ds \ge \beta(\alpha^{-1}(\varepsilon))m(J).$$

Hence we can choose a $\theta\in\mathcal{K}$ so that (1) holds by setting $k := \log[3\alpha(M)/\gamma A]/\gamma$ and $\varepsilon := \min\{\alpha(M)/2, A/3k\}$. Therefore,

$$\begin{aligned} d(x_t) &\le p(|x(t)|) + W\biggl(\theta\biggl(\int_{\tau}^{t}\beta(|x(s)|)\,ds\biggr) \\ &\qquad\qquad + e^{\gamma(\tau-t)}\int_{-\infty}^{0} e^{\gamma s}\alpha(|x(\tau+s)|)\,ds\biggr) \\ &\le r(\lambda(x;\tau,t), e^{\gamma(\tau-t)}\,d(x_\tau)), \end{aligned}$$

where $r\in\mathcal{K}$ is given by $r(u,v) := p(q^{-1}(u)) + W(\theta(u)+v)$.

Now consider the case (ii). Since we have

$$\int_{-h}^{0} \beta(|\varphi(s)|)\,ds \ge \beta(\inf{}_{-\xi\le s\le 0}|\varphi(s)|)\xi \quad \text{if } h \ge \xi,$$

we have

$$\begin{aligned} d(x_t) &\le \inf_{-\xi\le u\le 0} p(|x(t+u)|) + W\biggl(\theta\biggl(\int_\tau^t \beta(|x(s)|)\,ds\biggr) \\ &\qquad + e^{\gamma(\tau-t)}\int_{-\infty}^{u} e^{\gamma s}\alpha(|x(\tau+s)|)\,ds\biggr) \\ &\le r(\lambda(x;\tau,t), e^{\gamma(\tau-t)}\,d(x_\tau)) \end{aligned}$$

by setting $r(u,v) := p(\beta^{-1}(u/\xi)) + W(\theta(u)+v)$.

Finally, consider the case (iii). Let $x \in X_\tau C^L$, and put

$$A := \sup_{\tau-t\le s\le 0} e^{\gamma s}|x(t+s)| =: e^{\gamma\sigma}|x(t+\sigma)|$$

for a given $t > \tau$. Then $|x(t+\sigma)| \ge A$, and $|x(t+s)| \ge A/2$ if $|s-\sigma| \le A/2L$. Thus, we have

$$\int_\tau^t \beta(|x(s)|)\,ds \ge \beta(A/2)\cdot\min\{t-\tau, A/2L\}.$$

On the other hand, if $A \ge 2e^{\gamma(\tau-t)}|x(\tau)|$, then

$$\begin{aligned} A &= e^{\gamma\sigma}|x(t+\sigma)| \\ &\le |x(t+\sigma)| \le |x(\tau)| + L|t+\sigma-\tau| \\ &\le |x(\tau)| + L|t-\tau| \\ &\le e^{\gamma(t-\tau)}A/2 + L|t-\tau|, \end{aligned}$$

which implies that $t-\tau \ge \xi(A)$, where $\xi(A)$ is a solution of

$$L\xi + e^{\gamma\xi}A/2 = A.$$

Obviously, $\xi \in \mathcal{K}$. Therefore, we can see that

$$d(x_t) \le \theta^{-1}\biggl(\int_\tau^t \beta(|x(s)|)\,ds\biggr) + 2e^{\gamma(\tau-t)}\,d(x_\tau),$$

where θ is defined by $\theta(A) := \beta(A/2)\min\{\xi(A), A/2L\}$ and obviously which belongs to $\mathcal{K}$.

Comparing the stabilities associated with the choice of the functional λ, we have the following proposition.

Proposition 6. *Let d be an invariant distance on X and*

$$\lambda(\cdot;\tau,t), \lambda^*(\cdot;\tau,t) : X^d_\tau C|_{(-\infty,t]} \longrightarrow [0,\infty)$$

be functionals, and assume that

$$\lambda^*(x;\tau,t) \geq a(\lambda(x;\tau,t)) \tag{2}$$

when $t-\tau \geq b(\lambda(x;\tau,t))$ and

$$b(\lambda(x;\tau,t)) \leq c(t-\tau, d(x_\tau)) \tag{3}$$

if x is a solution of (E) *defined on $[\tau,t]$, where $a \in \mathcal{K}$, $b \in \mathcal{K}$ with $b(u) \leq B$ or $b(u) \equiv 0$ on a $U \in \mathcal{U}^1_+$, and c is a nondecreasing continuous function such that $c(0,0) = 0$.*

(i) *Then the (d,λ^*)-(uniform, asymptotic) stabilities for the zero solution imply the (d,λ)-stability of the same category.*

(ii) *Suppose that*

$$\lambda^*(x;\tau,t) \geq \lambda^*(x;\tau,s) + \lambda^*(x;s,t) \tag{4}$$

if $\tau \leq s \leq t$ and that the pair (d,λ) satisfies the relation (D). *If the zero solution is (d,λ^*)-uniformly stable, then it is (d,λ)-uniformly asymptotically stable.*

Remark. By setting $\lambda^* = \lambda$ in (ii) and referring to Proposition 5, we have

(ii*) If λ satisfies the relation (4) and if the pair (d,λ) admits the axiom (D), then the zero solution of (E) is (d,d)-uniformly asymptotically stable whenever it is (d,λ)-uniformly stable.

Proof. The assertion (i) is obvious when $b(u) \equiv 0$ on a $U \in \mathcal{U}^1_+$. Suppose that $b \in \mathcal{K}$, and assume that the zero solution of (E) is (d,λ^*)-stable, that is, there exists a $\delta \in \mathcal{K}$ for which $(S_{\lambda *})$ holds. Now, by the relation (3) we can choose $\sigma, \eta \in \mathcal{K}$ so that

$$b^{-1}(c(t,u)) < \varepsilon \quad \text{if } t \leq \sigma(\varepsilon) \quad \text{and } u \leq \eta(\varepsilon) \tag{5}$$

for any $\varepsilon > 0$. Let $d(x_\tau) < \min\{\eta(\varepsilon), \delta(a(\gamma(\varepsilon)))\}$, where $\gamma(\varepsilon)$ is so small that $\gamma(\varepsilon) < \varepsilon$ and $b(\gamma(\varepsilon)) < \sigma(\varepsilon)$. First of all, the relation (2) shows that

$$a(\lambda(x;\tau,t)) \le \lambda^*(x;\tau,t)) < a(\gamma(\varepsilon)), \tag{6}$$

that is, $\lambda(x;\tau,t) < \gamma(\varepsilon)$ for all $t \ge \tau + B$. Suppose that there is a $t^* \in [\tau + b(\gamma(\varepsilon)), \tau + B)$ for which $\lambda(x;\tau,t^*) = \gamma(\varepsilon)$. Then, again, the relation (2) implies (6) at $t = t^*$ because $t^* - \tau \ge b(\gamma(\varepsilon)) = b(\lambda(x;\tau,t^*))$, which yields a contradiction. Since $\lambda(x;\tau,t)$ is continuous in t, this implies that $\lambda(x;\tau,t) < \gamma(\varepsilon)$ for all $t \ge \tau + b(\gamma(\varepsilon))$. On the other hand, $b(\lambda(x;\tau,t)) \le c(t-\tau, d(x_\tau))$ by (3), and hence $\lambda(x;\tau,t) < \varepsilon$ for all $t \in [\tau, \tau + \sigma(\varepsilon)]$ and, hence, for all $t \in [\tau, \tau + b(\gamma(\varepsilon))]$. Therefore, the zero solution of (E) is (d,λ)-stable. In order to see the (d,λ)-asymptotic stability, no problem will arise if $T(\varepsilon,\tau)$ in (AS_λ) is greater than B.

Suppose that the hypotheses in (ii) hold. The (d,λ^*)-uniform stability implies the (d,λ)-uniform stability by (i) and, hence, the (d,d)-uniform stability under the condition (D) by Proposition 5. Hence, there is a $\delta(\varepsilon) > 0$, for $\varepsilon > 0$, such that

$$d(x_t) < \varepsilon \quad \forall\, t \ge \tau \quad \text{if } d(x_\tau) < \delta(\varepsilon).$$

Especially, we can choose a $\gamma > 0$ so that

$$d(x_\tau) < \gamma \quad \text{implies} \quad d(x_t) < v \quad \text{and} \quad \lambda^*(x;\tau,t) < v$$

for all $t \ge \tau$ and a given $v > 0$.

Let x be a solution of (E) satisfying $d(x_\tau) < \gamma$, and let $\varepsilon > 0$ be given. Choose $T(\varepsilon) \ge 0$ so that P^* in (D) satisfies

$$P^*(T(\varepsilon), 0, v) < \delta(\varepsilon) \quad \text{and} \quad T(\varepsilon) \ge \max\{B, b^{-1}(a^{-1}(v))\}.$$

Then we can find an $\alpha(\varepsilon) > 0$ so that

$$\lambda(x;\sigma,\sigma + T(\varepsilon)) \ge a^{-1}(\alpha(\varepsilon))$$

if $d(x_t) \ge \delta(\varepsilon)$ on $[\sigma, \sigma + T(\varepsilon)]$.

Now, assume that $d(x_t) \geq \delta(\varepsilon)$ on $[\tau, \tau + mT(\varepsilon)]$. Then, we have

$$\begin{aligned} v > \lambda^*(x; \tau, \tau + mT(\varepsilon)) &\geq \sum_{k=1}^{m} \lambda^*(x; \tau + (k-1)T(\varepsilon), \tau + kT(\varepsilon)) \\ &\geq \sum_{k=1}^{m} a(\lambda(x; \tau + (k-1)T(\varepsilon), \tau + kT(\varepsilon))) \\ &\geq m\alpha(\varepsilon) \end{aligned}$$

since $T(\varepsilon) \geq B \geq b(\lambda(x; \tau + (k-1)T(\varepsilon), \tau + kT(\varepsilon)))$ for any $k \leq m$, which yields a contradiction for $m > v/\alpha(\varepsilon)$. Therefore, there is a $\sigma \in [\tau, \tau + vT(\varepsilon)/\alpha(\varepsilon)]$ such that $d(x_\sigma) < \delta(\varepsilon)$, and hence $d(x_t) < \varepsilon$ for all $t \geq \sigma$, especially for all $t \geq \tau + vT(\varepsilon)/\alpha(\varepsilon)$. □

Remark. If the relations (2) and (3) are certified for the solutions of the perturbed equations under consideration with small perturbations, then obviously the assertion (i) holds even for the total stability.

Consider the case where

$$\lambda(x; \tau, t) := \int_\tau^t \alpha(|x(s)|)\, ds \quad \text{for an } \alpha \in \mathcal{K},$$

which is a typical example satisfying the property (4). The (d, λ)-stability is called (d, L_α)-stability, especially (d, L^1)-stability if $\alpha(u) = u$. The (d, R^n)-stability and the (d, L^1)-stability are different concepts. However, we have the following:

Corollary. *Let $d(\varphi) = \mu(\varphi)$, and assume that there is an $L(u)$ for which $|f(t, \varphi)| \leq L(\mu(\varphi))$, and let (X, μ) admit a UFM. If the zero solution of* (E) *is (d, L_α)-uniformly stable for an $\alpha \in \mathcal{K}$, then it is (d, d)-uniformly asymptotically stable.*

Proof. In order to prove this corollary it is sufficient to show that the relations (2) and (3) hold for

$$\lambda^*(x; \tau, t) := \int_\tau^t \alpha(|x(s)|)\, ds$$

and

$$\lambda(x;\tau,t) := |x|_{[\tau,t]},$$

that is, we shall show that

$$\lambda^*(x;\tau,t) \geq \beta(|x|_{[\tau,t]}) \quad \text{if } t-\tau \geq \gamma(|x|_{[\tau,t]}) \tag{7}$$

and

$$|x|_{[\tau,t]} \leq c(t-\tau, \mu(x_\tau)) \tag{8}$$

for some β, $\gamma \in \mathcal{K}$ and a nondecreasing continuous function c such that $c(0,0) = 0$.

Since every solution x with $\mu(x_t) \leq \varepsilon_0$ for an $\varepsilon_0 > 0$ and all $t \geq \tau$ satisfies

$$|x(t) - x(s)| \leq L|t-s| \quad \text{for all } t, s \geq \tau,$$

because $|f(t,\varphi)| \leq L := L(\varepsilon_0)$ if $\mu(\varphi) \leq \varepsilon_0$, we have (7) if we put $\beta(u) := \alpha(u/2)\gamma(u)/2L$ and $\gamma(u) := \min\{1, u/2L\}$, since $|x(s)-x(\sigma)| \leq |x|_{[\tau,t]}/2$ if $|x(\sigma)| = |x|_{[\tau,t]}$ and $|s-\sigma| \leq |x|_{[\tau,t]}/2L$. While we have

$$\begin{aligned} |x|_{[\tau,t]} &\leq |x(\tau)| + L(t-\tau) \\ &\leq p^{-1}(\mu(x_\tau)) + L(t-\tau), \end{aligned}$$

which shows the existence of c for which the relation (8) holds. $\square$

Consider two invariant metrics

$$d, d^* : X \to [0,\infty) \quad \text{with the domains } X^d, X^{d*}.$$

Then we shall say that d^* is *stronger than d relatively in* (X,μ), $d \leq_\mu d^*$, if $\eta(\varepsilon) > 0$ can be chosen for $\varepsilon > 0$ so that for any $\varphi \in X^{d*}$ with $d^*(\varphi) < \eta(\varepsilon)$ there is a $\psi \in X^d$ satisfying

$$\varphi(0) = \psi(0), d(\psi) < \varepsilon \quad \text{and} \quad \mu(\varphi - \psi) < \varepsilon.$$

Obviously, we have $d \leq_\mu d^*$ if there is an $a \in \mathcal{K}$ such that $d(\varphi) \leq a(d^*(\varphi))$ or $d(\varphi) \leq a(\mu(\varphi))$, and also we have

Proposition 7. *The relation $\leq_\mu$ is an order relation.*

Proof. Suppose that $d \leq_\mu d^*$ and $d^* \leq_\mu d^{**}$, and let $\eta(\varepsilon)$ and $\eta^*(\varepsilon)$ be the associated numbers. Then, clearly for a given φ satisfying $d(\varphi) < \eta(\min\{\eta^*(\varepsilon/2), \varepsilon/2\})$ there is a ψ such that $\psi(0) = \varphi(0)$, $d^*(\psi) < \eta^*(\varepsilon/2)$ and $\mu(\varphi - \psi) < \varepsilon/2$, and then there exists a ξ such that $\xi(0) = \psi(0)$, $d^{**}(\xi) < \varepsilon$ and $\mu(\psi - \xi) < \varepsilon/2$, which implies $\mu(\varphi - \xi) \leq \mu(\varphi - \psi) + \mu(\psi - \xi) < \varepsilon$. Thus, we have $d \leq_\mu d^{**}$. □

Example 6. Let (X, μ) be a phase space. Then, the metric $|\cdot|_{(-\infty,0]}$ on the BC-space is obviously stronger than ρ on the CC-space since $\rho(\varphi) \leq |\varphi|_{(-\infty,0]}$. Conversely, on a BC-ball $X^* := \{\varphi \in X \cap BC : |\varphi|_{(-\infty,0]} \leq M\}$ for an $M > 0$ the metric ρ is stronger than $|\cdot|_{(-\infty,0]}$ relatively in (X, μ). In fact, by Proposition 4 there is a $T(\varepsilon) \geq 0$ for which

$$\mu(\varphi) < \varepsilon \quad \text{if } \varphi \in X^* \quad \text{and} \quad \varphi|_{[-T(\varepsilon),0]} = 0$$

by noting that $\mu(\varphi_s) \leq P(0, M, 0)$ for all $s \leq 0$. Therefore, for φ with $\rho(\varphi) < \eta(\varepsilon)$ the function ψ defined by

$$\psi(s) := \begin{cases} \varphi(s) & \text{for } -T(\varepsilon) \leq s \leq 0 \\ \varphi(-T(\varepsilon)) & \text{for } s \leq -T(\varepsilon) \end{cases}$$

satisfies $|\psi|_{(-\infty,0]} < \varepsilon$ and $\mu(\varphi - \psi) < \varepsilon$, where $\eta(\varepsilon) > 0$ is chosen so that $\rho(\varphi) < \eta(\varepsilon)$ implies $|\varphi|_{[-T(\varepsilon),0]} < \varepsilon$.

It is obvious that the BC-ball X^* can be replaced by $\{\varphi \in X \cap BC : \mu(\varphi_s) \leq M \text{ for all } s \leq 0\}$.

Proposition 8. *Suppose that $d^* \leq_\mu d$ and that*

(c) *$f(t, \varphi)$ is uniformly continuous in φ on $[0, \infty) \times U$ for a neighborhood $U \subset X$ of* 0.

If the zero solution of (E) *is (d, λ)-totally stable, then it is (d^*, λ)-totally stable.*

Proof. By the assumptions there is a $\delta(\varepsilon) > 0$ for a given $\varepsilon > 0$ such that $\lambda(y; \tau, t) < \varepsilon$ if $d(y_\tau) < \delta(\varepsilon)$ and $|p|_{[\tau,\infty)} < \delta(\varepsilon)$, where $y(t)$ is a solution of

$$\text{(P)} \qquad \dot{y}(t) = f(t, y_t) + p(t),$$

and also we can find a $\gamma(\varepsilon) > 0$ for which

$$|f(t,\varphi) - f(t,\psi)| < \delta(\varepsilon)/2 \quad \text{if } \mu(\varphi - \psi) < \gamma(\varepsilon).$$

Let $z(t)$ be a solution of

$$\dot{z}(t) = f(t, z_t) + q(t)$$

satisfying $d^*(z_\tau) < \eta(\min\{P_v^{-1}(0,0,\gamma(\varepsilon)), \delta(\varepsilon)\})$ and $|q|_{[\tau,\infty)} < \delta(\varepsilon)/2$, where P is in the axiom (A) (ii) and P_v^{-1} denotes the inverse function of P with respect to the last argument. Then there is a ψ such that $\psi(0) = z(\tau)$, $d(\psi) < \delta(\varepsilon)$ and $\mu(\psi - z_\tau) < P_v^{-1}(0,0,\gamma(\varepsilon))$, and set

$$y(t) := \begin{cases} z(t) & \text{for } t \geq \tau \\ \psi(t-\tau) & \text{for } t \leq \tau. \end{cases}$$

Obviously, $z(t)$ is a solution of (P) with

$$p(t) := q(t) + f(t, z_t) - f(t, y_t),$$

which satisfies $|p|_{[\tau,\infty)} \leq |q|_{[\tau,\infty)} + |f(t,z_t) - f(t,y_t)| < \delta(\varepsilon)$ since $\mu(z_t - y_t) \leq P(0, |z-y|_{[\tau,t]}, \mu(z_\tau - y_\tau)) = P(0,0,\mu(z_\tau - \psi)) < \gamma(\varepsilon)$. Thus, we can conclude that $\lambda(z;\tau,t) = \lambda(y;\tau,t) < \varepsilon$ for all $t \geq \tau$ since $d(y_\tau) = d(\psi) < \delta(\varepsilon)$, namely the zero solution is (d^*, λ)-totally stable. □

Remark. Here we only consider the stability of the zero solution. However, for an arbitrary solution u of (E) the transformation

$$x \to y : x(t) = u(t) + y(t),$$

provides the concept of the stability of u according to the stability of the zero solution of

$$\dot{y}(t) = g(t, y_t),$$

where $g(t,\varphi)$ is defined by

$$g(t,\varphi) := f(t, u_t + \varphi) - f(t, u_t). \tag{9}$$

If $u \in X_\tau C_M^L$ for constants M and L, then $\Gamma(u) := \{u_t : t \geq \tau\}$ is relatively compact by Proposition 3. Hence g retains the same property as f. For example, we can state precisely the following

Proposition 9. *If $f(t,\varphi)$ satisfies the condition*

(c*) *$f(t,\varphi)$ is uniformly continuous in φ on $[0,\infty)\times U$ for a suitable neighborhood U of any $\xi \in X$,*

and if $u : [0,\infty) \to \Gamma$ is continuous for a compact set $\Gamma \subset X$, then the function $g(t,\varphi)$ defined by $g(t,\varphi) := f(t, u(t)+\varphi)$ satisfies the condition (c).

Proof. First of all, we shall prove that the condition (c*) guarantees the case where U in the (c*) can be replaced by a suitable neighborhood of the compact set $\Gamma \subset X$. Let $U(\xi, r(\xi))$ be a neighborhood of ξ mentioned in (c*), and let $\{U(\xi_k, r(\xi_k)/2)\}_{k=1}^N$ be a covering of Γ. Then $U(\Gamma, r)$, $r := \min_{k=1}^N r(\xi_k)/2$ is a desirable neighborhood. Hence, we can choose a $\delta \in \mathcal{K}$ so that

$$|f(t,\varphi) - f(t,\psi)| \leq \varepsilon$$

if $\varphi, \psi \in U(\Gamma, r)$ and $\mu(\varphi - \psi) < \delta(\varepsilon)$, which obviously implies

$$|g(t,\varphi) - g(t,\psi)| \leq \varepsilon$$
$$\text{if } \varphi,\psi \in U(0,r) \quad \text{and} \quad \mu(\varphi-\psi) < \delta(\varepsilon). \qquad \square$$

The following is a simple consequence of Example 6 and Proposition 8.

Corollary (cf. [**16**]). *Consider the equation* (E) *on the space (X^*, μ), $X^* := \{\varphi \in X \cap BC : |\varphi|_{(-\infty,0]} \leq M\}$ for an M. Then, under the condition* (c) *the zero solution of* (E) *is (ρ, R^n)-totally stable if and only if it is $(|\cdot|_{(-\infty,0]}, R^n)$-totally stable.*

5. Liapunov function. The Liapunov second method based on Liapunov functions is a most systematical and effective method in the study of the stability not only as a sufficient condition but also as a necessary condition.

Consider the equation

$$\dot{x}(t) = f(t, x_t) \tag{E}$$

with a completely continuous function $f(t, \varphi)$ defined on a phase space (X, μ). A Liapunov function is a continuous function

$$V : [0, \infty) \times X \longrightarrow [0, \infty)$$

and the derivative of V along the solution of (E) is defined by

$$\dot{V}(t, \varphi) := \overline{\lim_{h \to +0}} \sup_{x} \frac{V(t + h, x_{t+h}) - V(t, \varphi)}{h},$$

where the sup ranges over the solution x of (E) through (t, φ).

The following results are well-known for functional differential equations defined on the phase space $(C([-h, 0], R^n), |\cdot|_{[-h,0]})$, [**1, 2, 12, 19**]. In addition to the conditions

$$a(|\varphi(0)|) \leq V(t, \varphi) \quad \text{and} \quad \dot{V}(t, \varphi) \leq -c(|\varphi(0)|),$$

we assume either

$$|f(t, \varphi)| \leq L(|\varphi|_{[-h,0]}) \quad \text{and} \quad V(t, \varphi) \leq b(|\varphi|_{[-h,0]}); \tag{I}$$

or

$$V(t, \varphi) \leq b_1(|\varphi(0)|) + b_2\left(\int_{-h}^{0} |\varphi(s)|\, ds\right), \tag{II}$$

where $a, b, b_1, b_2, c \in \mathcal{K}$. Then the uniform asymptotic stability can be concluded.

The key ideas in these results will be clarified in the following way.

Proposition 10. *Assume that there exist two Liapunov functions U and V defined on $[0, \infty) \times \Omega$, $\Omega \subset X$ being a neighborhood of* 0*, and satisfying*

$$a(|\varphi(0)|) \leq U(t, \varphi) \leq b(d(\varphi)), \qquad \dot{U}(t, \varphi) \leq 0, \tag{U}$$

and

$$\text{(V)} \qquad 0 \le V(t,\varphi) \le b^*(d^*(\varphi)), \qquad \dot V(t,\varphi) \le -c(|\varphi(0)|),$$

where $a, b, b^*, c \in \mathcal{K}$ *and* d, d^* *are invariant metrics, and suppose that the pair* (d,λ) *satisfies the axiom* (D) *on* $X^d_\tau C_M$*, or on* $X^d_\tau C^L_M$ *if* $|f(t,\varphi)| \le L$*, for* $M > 0$ *and* $L > 0$ *for the functional*

$$\lambda(x;\tau,t) := \int_\tau^t c(|x(s)|)\,ds.$$

Then the zero solution of (E) *is* (d^{**}, R^n)*-uniformly asymptotically stable where* $d^{**}(\varphi) := \max\{d(\varphi), d^*(\varphi)\}$*.*

Remark. The case (I) corresponds to the case of $d(\varphi) := |\varphi|_{[-h,0]}$ while (II) corresponds to the case of $d(\varphi) = b_1(|\varphi(0)|) + b_2(\int_{-h}^0 |\varphi(s)|)$. Refer to Example 5 (iii) and (ii) for the UFM property.

Proof. Obviously the relation (U) implies that the zero solution of (E) is (d, R^n)-uniformly stable, while the relation (V) shows that it is (d^*, λ)-uniformly stable, that is, there are $\delta_U, \delta_V \in \mathcal{K}$ such that

$$(10) \qquad d(x_\tau) < \delta_U(\varepsilon) \quad \text{implies} \quad |x(t)| < \varepsilon$$

and

$$d^*(X_\tau) < \delta_V(\varepsilon) \quad \text{implies} \quad \lambda(x;\tau,t) < \varepsilon$$

for the solutions x and all $t \ge \tau$, from which the (d^{**}, R^n)-uniform stability is obvious. Now we shall prove the (d^{**}, R^n)-uniform asymptotic stability. Put $\gamma := \min\{\delta_U(M), \delta_V(\varepsilon_0)\}$ for an $\varepsilon_0 > 0$. Then we have

$$|x(t)| < M \quad \text{and} \quad d(x_t) \le P^*(0,\varepsilon_0,\gamma) =: v \\ \forall\, t \ge \tau$$

if $d^{**}(x_\tau) < \gamma$, where P^* is the function in (D), and moreover we can assert that there exists $\eta \in \mathcal{K}$ and nonincreasing function $T(\varepsilon)$ such that

$$d(x_t) < \varepsilon \text{ if } t - \sigma \ge T(\varepsilon) \quad \text{and} \quad \lambda(x;\tau,t) \le \eta(\varepsilon)$$

whenever $d(x_\sigma) \le v$ for a $\sigma \ge \tau$ by noting that $|x(t)| < M$ for all $t \ge \tau$. Then, by the same arguments as in the proof of Proposition 6 (ii), we can see that for a given $\varepsilon > 0$ there is a $T^*(\varepsilon) \ge 0$ such that $d(x_\sigma) < \delta_U(\varepsilon)$ for a $\sigma \in [\tau, \tau + T^*(\varepsilon)]$, and hence $|x(t)| < \varepsilon$ for all $t \ge \tau + T^*(\varepsilon)$, that is, the zero solution is (d^{**}, R^n)-uniformly asymptotically stable. □

Remark. The evaluation of the solutions due to a Liapunov function like a V is considered in [**3**].

In Proposition 10 we can replace the property (D) for (d, λ) by the one for $(i(d), \lambda), i(d)(\varphi) := \inf_{-\xi \le s \le 0} d(\varphi_s)$ for a $\xi \ge 0$, when $d(x_t) \le q(d(x_\tau))$ on $X_\tau C_M$ for a q and $t \in [\tau, \tau + \xi]$.

Corollary [**1, 2**]. *Assume that there exists a Liapunov function V satisfying*

$$a(|\varphi(0)|) \le V(t, \varphi) \le b(|\varphi(0)|) + W\left(\int_{-h}^{0} |\varphi(s)|\right) ds\Big)$$

and

$$\dot V(t, \varphi) \le -c(|\varphi(0)|)$$

for an $h > 0$ and $a, b, c, W \in \mathcal{K}$. Then the zero solution is $(C([-h, 0], R^n), R^n)$-uniformly asymptotically stable.

Proof. Noting Example 5 (ii) the conclusion follows immediately from Proposition 10. □

REFERENCES

1. T.A. Burton, *Uniform asymptotic stability in functional differential equations*, Proc. Amer. Math. Soc. **68** (1978), 195–199.

2. ———, *Volterra integral and differential equations*, Math. Sci. Engrg. **167** (1983).

3. ——— and L. Hatvani, *On nonuniform asymptotic stability for nonautonomous functional differential equations*, Differential Integral Equations **3** (1990), 285–293.

4. B.D. Coleman and V.J. Mizel, *On the stability of solutions of functional differential equations*, Arch. Rational Mech. Anal. **30** (1968), 173–196.

5. J.R. Haddock, *Friendly spaces for functional differential equations with infinite delay*, in *Trends in the theory and practice of non-linear analysis*, Elsevier Sci. Publ., New York, 173–182, 1985.

6. ——— and W.E. Hornor, *Precompactness and convergence in norm of positive orbits in a certain fading memory space*, Funkcial. Ekvac. **31** (1988), 349–361.

7. ———, *A uniform asymptotic stability theorem for fading memory space*, J. Math. Anal. Appl. **155** (1991), 55–65.

8. J.K. Hale and J. Kato, *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac. **21** (1978), 11–41.

9. Y. Hino, S. Murakami and T. Naito, *Functional differential equations with infinite delay*, Lecture Notes Math. **1473** (1991).

10. T. Kaminogo, *Continuous dependence of solutions for integro-differential equations with infinite delay*, J. Math. Anal. Appl. **129** (1988), 307–314.

11. F. Kappel and W. Schappacher, *Some considerations to the fundamental theory of infinite delay equations*, J. Differential Equations **37** (1980), 141–183.

12. J. Kato, *Phase space for functional differential equations*, Colloq. Math. Soc. János Bolyai **53** (1988), 307–325.

13. V. Lakshmikantham, S. Leela and A.A. Martynyuk, *Stability analysis of nonlinear systems*, Marcel Dekker, Monticello, 1989.

14. G. Makay, *On the asymptotic stability in terms of two measures for functional differential equations*, Nonlinear Anal. Theor. Meth. Appl. **16** (1991), 721–727.

15. S. Murakami and T. Naito, *Fading memory spaces and stability properties for functional differential equations*, Funkcial. Ekvac. **32** (1989), 91–105.

16. S. Murakami and T. Yoshizawa, *Relationships between* BC*-stabilities and* ρ*-stabilities in functional differential equations with infinite delay*, Tohoku Math. J. **44** (1992), 45–57.

17. K. Schumacher, *Existence and continuous dependence for functional-differential equations with unbounded delay*, Arch. Rational Mech. Anal. **67** (1978), 315–335.

18. G. Seifert, *On Caratheodory conditions for functional differential equations with infinite delays*, Rocky Mountain J. Math. **12** (1982), 615–619.

19. T. Yoshizawa, *Stability theory by Liapunov's second method*, Publication #9, The Mathematical Society of Japan, 1966.

Tohoku University, Mathematical Institute, Sendai 980, Japan.