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QUOTIENTS OF G-STABLE CLOSED SUBSCHEMES, CARTESIAN DIAGRAMS, AND CLOSED IMMERSIONS

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ABSTRACT. Let T be a separated scheme of finite type over an algebraically closed field k, G a finite group acting on T, and $i : S \hookrightarrow T$ the inclusion of a G-stable closed subscheme (i.e., for all $g \in G$, the scheme-theoretic image of S under the composite map $S \overset{i}{\hookrightarrow} T \overset{g_T}{\to} T$ is equal to S). It then follows that S inherits a G-action. If the quotient T/G exists, then so does the quotient S/G; the universal property of the quotient gives rise to a map $i/G : S/G \to T/G$. We ask two questions: Is the commutative square formed by the maps i, i/G, and the quotient maps π_S, π_T, cartesian? Is the map i/G a closed immersion? In case G acts freely on T, we show that both answers are "yes." On the other hand, suppose $T = X^n \times X$ for X a quasiprojective variety, G is a symmetric group on n letters acting by permuting the factors of X^n, and S is a reduced closed subscheme of T supported on the locus whose k-points are all $((t_1, \dots, t_n), t)$ such that $t = t_j$ for some j, $1 \le j \le n$. Then i/G is a closed immersion, but the square is not in general cartesian (but is so when X is a nonsingular curve). This corrects an error in the paper, "The Secant Bundle of a Projective Variety," by R.L.E. Schwarzenberger [**11**].

1. Introduction. We fix an algebraically closed field k. Let T be a k-*scheme*, that is, a separated scheme of finite type over k, and let G be a finite group acting on T; for $g \in G$, we write $g_T : T \to T$ for the associated automorphism of T. Let $i : S \hookrightarrow T$ be the inclusion of a G-stable closed subscheme (i.e., for all $g \in G$, the scheme-theoretic image of S under the composite map $S \overset{i}{\hookrightarrow} T \overset{g_T}{\to} T$ is equal to S). It then follows that S inherits a G-action such that i is a G-morphism (for all $g \in G$, $i \circ g_S = g_T \circ i$). If the quotient T/G exists, then so does the quotient S/G; the universal property of the quotient then gives rise to the map i/G in the following commutative square (in which the vertical

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arrows are the quotient maps):

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "\pi_S"'] & T \arrow[d, "\pi_T"] \\ S/G \arrow[r, "i/G"'] & T/G \end{tikzcd}$$

In this paper we consider two questions concerning this diagram: Is the square cartesian? Is the map i/G a closed immersion? We were led to consider these questions by the discovery of an erroneous assertion in Schwarzenberger's paper [**11**]; it was therein claimed without proof that a particular case of the diagram is cartesian when in fact it is not (see Sections 6 and 7); for this case, the map i/G is a closed immersion. In case G acts freely on T, both questions have affirmative answers; we prove this in Section 5, as well as the very elementary fact that i/G is always a closed immersion in characteristic 0. It is also possible for both answers to be negative, as shown by the following.

Example 1.1. Let $T = \operatorname{Spec}(k[x_1, x_2])$, G the symmetric group on two letters acting by interchanging x_1 and x_2, $S = \operatorname{Spec}(k[x])$, and $i : S \hookrightarrow T$ the diagonal morphism (the comorphism $i^* : k[x_1, x_2] \to k[x]$ is defined by $x_1 \mapsto x$ and $x_2 \mapsto x$). It is clear that S is pointwise fixed (and so G-stable) under the G-action on T; the induced G-action on S is therefore the trivial action, so that $S/G = S$. We have that $T/G = \operatorname{Spec}(k[x_1, x_2]^G) = \operatorname{Spec}(k[\sigma_1, \sigma_2])$, where $\sigma_1 = x_1 + x_2$ and $\sigma_2 = x_1 \cdot x_2$ are the elementary symmetric polynomials in two variables. The diagram above thus corresponds to the following diagram of affine rings:

$$\begin{tikzcd} k[x] & k[x_1, x_2] \arrow[l, "i^*"'] \\ k[x] \arrow[u, "\pi_{S^*} = \mathrm{id}"] & k[\sigma_1, \sigma_2] \arrow[l, "(i/G)^*"] \arrow[u, "\pi_{T^*}"'] \end{tikzcd}$$

Note that $(i/G)^*(\sigma_1) = 2 \cdot x$, $(i/G)^*(\sigma_2) = x^2$; therefore, in characteristic two, $(i/G)^*$ is not surjective, which implies that i/G is not a closed immersion. Using Lemma 7.1, we find that

$$k[x] \otimes_{k[\sigma_1, \sigma_2]} k[x_1, x_2] = k[x, x_1, x_2]/(2x - (x_1 + x_2), x^2 - x_1 x_2);$$

in all characteristics not equal to 2, the latter ring is isomorphic to $k[x_1, x_2]/(x_1 - x_2)^2$, and in characteristic two, it is isomorphic to $k[x, x_1]/(x - x_1)^2$—since neither of these is isomorphic to $k[x] = k[x_1, x_2]/(x_1 - x_2)$, our diagram fails to be cartesian in all characteristics. (I do not know of an example for which the diagram is cartesian and the map i/G is not a closed immersion.)

We now discuss the organization of the paper. We begin in the next section by recalling some needed preliminaries on scheme-theoretic images and radicial maps. We develop some of the basic theory of G-stable subschemes in Section 3; in particular, we give several simple criteria for G-stability. We discuss quotients in Section 4 and establish some general technical propositions needed in the sequel; the first fruits thereof are harvested in Section 5, where, as mentioned above, we prove that our two basic questions have affirmative answers if G acts freely on T. Beginning with Section 6, our focus narrows to (a generalization of) the situation discussed in Schwarzenberger's paper [**11**]: let $T = X^n \times X$ for X a quasiprojective variety, G the symmetric group on n letters acting by permuting the factors of X^n, and S the reduced closed subscheme of T supported on the locus whose k-points are all $((t_1, \dots, t_n), t)$ such that $t = t_j$ for some j, $1 \leq j \leq n$. We study this situation in detail in Section 6; in particular, we compute the quotient S/G and verify that i/G is a closed immersion. In Section 7 we show that Schwarzenberger's general claim of cartesianity is false by direct computation for the case when $X = P_k^2$, the projective plane/k, and $n = 2$. However, all is not lost; we conclude the paper in Section 8 by proving that the diagram is cartesian when X is an irreducible and nonsingular curve.

In addition to those established above, we adopt the following conventions. A *variety* is a reduced and irreducible k-scheme. If $f : Z \to X$ is a map of k-schemes, the image of the k-point $z \in Z$ is denoted by $f(z)$, and the associated map of sheaves is denoted by $\theta_f : \mathcal{O}_X \to f_*(\mathcal{O}_Z)$. If $Z = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine schemes, then the ring map (or comorphism) associated to f is denoted $f^* : B \to A$.

2. Preliminaries. Our purpose in this section is to recall what we need of the theory of scheme-theoretic images and of radicial maps; see [**4**, 6.10, pp. 324–325 and 3.7, pp. 246–249], respectively, for the definitive expositions.

2.1 *Scheme-theoretic images.* Let $f : Z \to X$ be a morphism of k-schemes. Roughly speaking, the *scheme-theoretic image* of f is the smallest closed subscheme Y of X through which f factors; more precisely, Y is the unique closed subscheme of X satisfying the following universal property: f factors as $Z \xrightarrow{f_\circ} Y \overset{i}{\hookrightarrow} X$, where i denotes the inclusion of Y in X, and whenever f factors through a closed immersion $j : V \hookrightarrow X$, then i factors through j as well. This definition is summarized by the following commutative diagram.

$$\begin{tikzcd}
Z \arrow[rr, "f"] \arrow[dr, "f_o"] \arrow[ddr, "f'"'] & & X \\
& Y \arrow[ur, "i"'] \arrow[d, "i'"] & \\
& V \arrow[uur, "j"'] &
\end{tikzcd}$$

Note that the maps f_o, f' and i' are unique, since (closed) immersions are categorical monomorphisms [**4**, 4.2.1, p. 260]; furthermore, i' is a closed immersion, by [**4**, 4.3.6 (iv), p. 265], and f_o is scheme-theoretically dominant (i.e., the sheaf map $\theta_{f_o} : \mathcal{O}_Y \to f_{o*}(\mathcal{O}_Z)$ associated to f_o is injective), by [**7**, Proposition 6.10.5, p. 325].

Y is in fact the closed subscheme of X defined by the kernel $\mathcal{I}_f$ of the map $\theta_f : \mathcal{O}_X \to f_*(\mathcal{O}_Z)$; this kernel is quasi-coherent since $f_*(\mathcal{O}_Z)$ is, Z being noetherian [**5**, Proposition 5.8, p. 15]. (In particular, if $Z = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine, then $Y = \operatorname{Spec}(B/I)$, where I is the kernel of the map of rings $f^* : B \to A$ induced by f.) The underlying space of Y is the closure of the image of Z in X; if Z is reduced, then Y is the reduced closed subscheme of X supported on this locus. Finally, we note that formation of the scheme-theoretic image is transitive: if we compose f with a map $g : X \to X'$ of k-schemes, then the scheme-theoretic image of the composition $g \circ f$ is equal to the scheme-theoretic image of the map $g \circ i : Y \to X'$.

2.2. *Radicial morphisms.* Recall that a morphism of schemes $f : X \to Y$ is called *radicial* provided that f is universally injective; that is, f is injective, and for any base extension $Y' \to Y$, the pullback $f_{Y'} : X \times_Y Y' \to Y'$ is injective. Equivalently, f is radicial provided that, for any field K, the induced map of K-points $X(K) \to Y(K)$ is injective; consequently, all categorical monomorphisms, and therefore all immersions, are radicial [**4**, Proposition 3.7.1, p. 246; Proposition 3.7.3, p. 248]. For maps of k-schemes (recall that k is an algebraically closed field), we have the following simple criterion for radiciality:

Proposition 2.1. *Let $f : X \to Y$ be a map of k-schemes. If f induces an injection $X(k) \to Y(k)$ of k-points, then f is radicial.*

Proof. According to [**4**, Proposition 3.7.1, p. 246], f is radicial if and only if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is surjective, so we seek to establish the latter condition; since Δ_f is a morphism of schemes of finite type/k, it suffices to prove surjectivity on k-points. Consider therefore a k-point (x_1, x_2) of $X \times_Y X$; that is, x_1 and x_2 are k-points of X such that $f(x_1) = f(x_2)$ in Y. Since by hypothesis f induces an injection on k-points, we conclude that $x_1 = x_2 = x$, and therefore $(x_1, x_2) = \Delta_f(x)$. It follows that Δ_f is surjective on k-points, as desired. $\square$

3. G-stable closed subschemes. Let T be a k-scheme, G a finite group acting on T, and $i : S \hookrightarrow T$ the inclusion of a closed subscheme. We say that S is G-stable provided that, for all $g \in G$, we have that S is the scheme-theoretic image of the composite morphism $S \overset{i}{\hookrightarrow} T \overset{g_T}{\hookrightarrow} T$. In this case the universal property of the scheme-theoretic image implies that each map $g_T \circ i$ factors through i by a unique map $g_S : S \to S$, as shown:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "g_S"'] & T \arrow[d, "g_T"] \\ S \arrow[r, "i"'] & T \end{tikzcd}$$

It follows that an action of G on S is induced such that the inclusion map i is a G-morphism. Conversely, one easily sees that if G acts on S such that i is a G-morphism, then S is G-stable.

In the remainder of this section, we gather together a variety of simple criteria for demonstrating that a closed subscheme is G-stable. The situation for *reduced* subschemes is particularly simple:

Proposition 3.1. *Let T be a k-scheme, G a finite group acting on T, and $i : S \hookrightarrow T$ the inclusion of a reduced closed subscheme. Then S is G-stable if and only if S is pointwise stable under G, that is, for each $g \in G$ and each k-point $s \in S$, we have that $g_T(s) \in S$.*

Proof. One direction is obvious: S is G-stable implies that S is pointwise stable under G. Conversely, suppose that S is pointwise stable under G. Then one easily sees that the (closure of the) image of each of the maps $g_T \circ i$ is equal to S; since S is reduced, it follows that the scheme-theoretic image of each of the maps $g_T \circ i$ is again equal to S; that is, S is G-stable. $\square$

It is evident that a G-stable closed subscheme of an affine k-scheme is cut out by a G-stable ideal, and conversely; we record this observation as

Proposition 3.2. *Let $T = \operatorname{Spec}(B)$ be an affine k-scheme, G a finite group acting on T, and $S = \operatorname{Spec}(B/I)$ a closed subscheme of T. Then S is G-stable if and only if $(g_{T^*})^{-1}(I) = I$ for all $g \in G$, where $g_{T^*} : B \to B$ is the map of rings corresponding to the morphism $g_T : T \to T$.*

Proof. Recalling from Section 2.1 that the scheme-theoretic image of a morphism of affine k-schemes is defined by the kernel of the induced map of affine rings, we see that the scheme-theoretic image of $g_T \circ i : S \to T$ is defined by the ideal $(g_{T^*})^{-1}(I)$; the proposition follows immediately from this observation. $\square$

Next we show that the scheme-theoretic image of a G-morphism is G-stable:

Proposition 3.3. *Let $\tilde{S}$ and T be k-schemes on which the finite group G acts, $\phi : \tilde{S} \to T$ a G-morphism and $i : S \hookrightarrow T$ the inclusion of the scheme-theoretic image of ϕ. Then S is a G-stable closed subscheme of T. Moreover, the map $\phi_o : \tilde{S} \to S$, by which ϕ factors through i, is a G-morphism for the induced G-action on S.*

Proof. We begin by showing that S inherits a G-action such that i is a G-morphism; as noted earlier, this is equivalent to the G-stability of S. To do this, we consider the following commutative diagram of solid arrows, in which S' denotes the product $S \times_T T$ with projection maps p_1 and p_2:

$$\begin{tikzcd}
\tilde{S} \arrow[rr, "\phi"] \arrow[dr, "\phi_o"] \arrow[ddr, dashed, "\phi'"'] \arrow[dddd, "g_{\tilde{S}}"'] & & T \arrow[dddd, "g_T"] \\
& S \arrow[ur, "i"'] \arrow[d, dashed, "i'"] & \\
& S' \arrow[uur, "p_2"'] \arrow[d, "p_1"] & \\
& S \arrow[dr, "i"] & \\
\tilde{S} \arrow[ur, "\phi_o"] \arrow[rr, "\phi"'] & & T
\end{tikzcd}$$

The dotted arrow ϕ' is induced by the pair of maps $(\phi, \phi_o \circ g_{\tilde{S}})$, since $g_T \circ \phi = \phi \circ g_{\tilde{S}} = i \circ (\phi_o \circ g_{\tilde{S}})$. The dotted arrow i' is now induced by the universal property of the scheme-theoretic image, since p_2, the pullback of the closed immersion i, is a closed immersion through which ϕ factors. We therefore obtain a (necessarily unique) map $g_S = p_1 \circ i' : S \to S$ satisfying $g_T \circ i = i \circ g_S$. One verifies easily that the maps g_S define a G-action on S (for which i is a G-morphism); whence, S is G-stable. To prove the second assertion, we simply note that, for each $g \in G$,

$$\begin{aligned}
g_T \circ \phi = \phi \circ g_{\tilde{S}} \quad &\Longrightarrow \\
g_T \circ i \circ \phi_o = i \circ \phi_o \circ g_{\tilde{S}} \quad &\Longrightarrow \\
i \circ g_S \circ \phi_o = i \circ \phi_o \circ g_{\tilde{S}} \quad &\Longrightarrow \\
g_S \circ \phi_o = \phi_o \circ g_{\tilde{S}}, \quad &
\end{aligned}$$

since i is a categorical monomorphism. $\square$

Finally, we show that the pullback of a G-stable closed subscheme by a G-morphism is (up to isomorphism) a G-stable closed subscheme. To do this, we need

Lemma 3.1. *Let G be a finite group acting on k-schemes S, T, T' and Z, and let $\phi : S \to T$, $\tau : T' \to T$ be G-morphisms. Then the product $S' = S \times_T T'$ inherits a unique G-action such that the* *projections $p_1 : S' \to S$, $p_2 : S' \to T'$ are G-morphisms. Moreover, given G-morphisms $\alpha : Z \to S$, $\beta : Z \to T'$ such that $\phi \circ \alpha = \tau \circ \beta$, the induced map $(\alpha, \beta) : Z \to S'$ is a G-morphism.*

Proof. For each $g \in G$, we consider the pair of maps $g_S \circ p_1 : S' \to S$, $g_{T'} \circ p_2 : S' \to T'$. Since, as is easily seen, $\phi \circ (g_S \circ p_1) = \tau \circ (g_{T'} \circ p_2)$, the universal property of the product yields a unique map $g_{S'} : S' \to S'$ such that $p_1 \circ g_{S'} = g_S \circ p_1$ and $p_2 \circ g_{S'} = g_{T'} \circ p_2$; furthermore, the uniqueness implies that $g_{S'} \circ h_{S'} = (gh)_{S'}$ for all $g, h \in G$. The first assertion follows immediately, and the second results from a routine verification that $p_i \circ (g_{S'} \circ (\alpha, \beta)) = p_i \circ ((\alpha, \beta) \circ g_Z)$, $i = 1, 2$. □

We may now prove

Proposition 3.4. *Let G be a finite group acting on k-schemes T and T', $\tau : T' \to T$ a G-morphism, and $i : S \hookrightarrow T$ the inclusion of a G-stable closed subscheme. Let S' denote the product $S \times_T T'$ and p_i, $i = 1, 2$, the projections, as shown in the following diagram:*

$$\begin{tikzcd} S' \arrow[r, "p_2"] \arrow[d, "p_1"'] & T' \arrow[d, "\tau"] \\ S \arrow[r, "i"'] & T \end{tikzcd}$$

Then p_2 maps S' isomorphically onto a G-stable closed subscheme of T', and p_1 is a G-morphism for the resulting G-action induced on S'.

Proof. By Lemma 3.1, we know that S' inherits a unique G-action such that p_1 and p_2 are G-morphisms; consequently, Proposition 3.3 implies that the scheme-theoretic image of p_2 is a G-stable closed subscheme of T'. On the other hand, p_2, the pullback of the closed immersion i, is itself a closed immersion, and therefore maps S' isomorphically onto its scheme-theoretic image. The proposition follows. □

4. Quotients. Let T be a k-scheme and G a finite group acting on T. Recall that a *quotient* for the action of G on T is a pair $(T/G, \pi_T)$, where T/G is a k-scheme and $\pi_T : T \to T/G$ is a finite surjective morphism satisfying the following universal property: π_T is G-invariant, that is, $\pi_T \circ g_T = \pi_T$ for all $g \in G$, and any other G-invariant map $f : T \to Y$ factors uniquely through π_T, as shown:

$$f = \bar{f} \circ \pi_T, \qquad T \xrightarrow{\pi_T} T/G \xrightarrow{\bar{f}} Y.$$

It follows that a quotient, if one exists, is unique to unique isomorphism; the quotient map π_T is evidently a categorical epimorphism, and therefore scheme-theoretically dominant [**4**, 5.4.6, p. 285].

Turning to the question of existence, we recall that the quotient exists if and only if the G-action on T has the following property: *for every G-orbit in T, there exists an affine open subscheme of T containing that orbit.* It follows that every G-action on a quasiprojective k-scheme has a quotient, since any such T satisfies the following stronger property [**10**, p. 69]:

(1) for every finite subset $\mathcal{S}$ of points of T, there exists an affine open subscheme of T containing $\mathcal{S}$.

In particular, the quotient always exists when $T = \operatorname{Spec}(A)$ is affine, and in this case we have that $T/G = \operatorname{Spec}(A^G)$, where A^G denotes the subring of G-invariants of A. (References for these well-known assertions include [**10**, pp. 66–69] or [**12**, pp. 57–59]; the proofs given there for varieties carry over to k-schemes.)

Suppose now that the G-action on T has a quotient, and that $i : S \hookrightarrow T$ is the inclusion of a G-stable closed subscheme. We then have that the induced G-action on S has a quotient; indeed, any G-orbit in S lies in an affine open subscheme $U \subseteq T$, but then $S \cap U$, being a closed subscheme of U, is an affine open subscheme of S containing the given G-orbit. We may therefore form the quotients T/G and S/G; the universal property of the quotient then gives rise to the map i/G in the following commutative square:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "\pi_S"'] & T \arrow[d, "\pi_T"] \\ S/G \arrow[r, "i/G"'] & T/G \end{tikzcd} \tag{2}$$

This diagram is the focus of our attention in this paper; in particular, we are interested in the following two questions:

Question 4.1. Under what conditions is the diagram cartesian?

Question 4.2. Under what conditions is the map i/G a closed immersion?

In the next section we will consider these questions in the special case where G acts freely on T; in this case, we will show that the diagram is always cartesian and that the map i/G is always a closed immersion. In subsequent sections we will restrict attention to the situation presented in Schwarzenberger's paper [**11**]; we will show that the claim of cartesianity made therein is true when the variety X is a nonsingular curve, but false in general.

Much of our work rests on an analysis (summarized in Proposition 4.1) of the following more general situation: Let S and T be k-schemes, G a finite group acting on S and T such that both quotients exist, and $\phi : S \to T$ a G-morphism. Consider diagram (3) in which S' denotes the product $S/G \times_{T/G} T$, with projections p_1 and p_2, and δ the map induced by the pair of maps π_S, ϕ. Viewing π_T and ϕ/G as G-morphisms for the trivial G-actions on S/G and T/G, we deduce from Lemma 3.1 that S' inherits a unique G-action such that p_1 is G-invariant and p_2 and δ are G-morphisms. We note that the quotient S'/G for this action exists; indeed, given a k-point x of S', we can find an affine open subscheme U of T such that the orbit of $p_2(x)$ lies in U. Replacing U by the intersection of its translates, we may assume that U is G-stable. It follows that U/G is an affine open subscheme of T/G; letting V denote an affine open subscheme of S/G such that $p_1(x) \in V \subseteq (\phi/G)^{-1}(U/G)$, we have that $V \times_{U/G} U \subseteq S'$ is an affine open subscheme containing the orbit of x. The universal property of the quotient now gives rise to the maps $\bar{p_1}$, δ/G and p_2/G, as shown. Note that the outermost square in the diagram is cartesian if and only if the map δ is an isomorphism.

$$
\begin{tikzcd}
S \arrow[rr, "\phi"] \arrow[dr, "\delta"] \arrow[ddd, "\pi_S"'] & & T \arrow[ddd, "\pi_T"] \\
& S' \arrow[ur, "p_2"'] \arrow[d, "\pi_{S'}"] \arrow[ddl, "p_1"'] & \\
& S'/G \arrow[dl, shift right, "\bar{p_1}"'] \arrow[dr, "p_2/G"] & \\
S/G \arrow[ur, shift right, "\delta/G"'] \arrow[rr, "\phi/G"'] & & T/G
\end{tikzcd}
\tag{3}
$$

Proposition 4.1. *Under the hypotheses of the previous paragraph, we have*

(1) *the map δ is a surjection on k-points, and hence a surjection*;

(2) *the map δ/G is a surjective closed immersion*;

(3) *if ϕ is a radicial map, then so too are δ (hence δ is a bijection on k-points) and ϕ/G*;

(4) *if ϕ is a closed immersion, then δ is a surjective closed immersion*;

(5) *if ϕ is a closed immersion and S' is reduced, then δ is an isomorphism*; *consequently, the outermost square in diagram* (3) *is cartesian.*

Proof. (1) Let $s' \in S'$ be a k-point. Since π_S is surjective and ϕ is a G-morphism, a routine diagram chase yields a k-point $s \in S$ such that $\pi_S(s) = p_1(s')$ and $\phi(s) = p_2(s')$; the universal property of the product now implies that $\delta(s) = s'$.

(2) Since $\pi_{S'} \circ \delta = \delta/G \circ \pi_S$ is surjective, it follows that the same is true of the map δ/G. Moreover, one easily checks that $\bar{p_1} \circ \delta/G \circ \pi_S = \pi_S$, which implies that $\bar{p_1} \circ \delta/G = \mathrm{id}_{S/G}$; since the identity map is a closed immersion, it follows from [**4**, p. 265] that δ/G is a closed immersion.

(3) Since $\phi = p_2 \circ \delta$ is by hypothesis radicial, it follows from [**1**, p. 120] that δ is radicial and therefore injective on k-points; in view of (1), we conclude that δ is a bijection on k-points. Furthermore, one easily sees that ϕ/G is injective on k-points (since ϕ is), and therefore radicial, by Proposition 2.1.

(4) Since $\phi = p_2 \circ \delta$ is by hypothesis a closed immersion, it follows as in the proof of (2) that δ is a (surjective, by assertion (1)) closed immersion.

(5) By (4), δ is a surjective closed immersion. Recalling that such a map is defined by a nilpotent sheaf of ideals, we see at once that if S' is reduced, then δ is an isomorphism, and we are done. □

Remark 4.1. In Section 7 we will give an example showing that the surjective closed immersion δ/G need not be an isomorphism, even when ϕ is a closed immersion.

Corollary 4.1. *The map i/G in diagram* (2) *is always a radicial map.*

Proof. Since the closed immersion $i : S \hookrightarrow T$ is radicial (Section 2.2), the corollary results immediately from assertion (3) of the proposition. □

We end this section with a brief exploration of the relationship between quotients and scheme-theoretic images. Let $\tilde{S}$ and T be k-schemes with G-action such that both quotients exist, $\phi : \tilde{S} \to T$ a G-morphism, and $i : S \hookrightarrow T$ the inclusion of the scheme-theoretic image of ϕ. Proposition 3.3 yields that S is a G-stable closed subscheme of T; moreover, as noted above, the existence of T/G implies that of S/G. Let $\phi_0 : \tilde{S} \to S$ be the map by which ϕ factors through i, and $j : Q \hookrightarrow T/G$ the inclusion of the scheme-theoretic image of the induced map $\phi/G : \tilde{S}/G \to T/G$. These maps fit into the following commutative diagram of solid arrows:

$$\begin{tikzcd}
\tilde{S} \arrow[rr, "\phi"] \arrow[dr, "\phi_o"'] \arrow[dddd, "\pi_{\tilde{S}}"'] & & T \arrow[dddd, "\pi_T"] \\
& S \arrow[ur, "i"'] \arrow[d, "\pi_S"] & \\
& S/G \arrow[d, dashed, "\rho"] \arrow[ddr, "i/G"] & \\
& Q \arrow[dr, "j"] & \\
\tilde{S}/G \arrow[uur, "\phi_o/G"] \arrow[ur, "(\phi/G)_o"'] \arrow[rr, "\phi/G"'] & & T/G
\end{tikzcd}$$

We claim that there exists a map $\rho : S/G \to Q$ which renders the entire diagram commutative. To see this, first note that the transitivity of scheme-theoretic images (Section 2.1) implies that Q is the scheme-theoretic image of the composite map $(\phi/G) \circ \pi_{\tilde{S}}$, since the scheme-theoretic image of a quotient map, being scheme-theoretically dominant, is clearly equal to its target. By appealing twice more to the transitivity of scheme-theoretic images, we infer in turn that Q is the scheme-theoretic image of the maps $\pi_T \circ i : S \to T/G$ and $i/G : S/G \to T/G$; the latter map therefore factors as

$$S/G \xrightarrow{(i/G)_o} Q \xrightarrow{j} T/G.$$

It is a straightforward exercise to show that the diagram remains commutative after the map $\rho = (i/G)_o$ is inserted. We may now prove

Proposition 4.2. *Let $\tilde{S}, T, \phi$, etc., be as in the preceding paragraph. If, in addition, we are given that the map $\phi/G : \tilde{S}/G \to T/G$ is a closed immersion, then the map $\phi_o/G : \tilde{S}/G \to S/G$ is an isomorphism; briefly, $\tilde{S}$ and S have the same quotient. Consequently, the map i/G is a closed immersion as well.*

Proof. Since ϕ/G is a closed immersion, it maps $\tilde{S}/G$ isomorphically onto its scheme-theoretic image; that is, the map $(\phi/G)_o$ is an isomorphism. It is routine to check that the map $(\phi/G)_0^{-1} \circ \rho : S/G \to \tilde{S}/G$ is the inverse of ϕ_o/G; whence, the latter is an isomorphism and $i/G = \phi/G \circ (\phi_0/G)^{-1}$ is a closed immersion. $\square$

5. The case of free G-actions. Let T be a k-scheme and G a finite group acting on T. Recall that the action of G on T is said to be **free**, or that G is acting **freely** on T, provided that $g_T(t) \neq t$ for all closed points $t \in T$ and all $g \in G - \{1\}$. In this case (assuming the quotient exists) the quotient map $\pi_T : T \to T/G$ is étale [**10**, p. 66]; this fact is a key ingredient for the proofs of the results claimed in the last section. We begin with

Theorem 5.1. *Let S and T be k-schemes, G a finite group acting on S and T such that both quotients exist, and $\phi : S \to T$ a radicial G-morphism. If G acts freely on T, then the outermost square in diagram* (3) *is cartesian.*

Proof. Referring to diagram (3), we recall that it suffices to prove that the map $\delta : S \to S'$ is an isomorphism, where $S' = S/G \times_{T/G} T$. By hypothesis, G acts freely on T and therefore on S; as noted above this implies that the quotient maps π_S and π_T are étale. It follows that the projection map $p_1 : S' \to S/G$, the pullback of π_T, is étale, and consequently δ is étale [**1**, p. 116]. On the other hand, δ is both a surjection and radicial, by assertions (1) and (3) of Proposition 4.1. Since δ is both étale and radicial, [**1**, p. 121] yields that δ is an open immersion; since δ is also a surjection, we conclude that δ is an isomorphism, as desired. $\square$

Corollary 5.1. *Let G be a finite group acting on a k-scheme T such that the quotient T/G exists and $i : S \hookrightarrow T$ the inclusion of a G-stable closed subscheme. If G acts freely on T, then the diagram* (2) *is cartesian.*

Proof. Since i is a closed immersion, it is radicial, by [**4**, p. 260]; the result therefore follows immediately from the theorem. $\square$

Our next goal is to show that the map $i/G : S/G \to T/G$ in diagram (2) is a closed immersion whenever G acts freely on T. To do this, we need the following

Lemma 5.1. *Suppose given a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\alpha_X"'] & Y' \arrow[d, "\alpha_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

of k-schemes and morphisms thereof, such that α_Y is flat and f is affine and scheme-theoretically dominant. Then f' is scheme-theoretically dominant.

Proof. By definition, we are given that the map of sheaves $\theta_f : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective, and we want to show the same is true of the map $\theta_{f'} : \mathcal{O}_{Y'} \to f'_*\mathcal{O}_{X'}$. However, these maps fit into the following commutative diagram, in which χ is the natural map defined in [**3**, II, 1.5.2]:

$$\begin{tikzcd} {\alpha_Y}^*\mathcal{O}_Y \arrow[r, "{\alpha_Y}^*(\theta_f)"] \arrow[dd, "\cong"'] & {\alpha_Y}^*f_*\mathcal{O}_X \arrow[d, "\chi"] \\ & f'_*{\alpha_X}^*\mathcal{O}_X \arrow[d, "\cong"] \\ \mathcal{O}_{Y'} \arrow[r, "\theta_{f'}"'] & f'_*\mathcal{O}_{X'} \end{tikzcd}$$

In view of our hypotheses, the main result of the cited passage [**3**, II, 1.5.2] implies that χ is an isomorphism. Furthermore, since α_Y is flat, the functor ${\alpha_Y}^*$ is exact, and therefore ${\alpha_Y}^*(\theta_f)$ is injective. From this it follows immediately that $\theta_{f'}$ is injective, as desired. $\square$

Theorem 5.2. *Let T be a k-scheme, G a finite group acting freely on T such that the quotient exists, and $i : S \hookrightarrow T$ the inclusion of a G-stable closed subscheme. Then the induced map $i/G : S/G \to T/G$ is a closed immersion.*

Proof. Let $j : Q \hookrightarrow T/G$ denote the inclusion of the scheme-theoretic image of the map i/G and $s = (\phi/G)_o : S/G \to Q$ the map by which i/G factors through j. As noted in Section 4, the transitivity of scheme-theoretic images implies that Q is the scheme-theoretic image of the map $i/G \circ \pi_S = \pi_T \circ i$. The support of Q is therefore the image of S under $\pi_T \circ i$ (which is closed since π_T is a finite map); it follows from this that the map s is surjective. Let $Q' = Q \times_{T/G} T$, and consider the following diagram, in which the map s' is defined by the pair of maps $(s \circ \pi_S, i)$ and p_1, p_2 denote the projection maps.

$$\begin{tikzcd} S \arrow[rr, "i"] \arrow[dr, "s'"] \arrow[dd, "\pi_S"'] & & T \arrow[dd, "\pi_T"] \\ & Q' \arrow[ur, "p_2"'] \arrow[d, "p_1"] & \\ S/G \arrow[r, "s"'] & Q \arrow[r, "j"'] & T/G \end{tikzcd} \tag{4}$$

We are out to show that the map s is an isomorphism; since we already know that s is surjective, it will suffice to prove that s is both étale and radicial, and therefore an open immersion by [**1**, p. 121]. That s is radicial is immediate: Corollary 4.1 implies that $i/G = j \circ s$ is radicial; whence, s is radicial by [**1**, p. 120].

It therefore remains to show that s is étale. Suppose for the moment that we have shown that the map s' is an isomorphism. Then we may focus attention on the left of diagram (4), which reduces to the following triangle of maps:

$$\begin{tikzcd} & S \arrow[dl, "\pi_S"'] \arrow[dr, "p_1 \circ s'"] & \\ S/G \arrow[rr, "s"'] & & Q \end{tikzcd}$$

As in the proof of Theorem 5.1, we have that the maps π_T, π_S and p_1 are étale, therefore the diagonal arrows in the triangle are étale. Let x be a k-point of S, and let $y = \pi_S(x)$, $z = (p_1 \circ s')(x)$. Since the diagonal arrows are étale, they induce isomorphisms on completions

$$\hat{\pi}_S : \hat{\mathcal{O}}_{S/G,y} \to \hat{\mathcal{O}}_{S,x} \qquad (p_1 \hat{\circ} s') : \hat{\mathcal{O}}_{Q,z} \to \hat{\mathcal{O}}_{S,x};$$

since $(p_1 \hat{\circ} s') = \hat{\pi}_S \circ \hat{s}$, it follows that $\hat{s} : \hat{\mathcal{O}}_{Q,z} \to \hat{\mathcal{O}}_{S/G,y}$ is an isomorphism; whence, s is étale at y [**1**, p. 116]. Since this argument applies to any k-point $y \in S/G$, we conclude from [**1**, p. 116] that s is étale (everywhere), as desired.

We are reduced to showing that the map s' is an isomorphism; to do this, we will show that s' is both a closed immersion and scheme-theoretically dominant since any such map must be an isomorphism [**4**, p. 283]. That s' is a closed immersion follows immediately from [**4**, p. 264], since $i = p_2 \circ s'$ is a closed immersion. To show that s' is scheme-theoretically dominant, we apply Lemma 5.1 to the leftmost subsquare of diagram (4): Note that this subsquare is cartesian by abstract nonsense, since the outermost and rightmost subsquares of the diagram are cartesian; furthermore, the map p_1, being étale, is flat, and $s = (\phi/G)_o$ is scheme-theoretically dominant by the generalities on scheme-theoretic images. We claim in addition that s is an affine morphism; indeed, given an affine open subscheme $U \subseteq Q$, we have that $(p_1 \circ s')^{-1}(U) = V$ is a G-stable affine open subscheme of S (since p_1, the pullback of π_T, is affine, s' is a closed immersion, and $p_1 \circ s' = s \circ \pi_S$ is G-invariant); therefore, by the construction of the quotient, it follows that $s^{-1}(U) = V/G$ is an affine open subscheme of S/G. Lemma 5.1 now yields that s' is scheme-theoretically dominant, whence an isomorphism, and the proof is complete. □

In [**12**, p. 62], Serre gives an example of a G-stable subvariety S of a variety T for which the inclusion $i/G : S/G \to T/G$ is not a closed immersion (Example 1.1 is a special case of this example). He had previously asserted without proof [**12**, p. 60] that the map i/G is always a closed immersion whenever the characteristic is zero or the group acts freely. Theorem 5.2 extends the second part of this assertion from varieties to k-schemes. The following elementary result, included for completeness, similarly extends the first part:

Proposition 5.1. *Let T be a k-scheme with* $\operatorname{char}(k) = 0$, *$G$ a finite group acting on T such that T has a quotient, and $i : S \hookrightarrow T$ the inclusion of a G-stable closed subscheme. Then the induced map $i/G : S/G \to T/G$ is a closed immersion.*

Proof. One easily reduces to the case where $T = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(B/I)$ are affine (here I is a G-stable ideal of B; see Proposition 3.2). The comorphism of i/G is the map $(i/G)^* : B^G \to (B/I)^G$, which sends $b \in B^G$ to the G-invariant coset $b + I$; we must show that this map is surjective. But given a G-invariant coset $c + I$, we have that

$$g_T{}^*(c) + I = c + I, \qquad \forall\, g \in G;$$

that is,

$$c = g_T{}^*(c) + \alpha_g, \qquad \alpha_g \in I,\ \forall\, g \in G.$$

Summing over $g \in G$, we obtain

$$|G| \cdot c = \sum_{g \in G} g_T{}^*(c) + \sum_{g \in G} \alpha_g,$$

therefore (char 0!)

$$c + I = (i/G)^* \left(\frac{1}{|G|} \sum_{g \in G} g_T{}^*(c) \right),$$

and we are done. $\square$

6. Cartesian and symmetric products. Let X be a k-scheme, X^n the n-fold cartesian product of X, and G_n the symmetric group on n letters (we view the elements of G_n as permutations of $\{1, \dots, n\}$). Let $p_{n,j} : X^n \to X$ denote the j-th projection map. The group G^n acts on X^n by permuting the factors; more precisely, the action is given by the maps

$$g_{X^n} : X^n \to X^n, \qquad g \in G_n,$$

where g_{X^n} is defined by requiring that

$$p_{n,j} \circ g_{X^n} = p_{n,g^{-1}(j)}, \qquad 1 \leq j \leq n.$$

The quotient $X^n/G_n = X^{(n)}$ for this action (if it exists) is called the n-fold *symmetric product* of X; the quotient map is denoted $\pi_{X^n} : X^n \to X^{(n)}$. Referring to Section 4, we recall that the quotient exists provided that every G_n-orbit in X^n lies in an affine open subscheme of X^n. To ensure that this holds, it suffices to assume that X satisfies property (1), which we will do for the balance of the paper; recall that all quasiprojective k-schemes satisfy this property.

Now let X_j^n, $1 \leq j \leq n$, denote n disjoint copies of X^n, and $i_{n,j} : X_j^n \to X^n \times X$ the map defined by the pair of maps $(\mathrm{id}_{X^n}, p_{n,j})$, where id_{X^n} denotes the identity map on X^n. On k-points, we have

$$(x_1, \dots, x_n) \overset{i_{n,j}}{\longmapsto} ((x_1, \dots, x_n), x_j).$$

We remark that $i_{n,j}$ is a closed immersion; indeed, the pair $(X_j^n, i_{n,j})$ is the kernel [**4**, pp. 276, 278] of the pair of maps

$$p_{n,j} \circ p_1 : X^n \times X \longrightarrow X, \qquad p_2 : X^n \times X \longrightarrow X,$$

where p_1, p_2 denote the projections to X^n and X, respectively. Let $\tilde{\mathcal{X}}_n$ denote the disjoint union of the X_j^n and $\tilde{i_n} : \tilde{\mathcal{X}}_n \to X^n \times X$ the map induced by the maps $i_{n,j}$. Note that G_n acts on $X^n \times X$ by acting on the first factor; we define a compatible G_n-action on $\tilde{\mathcal{X}}_n$ using the maps

$$g_{\tilde{\mathcal{X}}_n} : \tilde{\mathcal{X}}_n \longrightarrow \tilde{\mathcal{X}}_n, \qquad g \in G_n,$$

where $g_{\tilde{\mathcal{X}}_n}$ is defined on each component X_j^n as follows:

$$g_{\tilde{\mathcal{X}}_n}|_{X_j^n} : X_j^n = X^n \overset{g_{X^n}}{\longrightarrow} X^n = X_{g(j)}^n, \qquad 1 \leq j \leq n.$$

It is then easy to check that the map $\tilde{i_n}$ is a G_n-morphism; that is, for all $g \in G_n$,

$$(g_{X^n} \times \mathrm{id}_X) \circ \tilde{i_n} = \tilde{i_n} \circ g_{\tilde{\mathcal{X}}_n}.$$

Let $i_n : \mathcal{X}_n \hookrightarrow X^n \times X$ denote the inclusion of the scheme-theoretic image of the map $\tilde{i_n}$. By Proposition 3.3, we have that $\mathcal{X}_n$ is a G_n-stable closed subscheme of $X^n \times X$, and that the map $(\tilde{i_n})_o : \tilde{\mathcal{X}}_n \to \mathcal{X}_n$, by which $\tilde{i_n}$ factors through i_n, is a G_n-morphism. The focus of our attention for the remainder of this paper is the diagram

$$\begin{tikzcd} \mathcal{X}_n \arrow[r, "i_n"] \arrow[d, "\pi_{\mathcal{X}_n}"'] & X^n \times X \arrow[d, "\pi_{X^n} \times \mathrm{id}_X"] \\ \mathcal{X}_n/G_n \arrow[r, "i_n/G_n"'] & X^{(n)} \times X, \end{tikzcd} \tag{5}$$

the specialization of diagram (2) to the present situation. In particular, we seek answers to our two basic questions:

Question 6.1. Is the diagram cartesian?

Question 6.2. Is the map i_n/G_n a closed immersion?

We will show that the answer to the second question is always "yes," but the answer to the first question depends on X; in general, the answer is "no," but if X is a nonsingular and irreducible curve, the answer is "yes." As we proceed, we will exemplify several of the general results obtained in previous sections.

Our first goal is to compute the quotients $\tilde{\mathcal{X}}_n/G_n$ and $\mathcal{X}_n/G_n$ (which in fact turn out to be the same). For each j, $1 \leq j \leq n$, consider the action of G_{n-1} on $X^n_j = X^n$ given by viewing G_{n-1} as the subgroup of G_n consisting of the permutations of $\{1, \ldots, n\}$ which fix the integer j. It is clear that the quotient X^n_j/G_{n-1} for this action is isomorphic to $X^{(n-1)} \times X$; we denote the quotient map by $\pi_{n,j} : X^n_j \to X^{(n-1)} \times X$. One easily verifies that the following diagram commutes for all j, $1 \leq j \leq n$, and all $g \in G_n$:

$$\begin{tikzcd} X^n_j \arrow[rr, "g_{\tilde{\mathcal{X}}_n}|_{X^n_j}"] \arrow[dr, "\pi_{n,j}"'] & & X^n_{g(j)} \arrow[dl, "\pi_{n,g(j)}"] \\ & X^{(n-1)} \times X. & \end{tikzcd} \tag{6}$$

We write $\alpha_n : X^{(n-1)} \times X \to X^{(n)}$ for the map which "adds zero-cycles"—formally, α_n is the map induced on the quotient by the G_{n-1}-invariant map $\pi_{X^n} : X^{n-1} \times X = X^n \to X^{(n)}$; whence, $\alpha_n \circ \pi_{n,j} = \pi_{X^n}$ for each j, $1 \leq j \leq n$.

Proposition 6.1. *Let X, X^n, $\tilde{\mathcal{X}}_n$, G_n, etc., be as above. Then the quotient $\tilde{\mathcal{X}}_n/G_n$ is isomorphic to $X^{(n-1)} \times X$, and the quotient map $\pi_{\tilde{\mathcal{X}}_n} : \tilde{\mathcal{X}}_n \to X^{(n-1)} \times X$ is the map whose restriction to the j-th component X^n_j is the map $\pi_{n,j} : X^n_j \to X^{(n-1)} \times X$. Furthermore, the induced map $\tilde{i_n}/G_n : \tilde{\mathcal{X}}_n/G_n \to X^{(n)} \times X$ is a closed immersion.*

Proof. Let the map $\pi_{\tilde{\mathcal{X}}_n} : \tilde{\mathcal{X}}_n \to X^{(n-1)} \times X$ be defined as in the first assertion above; the commutativity of diagram (6) implies that this map is G_n-invariant. To complete the proof of the first assertion, we must show that $\pi_{\tilde{\mathcal{X}}_n}$ satisfies the universal property of the quotient; to this end, let $\phi : \tilde{\mathcal{X}}_n \to T$ be an arbitrary G_n-invariant map. Then, for each j, $1 \leq j \leq n$, the restriction $\phi|_{X^n_j} = \phi_j : X^n_j \to T$ is invariant for the "j-fixing" action of G_{n-1} on X^n_j defined above, hence induces a unique map $\bar{\phi}_j : X^{(n-1)} \times X \to T$ satisfying $\phi_j = \bar{\phi}_j \circ \pi_{n,j}$. We claim that the maps $\bar{\phi}_j$ are all equal to one another. Indeed, if $1 \leq j \neq l \leq n$, and $g \in G_n$ satisfies $g(j) = l$, then we have

$$\phi_j = \phi_l \circ g_{\tilde{\mathcal{X}}_n}|_{X^n_j} = \bar{\phi}_l \circ \pi_{n,l} \circ g_{\tilde{\mathcal{X}}_n}|_{X^n_j} = \bar{\phi}_l \circ \pi_{n,j},$$

where the last equality results from the commutativity of diagram (6); from the uniqueness of the maps $\bar{\phi}_j$, we now conclude that $\bar{\phi}_j = \bar{\phi}_l$, as claimed. It follows that the G_n-invariant map ϕ factors uniquely (via $\bar{\phi}_j$) through $\pi_{\tilde{\mathcal{X}}_n}$, which completes the proof of the first assertion.

We now prove the second assertion. We claim that the induced map

$$\tilde{i_n}/G_n : \tilde{\mathcal{X}}_n/G_n \to X^{(n-1)} \times X$$

is in fact the map $X^{(n-1)} \times X \to X^n \times X$ defined by the pair of maps (α_n, p_2), where α_n denotes the "addition map" and p_2 the second projection $X^{(n-1)} \times X \to X$. To see this, it suffices to verify that the diagram

$$\begin{tikzcd} X_j^n \arrow[r, "i_{n,j}"] \arrow[d, "\pi_{n,j}"'] & X^n \times X \arrow[d, "\pi_{X^n} \times \mathrm{id}_X"] \\ X^{(n-1)} \times X \arrow[r, "(\alpha_n, p_2)"'] & X^{(n)} \times X \end{tikzcd}$$

commutes for each j, $1 \leq j \leq n$, but this is an easy exercise. Since the map (α_n, p_2) is known to be a closed immersion [**6**, p. 22], the second assertion follows immediately. $\square$

Combining Propositions 4.2 and 6.1, we obtain the following result, which both computes the quotient $\mathcal{X}_n/G_n$ and gives an affirmative answer to Question 6.2.

Theorem 6.1. *Let X be a k-scheme satisfying property* (1), *and let $\tilde{\mathcal{X}}_n$, $\mathcal{X}_n$, G_n, etc., be as above. Then the induced map $(\tilde{i_n})_o/G_n : \tilde{\mathcal{X}}_n/G_n \to \mathcal{X}_n/G_n$ is an isomorphism; that is, $\mathcal{X}_n/G_n \approx X^{(n-1)} \times X$. Furthermore, the induced map i_n/G_n in diagram (5) is a closed immersion.*

We now turn our attention to Question 6.1. In case X is a reduced projective k-scheme, Schwarzenberger asserts without proof in [**11**, p. 375] that "[t]he definitions imply that $W_n = W_n' \times_{X_n'} X_n$" (which is equivalent to an affirmative answer to Question 6.1). Unfortunately, this assertion is false in general; we show in Section 7 that it fails for $X = P_k^2$, the projective plane over k. On the other hand, we show in Section 8 that the assertion is true when X is an irreducible and nonsingular curve. In consequence, [**11**, Proposition 3.2, p. 375], which depends on the erroneous assertion, is false in general, but holds whenever diagram (5) is cartesian.

Before leaving this section, we show that diagram (5), although not cartesian in general, is always "generically" cartesian. Indeed, let $X_\circ^n$ denote the open subscheme of X^n which is the complement of the multidiagonal; that is, the set of k-points of $X_\circ^n$ is

$$\{k\text{-points}(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } 1 \leq i \neq j \leq n\}.$$

It is clear that the G_n-action on X^n restricts to a free G_n-action on $X_\circ^n$; the quotient $X_\circ^n/G_n = X_\circ^{(n)}$ for this action can be identified with the open subscheme of $X^{(n)}$ which is the image of $X_\circ^n$ under π_{X^n}. Consider the following diagram, in which $\mathcal{X}'_n$ denotes the product $\mathcal{X}_n/G_n \times_{(X^{(n)}\times X)} (X^n \times X)$, p_1 and p_2 the projection maps, and δ the map induced by the pair of maps (π_{X^n}, i_n):

(7)
$$\begin{tikzcd}
\mathcal{X}_n \arrow[rr, "i_n"] \arrow[dr, "\delta"] \arrow[dd, "\pi_{\mathcal{X}_n}"'] & & X^n \times X \arrow[dd, "\pi_{X^n}\times \mathrm{id}_X"] \\
& \mathcal{X}'_n \arrow[ur, "p_2"'] \arrow[dl, "p_1"] & \\
\mathcal{X}_n/G_n \arrow[rr, "i_n/G_n"'] & & X^{(n)} \times X.
\end{tikzcd}$$

We wish to consider the pullback of this diagram over the open subscheme $X_\circ^{(n)} \times X \subseteq X^{(n)} \times X$; if we denote $i_n^{-1}(X_\circ^n \times X)$ and $p_2^{-1}(X_\circ^n \times X)$ by $\mathcal{X}_{n,\circ}$ and $\mathcal{X}'_{n,\circ}$, respectively, we see that this pullback is the diagram

(8)
$$\begin{tikzcd}
\mathcal{X}_{n,\circ} \arrow[rr, "i_n|_{\mathcal{X}_{n,\circ}}"] \arrow[dr, "\delta|_{\mathcal{X}_{n,\circ}}"] \arrow[dd, "\pi_{\mathcal{X}_{n,\circ}}"'] & & X_\circ^n \times X \arrow[dd, "\pi_{X_\circ^n}\times \mathrm{id}_X"] \\
& \mathcal{X}'_{n,\circ} \arrow[ur, "p_2|_{\mathcal{X}'_{n,\circ}}"'] \arrow[dl, "p_1|_{\mathcal{X}'_{n,\circ}}"] & \\
\mathcal{X}_{n,\circ}/G_n \arrow[rr, "(i_n|_{\mathcal{X}_{n,\circ}})/G_n"'] & & X_\circ^{(n)} \times X.
\end{tikzcd}$$

We may now state precisely and prove the "generic cartesianity" of diagram (5):

Proposition 6.2. *The outer square in diagram* (8) *is cartesian*; *equivalently, the map $\delta|_{\mathcal{X}_{n,o}}$ is an isomorphism.*

Proof. By Proposition 3.4, the map $i_n|_{\mathcal{X}_{n,o}}$ is the inclusion of a G_n-stable closed subscheme; therefore, since G_n acts freely on $X_\circ^n \times X$, Corollary 5.1 implies that the outer square in diagram (8) is cartesian. Since the inner square of diagram (8) is the pullback of a cartesian square, and therefore cartesian, it is clear that the outer square is cartesian if and only if the map $\delta|_{\mathcal{X}_{n,\circ}}$ is an isomorphism. $\square$

Corollary 6.1. *If X is a reduced and irreducible k-scheme, then the cartesian product $\mathcal{X}'_n$ in diagram* (7) *is generically reduced.*

Proof. The hypothesis implies that the product X^n is reduced (and irreducible); therefore, $\tilde{\mathcal{X}}_n$ (the disjoint union of n copies of X^n) is reduced, and consequently $\mathcal{X}_n$, the scheme-theoretic image of $\tilde{\mathcal{X}}_n$, is reduced. Since $\mathcal{X}_n$ and $\mathcal{X}'_n$ have the same underlying space, by (4) of Proposition 4.1, and since each irreducible component of $\mathcal{X}_n$ meets $\mathcal{X}_{n,\circ}$, Proposition 6.2 yields the result. $\square$

7. Case $X = P^2_k$. Our purpose in this section is to show that the answer to Question 6.1 is "no" when $X = P^2_k$, the projective plane/k. We begin with a simple observation which enables us to reduce the question to an affine computation. Returning briefly to the general situation of Section 4, we consider a k-scheme T with G-action such that the quotient T/G exists, a G-stable closed subscheme $i : S \hookrightarrow T$, and an open subscheme U of T/G. We observe that if diagram (2) is cartesian, then so is the following diagram obtained by pullback:

$$\begin{tikzcd} i^{-1}({\pi_T}^{-1}(U)) \arrow[r, "i|_{i^{-1}({\pi_T}^{-1}(U))}"] \arrow[d, "\pi_S|_{i^{-1}({\pi_T}^{-1}(U))}"'] & {\pi_T}^{-1}(U) \arrow[d, "\pi_T|_{{\pi_T}^{-1}(U)}"] \\ (i/G)^{-1}(U) \arrow[r, "i/G|_{(i/G)^{-1}(U)}"'] & U. \end{tikzcd}$$

Equivalently, if the latter diagram is not cartesian, then neither is diagram (2). We apply this to diagram (5) in case $X = P^2_k$, $n = 2$, and $U = V^{(2)} \times V \subseteq X^{(2)} \times X$, where $V \subseteq X$ is one of the standard affine patches; one easily checks that the pullback of (5) over $V^{(2)} \times V$ is the following diagram of affine k-schemes ($\mathcal{V}_2$ is to V as $\mathcal{X}_2$ is to X):

$$\begin{tikzcd} \mathcal{V}_2 \arrow[r, "i_2"] \arrow[d, "\pi_{\mathcal{V}_2}"'] & V_2 \times V \arrow[d, "\pi_{V^2} \times \mathrm{id}_V"] \\ \mathcal{V}_2/G_2 \arrow[r, "i_2/G_2"'] & V^{(2)} \times V. \end{tikzcd} \tag{9}$$

We will show that this last diagram is not cartesian by explicitly computing the tensor product of the affine rings of $\mathcal{V}_2/G_2$ and $V^2 \times V$ over the affine ring of $V^{(2)} \times V$, and showing that the result is not isomorphic to the affine ring of $\mathcal{V}_2$. We need the following

Lemma 7.1. *Let*

$$\begin{aligned} A &= k[x_1, \dots, x_n]/I, \qquad & I &= (f_1, \dots, f_r), \\ B &= k[y_1, \dots, y_m]/J, \qquad & J &= (g_1, \dots, g_s), \\ C &= k[z_1, \dots, z_p]/K, \qquad & K &= (h_1, \dots, h_t), \end{aligned}$$

be three finitely generated k-algebras, and suppose we have k-algebra maps $\phi : C \to A$, $\psi : C \to B$. Then the tensor product $A \otimes_C B$ is isomorphic to the k-algebra

$$D = k[x_1, \dots, x_n; y_1, \dots, y_m]/(f_1, \dots, f_r; g_1, \dots, g_s; \hat{z}_1, \dots, \hat{z}_p),$$

where, for $1 \le i \le p$, $\hat{z}_i$ is a polynomial in the x's and y's of the form

$$\hat{z}_i = (\textit{representative of } \phi(z_i)) - (\textit{representative of } \psi(z_i)).$$

Proof. Let $i_A : A \to D$ and $i_B : B \to D$ denote the obvious natural maps; since $i_A \circ \phi = i_B \circ \psi$, we have that D is a C-algebra and i_A and i_B are maps of C-algebras. Moreover, given any C-algebra E, and C-algebra maps $\alpha : A \to E$ and $\beta : B \to E$, it is evident that there exists a unique map $\xi : D \to E$ of k-algebras satisfying $\xi \circ i_A = \alpha$ and $\xi \circ i_B = \beta$; whence, ξ is a map of C-algebras as well. In other words, the triple (D, i_A, i_B) satisfies the universal property of the coproduct in the category of C-algebras; the lemma follows immediately. $\square$

We now set $V = \operatorname{Spec}(k[x, y])$, and identify the affine rings of the k-schemes in diagram (9). We write

$$V^2 \times V = \operatorname{Spec}(k[x_1, y_1, x_2, y_2; x, y]);$$

the affine ring of $V^{(2)} \times V$ is therefore given by

$$(k[x_1, y_1, x_2, y_2])^{G_2} \otimes_k k[x, y].$$

The subring $(k[x_1, y_1, x_2, y_2])^{G_2}$ of polynomials invariant under the interchange of (x_1, y_1) and (x_2, y_2) is generated by the following five polynomials [**7**, p. 687]:

$$z_1 = x_1 + x_2, \qquad z_2 = y_1 + y_2, \qquad z_3 = x_1 x_2,$$
$$z_4 = y_1 y_2, \qquad z_5 = x_2 y_1 + x_1 y_2,$$

modulo an ideal K of relations which we need not make explicit. Lemma 7.1 therefore yields

$$V^{(2)} \times V = k[z_1, z_2, z_3, z_4, z_5; x, y]/K.$$

By Theorem 6.1, we have that

$$\mathcal{V}_2/G_2 \approx V^{(2-1)} \times V = V \times V = \operatorname{Spec}(k[x_1', y_1', x_2', y_2']);$$

the ring morphism $(i_2/G_2)^*$ sends z_1 to $x_1' + x_2'$, z_2 to $y_1' + y_2'$, etc., and x to x_2', y to y_2'. We may therefore apply Lemma 7.1 to compute the affine ring D of the product $\mathcal{V}_2' = \mathcal{V}_2/G_2 \times_{(V^{(2)} \times V)} (V^2 \times V)$:

$$D = k[x_1', y_1', x_2', y_2', x_1, y_1, x_2, y_2, x, y]/J_1,$$

where

$$J_1 = \begin{pmatrix} \hat{z}_1 = (x_1' + x_2') - (x_1 + x_2), \\ \hat{z}_2 = (y_1' + y_2') - (y_1 + y_2), \\ \hat{z}_3 = (x_1' x_2') - (x_1 x_2), \\ \hat{z}_4 = (y_1' y_2') - (y_1 y_2), \\ \hat{z}_5 = (x_2' y_1' + x_1' y_2') - (x_2 y_1 + x_1 y_2), \\ \hat{x} = x_2' - x, \\ \hat{y} = y_2' - y \end{pmatrix};$$

since $x_2' \equiv x$ and $y_2' \equiv y$, this simplifies to

$$D = k[x_1', y_1', x_1, y_1, x_2, y_2, x, y]/J_2,$$

where

$$J_2 = \begin{pmatrix} (x_1' + x) - (x_1 + x_2), \\ (y_1' + y) - (y_1 + y_2), \\ (x_1' x) - (x_1 x_2), \\ (y_1' y) - (y_1 y_2), \\ (x y_1' + x_1' y) - (x_2 y_1 + x_1 y_2) \end{pmatrix}.$$

Now using the congruences $x_1' \equiv (x_1+x_2)-x$ and $y_1' \equiv (y_1+y_2)-y$, we further simplify to obtain

$$D = k[x_1, y_1, x_2, y_2, x, y]/J_3,$$

where

$$J_3 = \begin{pmatrix} \alpha = (x_1+x_2-x)x - (x_1x_2), \\ \beta = (y_1+y_2-y)y - (y_1y_2), \\ \gamma = (x(y_1+y_2-y) + (x_1+x_2-x)y) - (x_2y_1+x_1y_2) \end{pmatrix}.$$

On the other hand, the affine ring of $\mathcal{V}_2$ is given by $k[x_1, y_1, x_2, y_2, x, y]/I$, where I is the ideal of all polynomials vanishing on the locus of k-points

$$S = \{((p_1, p_2), p) \in V^2 \times V \mid p = p_1 \text{ or } p = p_2\}.$$

If, for $i = 1, 2$, we let $S_i = \{((p_1, p_2), p) \in V^2 \times V \mid p = p_i\}$, we have that the ideals of all polynomials vanishing on S_1 and S_2 are $I_1 = (x_1 - x, y_1 - y)$ and $I_2 = (x_2 - x, y_2 - y)$, respectively. Therefore,

$$I = I_1 \cap I_2 \supseteq I_1 \cdot I_2 = \begin{pmatrix} -\alpha = (x_1-x)(x_2-x), \\ g = (x_1-x)(y_2-y), \\ f = (y_1-y)(x_2-x), \\ -\beta = (y_1-y)(y_2-y) \end{pmatrix}$$

(it can in fact be shown that $I_1 \cap I_2 = I_1 \cdot I_2$). Observe that $\gamma = -(f+g)$; therefore, $J_3 \subseteq I$. Moreover, assertion (4) of Proposition 4.1 implies that $\mathcal{V}_2$ and $\mathcal{V}_2'$ have the same support, so that in fact $I = \sqrt{J_3}$ must hold. If diagram (9) were cartesian, we would have $J_3 = I$, but it is an easy exercise to show that $f \notin J_3$ and $g \notin J_3$; therefore, $J_3 \neq I$ and diagram (9) is not cartesian.

We may now present the example promised in Remark 4.1. We begin by noting that $\alpha \cdot \beta = f \cdot g$, and therefore

$$f^2 = f(f+g) - fg = f(-\gamma) - \alpha\beta \in J_3,$$

and

$$g^2 = g(f+g) - fg = g(-\gamma) - \alpha\beta \in J_3;$$

consequently, $I^2 \subseteq J_3$. In addition, we observe that, under the induced G_2-action on $D = k[x_1, y_1, x_2, y_2, x, y]/J_3$ (given by interchanging the pair (x_1, y_1) with (x_2, y_2)), the nonzero nilpotent elements represented by f and g are conjugate. Since $f + g = -\gamma \in J_3$, we have that $f \equiv -g (\operatorname{mod} J_3)$; therefore, in characteristic two, we find that $f \equiv -g = g$ represents a nilpotent invariant element for the action of G_2 on D. It follows that the quotient $\mathcal{V}_2'/G_2 = \operatorname{Spec}(D^{G_2})$ is a nonreduced scheme and therefore not isomorphic to $\mathcal{V}_2/G_2$ which, being the quotient of a reduced scheme, is reduced. Recalling assertion (2) of Proposition 4.1, we conclude that the map $\delta/G_2 : \mathcal{V}_2/G_2 \to \mathcal{V}_2'/G_2$ is a nontrivial surjective closed immersion in characteristic two.

8. Case $X =$ irreducible and nonsingular curve. We bring this paper to a close by showing that the answer to Question 6.1 is "yes" in at least one case:

Theorem 8.1. *Let X be an irreducible and nonsingular curve. Then diagram* (5) *is cartesian.*

Proof. Referring to diagram (7), we recall that the desired conclusion will follow if we can prove that the product $\mathcal{X}_n'$ is reduced, by assertion (5) of Proposition 4.1. Moreover, Corollary 6.1 implies that $\mathcal{X}_n'$ is generically reduced, so it suffices to prove, in addition, that $\mathcal{X}_n'$ has no embedded components [**1**, p. 132]; this we proceed to do.

It is well known that the n-fold symmetric product of an irreducible and nonsingular curve is a nonsingular variety (see, e.g., [**8**, p. 226]); therefore, $X^{(n)}$ and $\mathcal{X}_n/G_n \approx X^{(n-1)} \times X$ are nonsingular varieties. Since the map $\pi_{X^n} \times \mathrm{id}_X : X^n \times X \to X^{(n)} \times X$ is finite, and therefore quasi-finite, it follows from [**1**, p. 95] that this map is flat; consequently, the pullback $p_1 : \mathcal{X}_n' \to \mathcal{X}_n/G_n$ is finite and flat as well. It is clear that the generic points of the irreducible components of $\mathcal{X}_n'$ lie over the generic point ξ of $\mathcal{X}_n/G_n$, which is the only associated point of the latter. Since the map p_1 is flat, it follows from [**9**, p. 41] that any other associated point of $\mathcal{X}_n'$ must lie over ξ as well, but, since p_1 is finite, this is impossible [**2**, p. 61]. We conclude that $\mathcal{X}_n'$ has no associated points other than the generic points of its irreducible components, that is, $\mathcal{X}_n'$ has no embedded components, and the proof is complete. □

We recommend to the reader the instructive and amusing exercise of checking that the diagram (5) is cartesian in case $X = A^1_k$, the affine lines over k, using a direct computation similar to that given in Section 7.

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