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ASYMPTOTIC BEHAVIOR OF IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT. Based on integral inequalities and on the Banach's fixed point theorem, we study the problem of asymptotic equilibrium for impulsive differential equations at fixed times.

1. Introduction. The theory of impulsive differential equations has been developed over the last ten years (see [**2–5; 9–11**]). The evolution processes which, at certain points in time experience an abrupt change of state, are subjected to short-time perturbations of negligible lasting compared to that of the process. We assume that these perturbations act instantaneously, that is, in the form of impulses. These processes appear as a natural description of many models in medicine, biology, optimal control models in economics, etc.

In this paper we study the problem of the asymptotic equilibrium for a class of impulsive differential equations at fixed times, satisfying the following condition.

$[H_0]$ Let $\{t_i\}_{i=1}^{\infty} \subset I$ be an unbounded, strictly increasing sequence of times and $f(t,x)$ a continuous function in $I \times \mathbf{R}^n$, where $I = [t_0, \infty)$, with values in $\mathbf{R}^n$.

We consider the impulsive differential equation at fixed times [**2**].

$$\begin{aligned} x' &= f(t,x), \qquad t \neq t_i, \quad i = 1,2,\dots \\ \Delta x(t_i) &= \varphi_i(x(t_i)), \qquad i = 1,2,\dots, \end{aligned} \tag{1}$$

where the impulse functions φ_i, $i = 1,2,\dots$ are defined and continuous in $\mathbf{R}^n$ with values in $\mathbf{R}^n$, $\Delta x(t_i) = x(t_i + 0) - x(t_i)$ and

$$x(t_i + 0) = \lim_{\varepsilon \to 0^+} x(t_i + \varepsilon).$$

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Let $B(0,r) = \{x \in \mathbf{R}^n / |x| < r\}$, where $|\cdot|$ is any norm in $\mathbf{R}^n$.

Definition 1. We say that the impulsive differential equation (1) has *asymptotic equilibrium* if

(i) There exists a positive real number r such that for any $a \geq t_0$, the equation (1) with initial condition $x(a) = x_0$ has a solution $x(t)$ defined on $[a, \infty)$ and it satisfies

$$\lim_{t \to \infty} x(t) = \xi \tag{2}$$

for some $\xi \in \mathbf{R}^n$.

(ii) For all $\xi \in \mathbf{R}^n$ there exist $a \in I$ and a solution $x(t)$ of (1) defined on $[a, \infty)$ and satisfying (2).

We study this problem for the class of impulsive differential equations (1), defined by the following conditions over f and the sequence of impulses $\{\varphi_i\}_{i=1}^{\infty}$.

$[H_1]$ a) There exists an integrable function λ on I such that for all (t,x) in $I \times \mathbf{R}^n$

$$|f(t,x)| \leq \lambda(t)|x|.$$

b) There exists a summable sequence of nonnegative real numbers μ_i, $i = 1, 2, \ldots$ such that for all $x \in \mathbf{R}^n$

$$|\varphi_i(x)| \leq \mu_i |x|, \qquad i = 1, 2, \ldots$$

$[H_2]$ a) The function $f(t,0)$ is integrable on I, and there exists an integrable function $\hat{\lambda}$ on I such that for all $(t,x), (t,y)$ in $I \times \mathbf{R}^n$,

$$|f(t,x) - f(t,y)| \leq \hat{\lambda}(t)|x - y|.$$

b) The sequence $\{\varphi_i(0)\}_{i=1}^{\infty}$ is absolutely summable, and there exists a summable sequence of nonnegative real numbers $\{\hat{\mu}_i\}_{i=1}^{\infty}$ such that for all x, y in $\mathbf{R}^n$,

$$|\varphi_i(x) - \varphi_i(y)| \leq \hat{\mu}_i |x - y|$$

for $i = 1, 2, \ldots$.

For this class of impulsive differential equations we will prove that (1) has global asymptotic equilibrium, that is, the asymptotic equilibrium is valid for any $r > 0$. The technique used to study the asymptotic equilibrium is based on an impulsive generalization of the Gronwall-Bellman lemma [**2, 6**] and on the well-known Banach fixed point theorem. Similar techniques have been used in [**1, 2, 3, 7, 8**].

In the last two sections, we will give some applications to the theory of impulsive linear systems and some examples which illustrate the results obtained.

2. Main results. We need a few preliminary results for proving the asymptotic equilibrium of (1). The following result (see [**2, 10**]) show an equivalence between the impulsive problem (1) and one integral equation.

Lemma 1. *Let ψ be a piecewise continuous function defined in $[a, T) \subseteq [t_0, \infty)$, with discontinuities of the first kind and left continuous at t_i, $a \leq t_i < T$. Then ψ is a solution of* (1) *with initial condition $\psi(a+0) = x_0$ if and only if ψ is a solution of the integral equation*

$$x(t) = x(a+0) + \int_a^t f(s, x(s))\, ds + \sum_{a<t_i<t} \varphi_i(x(t_i)) \tag{3}$$

for all $t \in [a, T)$.

The following result (see [**2, 6**]) is an impulsive generalization of the Gronwall-Bellman lemma.

Lemma 2. *Assume that m is a piecewise continuous real function in I, with discontinuities of the first kind and left continuous at t_i, $i = 1, 2, \dots$. Moreover, if p is a nonnegative continuous function in I and*

$$m(t) \leq c + \int_{t_0}^t p(s) m(s)\, ds + \sum_{t_0<t_i<t} \beta_i m(t_i), \qquad t \geq t_0,$$

where c and $\{\beta_i\}_{i=1}^{\infty}$ are nonnegative constants, then for $t \geq t_0$, we have

$$m(t) \leq c\Pi_{t_0<t_i<t}(1+\beta_i) \exp\left[\int_{t_0}^t p(\sigma)\, d\sigma\right].$$

The next result shows that the class of impulsive differential equations (1) verifying hypothesis H_1 satisfies condition (i) of definition 1. The proof is based on Lemmas 1 and 2 and hypothesis $[H_1]$.

Theorem 1. *Assume that conditions* $[H_0]$ *and* $[H_1]$ *hold. Then every solution* $x(t)$ *of* (1) *with initial condition* $x(a) = x_0$, $a \geq t_0$, *is defined on* $[a, \infty)$ *and satisfies* (2) *for some* $\xi \in \mathbf{R}^n$.

Proof. If $x(t)$ is a solution of (1) with initial condition $x(a) = x_0$, $a \geq t_0$, defined on a finite subinterval $J \subset [a, \infty)$, then $x(t)$ is a solution of (3), and for all $t \in J$ we have

$$\begin{aligned} |x(t)| &\leq |x(a+0)| + \int_a^t |f(s, x(s))|\, ds \\ &\quad + \sum_{a<t_i<t} |\varphi_i(x(t_i))| \\ &\leq |x(a+0)| + \int_a^t \lambda(s)|x(s)|\, ds \\ &\quad + \sum_{a<t_i<t} \mu_i |x(t_i)| \end{aligned}$$

then by Lemma 2 we have

$$|x(t)| \leq |x(a+0)| \Pi_{a<t_i<t}(1+\mu_i) \exp\left[\int_a^t \lambda(s)\, ds\right].$$

We know that the summability of the sequence $\{\mu_i\}_{i=1}^{\infty}$ implies that the product $\Pi_{a<t_i<t}(1+\mu_i)$ converges. Now the integrability condition of λ in I proves that $x(t)$ is bounded on J, thus it can be continued beyond $\sup J$.

As $x(t)$ is a solution of (3), hypothesis $[H_1]$ implies that $x(t)$ is continuable to $[a, \infty)$ and bounded in this interval. Since $f(t, x(t))$ is integrable and the sequence $\{\varphi_i(x(t_i))\}_{i=1}^{\infty}$ is absolutely summable, then from (3) we deduce that $\lim_{t\to\infty} x(t)$ exists and hence (2) holds. So the proof is complete. $\square$

In the following theorem, we will prove the terminal value problem ii) of definition 1. For this, we apply Banach's fixed point theorem and hypothesis $[H_2]$.

Theorem 2. *Assume that conditions* $[H_0]$ *and* $[H_2]$ *hold. Then for each* $\xi \in \mathbf{R}^n$ *there exist* $a \in I$ *and a solution* $x(t)$ *of* (1) *defined on* $[a, \infty)$ *which verifies* (2).

Proof. Using hypothesis H_2, we can choose a sufficiently large real number $a \geq t_0$, so that

$$\alpha = \int_a^\infty \hat{\lambda}(s)\, ds + \sum_{t_i > a} \hat{\mu}_i < 1.$$

Let $\mathcal{B}$ be the Banach space of bounded functions defined on $[a, \infty)$ with values in $\mathbf{R}^n$. The norm is given by

$$||f|| = \sup\{|f(t)| : t \in [a, \infty)\}.$$

Let us take the operator $T : \mathcal{B} \to \mathcal{B}$ defined by

$$(Tx)(t) = \xi - \int_t^\infty f(s, x(s))\, ds - \sum_{t_i > t} \varphi_i(x(t_i)).$$

Integrability of $f(\cdot, 0)$ and summability of the sequence $\{\varphi_i(0)\}_{i=1}^\infty$ guarantee that T is an operator with values in $\mathcal{B}$. To show that T has a fixed point, it suffices to prove that T is a contradiction on $\mathcal{B}$. By using Lipschitz condition on f and φ_i, $i = 1, 2, \ldots$, we see that the following estimates are valid for all $x_1, x_2 \in \mathcal{B}$ and $t \in [a, \infty)$:

$$\begin{aligned}
|Tx_1(t) - Tx_2(t)| &= |\int_t^\infty [f(s, x_1(s)) - f(s, x_2(s))]\, ds \\
&\quad + \sum_{t_i > t} [\varphi_i(x_2(t_i)) - \varphi_i(x_1(t_i))]| \\
&\leq \Big[\int_t^\infty \hat{\lambda}(s)\, ds + \sum_{t_i > t} \hat{\mu}_i] ||x_1 - x_2|| \\
&\leq \alpha ||x_1 - x_2||,
\end{aligned}$$

and this shows that T is a contraction on $\mathcal{B}$. Therefore, Banach's fixed point theorem guarantees that T will have a unique fixed point x in $\mathcal{B}$, i.e.,

$$x(t) = \xi - \int_t^\infty f(s, x(s))\, ds - \sum_{t_i > t} \varphi_i(x(t_i)),$$

for all $t \geq a$. Clearly (2) is satisfied and

$$x(t) = \xi' + \int_a^t f(s, x(s))\, ds + \sum_{t_i < t} \varphi_i(x(t_i)),$$

where

$$\xi' = \xi - \int_a^\infty f(s, x(s))\, ds - \sum_{t_i > a} \varphi_i(x(t_i)),$$

i.e., $x(t)$ is a solution of (1). □

3. Applications to linear systems. In this section we assume the following hypothesis.

$[H_0']$ Let $\{t_i\}_{i=1}^\infty \subset I$ be an unbounded, strictly increasing sequence of times, A a continuous $n \times n$ matrix in $[t_0, \infty)$ and $\{D_i\}_{i=1}^\infty$ a sequence of constant $n \times n$ matrices called the impulse matrices.

We consider the impulsive differential system at fixed times t_i, $i = 1, 2, \dots$,

$$\begin{aligned} x' &= A(t)x, \qquad t \neq t_i, \quad i = 1, 2, \dots \\ \Delta x(t_i) &= D_i x(t_i), \qquad i = 1, 2, \dots . \end{aligned} \tag{4}$$

Corollary 1. *Assume that condition* $[H_0']$ *is fulfilled. If* $A(t)$ *is integrable on* $[t_0, \infty)$ *and the sequence* $\{D_i\}_{i=1}^\infty$ *is absolutely summable, then every solution of* (4) *with initial condition* $x(a) = x_0$, $a \geq t_0$, *is defined on* $[a, \infty)$ *and verifies* (2) *for some* $\xi \in \mathbf{R}^n$. *Conversely, for all* $\xi \in \mathbf{R}^n$ *there exists* $a \geq t_0$ *and a solution* $x(t)$ *of* (1) *defined on* $[a, \infty)$ *satisfying* (2).

Corollary 2. *Assume that condition* $[H_0']$ *is fulfilled. If* $R(t)$ *is a continuous* $n \times n$ *matrix on* $[t_0, \infty)$, $\Phi(t)$ *is a fundamental matrix of the linear system*

$$x' = A(t)x,$$

and the following conditions hold

i) $\Phi^{-1}(t)R(t)\Phi(t)$ *is integrable in* $[t_0, \infty)$.

ii) $\sum_{i=1}^{\infty} |\Phi^{-1}(t_i)D_i\Phi(t_i)|$ *is convergent.*

Then every solution $x(t)$ *with initial condition* $x(a) = x_0$, $a \geq t_0$, *of the impulsive differential system*

$$\begin{aligned} x' &= (A(t) + R(t))x, \qquad t \neq t_i, \quad i = 1, 2, \dots \\ \Delta x(t_i) &= D_i x(t_i), \qquad i = 1, 2, \dots \end{aligned}$$

is defined on $[a, \infty)$, *and there exists* $\xi \in \mathbf{R}^n$ *such that*

$$x(t) = \Phi(t)[\xi + o(1)] \tag{6}$$

as t *tends to* ∞. *Conversely, for all* $\xi \in \mathbf{R}^n$ *there exist* $a \geq t_0$ *and a solution* $x(t)$ *of* (5) *defined on* $[a, \infty)$ *such that* (6) *is verified.*

Proof. Consider the change of variable $x(t) = \Phi(t)u(t)$. Equation (5) becomes

$$\begin{aligned} u' &= \Phi^{-1}(t)R(t)\Phi(t)u, \qquad t \neq t_i, \quad i = 1, 2, \dots \\ \Delta u(t_i) &= \Phi^{-1}(t_i)D_i\Phi(t_i)u(t_i), \qquad i = 1, 2, \dots \end{aligned} \tag{7}$$

Now the result is a consequence of Theorems 1 and 2. $\square$

4. Examples. a) Consider the impulsive differential system at fixed times $t_i = i = 1, 2, \dots$

$$\begin{aligned} x' &= 0, \qquad t \neq t_i, t_i = i = 1, 2, \dots \\ \Delta x(t_i) &= a_i x(t_i), \qquad i = 1, 2, \dots \end{aligned} \tag{8}$$

in the interval $[0, \infty)$, where a_i are real numbers.

The solutions of $x' = 0$ are the constants; therefore the solution of (8) with initial condition $x(0) = x_0$ is defined by

$$x(t) = \begin{cases} x_0, & \text{if } t \in [0, 1] \\ x_0 \prod_{i=1}^{k} (1 + a_i), & \text{if } t \in (k, k+1]. \end{cases}$$

Clearly the limit of $x(t)$ as t approaches infinity exists if and only if the product $\prod_{i=1}^{\infty} (1 + a_i)$ converges. This product may be convergent even when the sequence $\{|a_i|\}_{i=1}^{\infty}$ is not summable. So, our conditions are only sufficient.

b) Consider the impulsive differential system for $t \geq 0$ given by

$$\begin{aligned} x' &= \frac{1}{(1+t)^2} y \\ y' &= e^{-t} x \qquad \text{for } t \neq t_i, \quad t_i = i = 1, 2, \dots \end{aligned} \tag{9}$$

with impulses at fixed times $\{t_i\}_{i=1}^{\infty}$ defined by

$$\Delta(x(t_i), y(t_i)) = (1/2^{i+1})(x(t_i), y(t_i)), \qquad i = 1, 2, \dots$$

Observe that

$$A(t) = \begin{pmatrix} 0 & \frac{1}{(t+1)^2} \\ e^{-t} & 0 \end{pmatrix}$$

is a continuous and integrable matrix on $[0, \infty)$. Moreover, for $a > 0$, for a suitable norm

$$\int_a^{\infty} |A(t)| \, dt = \int_a^{\infty} \frac{1}{(t+1)^2} \, dt = \frac{1}{a+1}.$$

In this example, with a suitable norm

$$|D_i| = \left| \begin{pmatrix} \frac{1}{2^{i+1}} & 0 \\ 0 & \frac{1}{2^{i+1}} \end{pmatrix} \right| = \frac{1}{2^{i+1}}, \qquad \text{for } i = 1, 2, \dots ,$$

thus

$$\sum_{i=1}^{\infty} |D_i| = \frac{1}{2}.$$

Then every solution of (9) converges as t approaches infinity. Moreover, if $a > 1$ then for all $\xi \in \mathbf{R}^2$, there exists a solution defined on $[a, \infty)$ such that (2) is verified.

c) Let $A(t)$ be a continuous and integrable $n \times n$ matrix defined on $I = [t_0, \infty)$ and h a K-Lipschitz function defined and with values in $\mathbf{R}^n$, where K is a positive constant. Consider a sequence $\{\varphi_i\}_{i=1}^{\infty}$ of impulses such that the hypothesis $H_1(b)$ is verified. Then every solution $x(t)$ of the impulsive differential system

$$\begin{aligned} x' &= A(t)h(x), \qquad t \neq t_i, \quad i = 1, 2, \dots \\ \Delta x(t_i) &= \varphi_i(x(t_i)), \qquad i = 1, 2, \dots \end{aligned} \tag{10}$$

with initial condition $x(a) = x_0$, $a \geq t_0$, is defined on $[a, \infty)$ and converges as t approaches infinity. Moreover, if hypothesis $H_2(b)$ is verified and a satisfies

$$\alpha = \int_a^\infty K|A(t)|\,dt + \sum_{t_i > a} \hat{\mu}_i < 1,$$

then for all $\xi \in \mathbf{R}^n$ there exists a solution $x(t)$ of (9) defined on $[a, \infty)$ such that (2) is verified.

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