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MODULE TYPES

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ABSTRACT. The direct sum decomposition of torsion free, i.e., nonsingular modules, into direct summands belonging to saturated classes of modules is developed here for the first time for arbitrary not necessarily torsion free modules. The previous classification of modules into types I, II, III, molecular, continuous molecular, and bottomless is extended to all modules from the previous torsion-free case. It is shown that there exists a contravariant functor Σ applicable to any associated ring R with identity—where $\Sigma(R)$ is a complete Boolean lattice. Each element $\Delta \in \Sigma(R)$ is a saturated class of right R-modules. The above six classes of modules are special examples of a general phenomenon—a universal saturated class. These have various functorial properties connected with the functor Σ. It is shown that there is a class of pairwise disjoint universal saturated classes one for each cardinal number.

0. Introduction. A saturated class of unital modules Δ over some ring R is a class of modules closed under isomorphic copies, submodules, direct sums, and injective hulls. In previous studies by Goodearl and Boyle [**23**], Rios and Tapia [**36**] and the author [**13, 14, 15**] the modules were required in addition to be torsion-free, that is, nonsingular. One of the objectives of this article will be to extend presently existing theory for torsion-free modules this theory to arbitrary modules including torsion and mixed. There are (infinitely) many examples. In fact, it is shown in this article that, for a fixed ring R and any given nonempty class of R-modules Υ, this class Υ generates a clearly describable unique saturated class (Proposition 2.5).

The saturated classes used here are special cases of the more general Wisbauer classes $\sigma[M]$ [**38**] as well as the natural classes used very recently by Page and Zhou [**33, 34, 35, 39**]. On the other hand, various special cases of torsion-free saturated classes have been used by different authors, for a long time, in different contexts without abstractly formalizing this concept [**32, 28, 23, 22, 3, 7, 8, 9**].

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The classes C and D of continuous and discrete modules [**9**] or types I, II and III are just a few examples. There are some others. However, the above examples have at least three remarkable additional features. First of all, they are definable over any ring (with unity) without exception. Secondly, every torsion-free injective module M over any ring decomposes uniquely as a direct sum into fully invariant direct summands, e.g., $M = M_C \oplus M_D$, or $M = M_I \oplus M_{II} \oplus M_{III}$. Surprisingly, the third property has been formulated and systematically analyzed in its full generality only very recently [**11, 13**–**17**] because to do so requires the concept of a universal saturated class. The latter was first defined in [**13, 14**]. And this third property is that the above classes (C, D or I, II and III) have various associated functorial properties. Suppose that $N = N_C \oplus N_D = N_I \oplus N_{II} \oplus N_{III}$ is an injective module over a ring S, and $\varphi : R \to S$ is a surjective ring homomorphism. Thus φ induces an R-module N_φ on the set $N = N_\varphi$. Then the previous direct sum decompositions of N as an S-module are also at the same time the same corresponding decompositions of N viewed as the R-module N_φ. There is one difficulty. As an R-module N_φ need not be injective. For this and many other reasons, the author has advocated the theory of saturated classes, and in particular types I, II and III for not necessarily injective modules [**14**].

The previous method of classification of operator algebras and torsion-free modules M into types I, II, III would have to be based on the central idempotents of the ring $\mathrm{End}_R\widehat{M}$ where $\widehat{M}$ is the injective hull of M [**31, 28, 3, 23**]. For actual concrete applications, frequently it is not possible to compute the injective hull $\widehat{M}$ from M. And even if this could be done, it may be impossible to calculate $\mathrm{End}_R\widehat{M}$. To circumvent these difficulties, here all the definitions and results will be for arbitrary modules, injective or not.

It was shown in [**11, 16**] that there is a contravariant functor Σ from rings $R, S, \ldots$ to complete Boolean lattices, where each point of $\Sigma(R)$ is a torsion-free class of saturated modules, and appropriate ring homomorphisms $\varphi : R \to S$ induce lattice homomorphisms $\varphi^* : \Sigma(S) \to \Sigma(R)$. There are two other functors Ξ and $\mathcal{I}$ which are isomorphic to Σ, but each having distinct advantages where each one can be used in ways that the others cannot [**11**–**14**]. For example, every ring R (with $1 \in R$) actually always contains a completely unique complete Boolean lattice $\mathcal{I}(\mathcal{R})$ ideals of R [**13**–**15**]. Even though

$\Sigma \cong \mathcal{I} \cong \Xi$, we can do things with $\mathcal{I}$ very simply which we cannot with Σ or Ξ, and of course conceptually "$\mathcal{I}$" is very different from Σ or Ξ. In [**16**] as well as [**17**] it was shown that if the torsion-free hypothesis was omitted, we still got a functor from rings to complete lattices. But then, and several years afterwards, it was not known whether these resulting lattices were even modular. All previous techniques of proof used in the torsion-free case failed. It is one of the main objectives of this article to show that the class $\Sigma(R)$ of all saturated classes of right R-modules is a complete Boolean lattice in a natural way (Theorem I). The partial order on $\Sigma(R)$ is class inclusion, the meet of $\Delta_1, \Delta_2 \in \Sigma(R)$ is $\Delta_1 \wedge \Delta_2 = \Delta_1 \cap \Delta_2$ just intersection. The join $\Delta_1 \vee \Delta_2$ is *not* the union, however.

The main results are labelled as propositions, theorems, and their corollaries. If Υ is any class of modules, the first Proposition 2.5 and Corollary 2.6 give explicit descriptions of the saturated class $\langle \Upsilon \rangle$ generated by Υ, i.e., the smallest saturated class containing Υ. Given any finite or infinite set Γ of saturated classes $\Gamma \subseteq \Sigma(R)$, the second Proposition 2.11 gives workable descriptions of infimum Γ and supremum Γ, i.e., $\wedge\Gamma$, $\vee\Gamma \in \Sigma(R)$. However, Theorem I is needed to prove that $\langle \oplus_{i\in I} N_i \rangle = \vee_{i \in I} \langle N_i \rangle \in \Sigma(R)$.

Suppose that $\Gamma \subset \Sigma(R)$ is any finite or infinite pairwise disjoint set with $\vee\Gamma = 1 \in \Sigma(R)$. Theorem II shows that for any R-module M, there is an essential submodule $\oplus_{\gamma\in\Gamma} M_{(\gamma)} \subseteq M$ with $M_{(\gamma)} \in \gamma \in \Gamma$, alternatively, $\widehat{M} = E(\oplus_{\gamma\in\Gamma} \widehat{M}_{(\gamma)})$ where "$\widehat{\ }$" and "E" denote right R-injective hulls, and $\widehat{M}_{(\gamma)} \in \gamma \in \Gamma$. In the general case the $M_{(\gamma)}$ are no longer unique as was the case when M was torsion-free [**15**, p. 58]. However, the $\widehat{M}_{(\gamma)}$ are superspective (in the sense of Mohamed and Müller [**32**, p. 12]).

The problem in Section 4 is to extend the definitions of types I, II, and III to all modules, not just the torsion-free injectives (as in [**23**]). In this, but even more so in related similar problems, it is not clear that there exist unique extensions—written types I, II, or III with small t—of the definitions to cover also torsion modules. The next problem is to show that the extended classes of type I, II and III are actually saturated classes. Both problems require another general method of building saturated classes (Theorem III). Perhaps later this theorem or some modification of it could be applied to other classes of modules which are not quite injective, but injective in some relative sense, e.g., as in [**33, 34, 35** and **39**]. Then in Section 4 in a similar way the definitions of continuous (C), discrete (D), molecular (A), continuous molecular (CA), and bottomless (B) modules are extended to the general not necessarily torsion-free case. The extended definitions result again in saturated classes, i.e., points of $\Sigma(R)$. The above five classes of modules, i.e., A, B, C, D and CA, where defined and studied in the torsion-free case in [**11**], where to the author's best understanding B, A and CA first appeared. They were discovered (and defined) by means of the functor $\Xi \cong \Sigma$. This again shows that parts of the older theory of types I, II and III are merely special cases of the more general theory of saturated classes. (Although $\Xi(R) \cong \Sigma(R)$, the points of $\Xi(R)$ could be called unsaturated classes.) The present approach in *no* way generalizes most of [**23**] or even attempts to do so, particularly all of Chapters VI–XIV of [**23**]. Also, it is beyond the scope of this article to describe the connection between the present article and the remarkable work [**36**]. There remain many open problems which arise from this article, and it is not yet clear whether the extended theory of saturated classes (including the torsion modules) has been cast in its optimal form. In particular, perhaps some of the present theory could also be applied or extended to the natural classes of Page and Zhou [**33, 34, 35** and **39**]. The author hopes to return to these questions later.

Section 5 shows that Σ is a contravariant functor (Theorem IV). As already stated earlier, for every ring R we have these classes, $I(R)$, $II(R)$, $III(R)$, $C(R)$, $D(R)$, $B(R)$, $A(R)$ and $CA(R)$. Section 6 shows that the functions I, II, III, C, D, B, A and CA are universal saturated classes. The direct sum decomposition property of the latter classes mentioned earlier is shown to be a special case of a general phenomenon of universal saturated classes (Theorem V). It happens when Σ is a finite direct sum, or a product of subfunctors corresponding to the given set of classes (e.g., $\Sigma = \Sigma_I \oplus \Sigma_{II} \oplus \Sigma_{III}$, or $\Sigma = \Sigma_C \oplus \Sigma_D$, or $\Sigma = \Sigma_{CA} \oplus \Sigma_D \oplus \Sigma_B$). Theorem V is proved for a finite for infinite class Δ_i, $i \in I$ of universal saturated classes such that for any ring R, $\{i \mid \Delta_i(R) \neq 0\}$ is a set, and $\{\Delta_i(R) \mid i \in I\} \subset \Sigma(R)$ is a pairwise orthogonal subset with $\sup\{\Delta_i(R) \mid i \in I\} = 1 \in \Sigma(R)$. Then any module M contains an essential direct sum $\oplus_{i\in I} M_i \ll M$, $M_i \in \Delta_i(R)$ as guaranteed by Theorem II, i.e., $\widehat{M} = E(\oplus_{i\in I}\widehat{M}_i)$, $\widehat{M}_i \in \Delta_i(R)$. Then also $\Sigma = \prod\{\Sigma_{\Delta_i} \mid i \in I\}$ where $\Sigma_{\Delta_i} \leq S$ is a subfunctor related to Δ_i. For a long time it was not known whether there were basically only a finite number of universal saturated classes or a set of these with a bounded cardinality, or an improper class. One new consequence of the theory in this article is that they form a proper class. For every cardinal $\aleph$, the class of modules $\Delta_\aleph$ of local Goldie dimension $\aleph$ is a universal saturated class containing both torsion and torsion-free modules (6.18). A module M is said to be of *local Goldie dimension* $\aleph$, if every nonzero submodule $0 \neq V \leq M$ contains a nonzero submodule $0 \neq W \leq V$ such that the Goldie dimension in the sense of [**18**] is $\aleph$.

1. Preliminaries.

A brief summary of some of the notations, terminology, conventions, and abbreviations used is given all at once in the same place.

Notation 1.1. Right unital R-modules are used over an arbitrary associative ring R. The notation "$M = M_R$" means that M is a right R-module and "$V = V_T$" or just V_T that V is a right module over a ring T, and similarly for left modules $V = {}_TV$. Submodules K of M are denoted by $K < M$ or $K \leq M$, or occasionally by $K \subset M$, $K \subseteq M$ when the set theoretic containment is to be emphasized; large submodules are denoted by "$\ll$," while $A <\not\ll B$ means that $A < B$ but A is not essential. A complement $P \leq Q$ is any submodule of P which does not have a proper essential extension inside Q; P is also said to be *closed* in Q.

For $x \in M$, $x^\perp = \{r \in R \mid xr = 0\} < R$, and the inverse image of K under the right R-map $R \to M$, $r \to xr$, is denoted by $x^{-1}K = \{r \in R \mid xr \in K\} = (x+K)^\perp \leq R$. Similarly, for any subset $X \subseteq M$, $X^\perp = \{r \mid Xr = 0\} \leq R$; and hence, $M^\perp \triangleleft R$ where "$\triangleleft$" stands for ideals not only in R but other rings, too.

Right R-injective hulls of modules are denoted by both "$\widehat{\ }$" and "E" as $\widehat{M} = EM = E(M)$, where the latter is used if M is given by a complicated formula. Thus $M \ll \widehat{M}$. The singular and second singular submodules are $Z(M) = ZM = \{x \in M \mid x^\perp \ll R\} \ll Z_2(M) = Z_2M \leq M$, where $Z[M/ZM] = Z_2M/ZM$. A module M is *torsion* if $Z_2M = M$ and *torsion-free*, abbreviated "t.f." if $Z_2M = 0$, which happens if and only if $ZM = 0$. For Z and Z_2 see [**11**, pp. 52–55], as well as [**16**, p. 748].

The complement closure $\overline{K}$ of K in M is defined only if $ZM \subseteq K$ by $Z(M/K) = \overline{K}/K$, where $K \ll \overline{K} \leq M$. If $K \triangleleft R$, then also $\overline{K} \triangleleft R$. For complement closures, see [**14**, pp. 101–104].

Internal and external direct sums of modules are denoted by "$\oplus$." If $A_i \cong A$, $i \in I$, write $\oplus\{A_i \mid i \in I\} = \oplus\{A \mid I\}$. Internal and usually nondirect sums of modules are denoted by "Σ." When the index set is surpassed in $\cup A_i$, $\cap A_i$, $\oplus A_i$ or ΣA_i, then it is understood that i ranges over the largest possible index set. The notation "$A \hookrightarrow B$" means that there exists some not necessarily unique embedding of the module A into the module B.

The cardinality of any set X is denote by $|X|$, while the set of all subsets $\mathcal{P}(X)$ of X is a Boolean lattice $\langle \mathcal{P}(X), \cup, \cap, \setminus, 0 = \varnothing, 1 = X\rangle$ and, completely equivalently, that is canonically, a Boolean ring, where lattice and ring ideals and homomorphisms coincide.

The category of right R-modules is written as $\mathcal{M}\mathrm{od}_R$, and the category of torsion-free modules, where the morphisms are R-homomorphisms with closed kernels as t.f. $\mathcal{M}\mathrm{od}_R$. Similarly, tor-$\mathcal{M}\mathrm{od}_R$ denotes the full subcategory of $\mathcal{M}\mathrm{od}_R$ consisting of all torsion modules.

Terminology 1.2. The terminology from Mohamed and Müller [**32**, p. 12, Definition 1.30] is used for direct summands $A, B \leq M$ of a module M; A and B are *perspective* if $M = A \oplus C = B \oplus C$ for some $C \leq M$; they are *superspective* if for any $D \leq M$, $M = A \oplus D \Leftrightarrow M = B \oplus D$. A decomposition of M as $M = A_1 \oplus A_2$ with certain properties (say $\mathcal{P}$) is *unique* up to *superspectivity* if for any other decomposition $M = B_1 \oplus B_2$ satisfying $(\mathcal{P})$, A_i and B_i are superspective for $i = 1$ and 2.

As in [**7**, p. 34], a module M is A-free if M does not contain a nonzero submodule isomorphic to a submodule of A.

Since the following simple argument has to be used repeatedly later on, and sometimes without mentioning that it is being used, it is stated here. Its proof does not require that any of the modules in it be torsion-free (see [**13**, p. 329] or [**14**, p. 101] or [**15**, p. 43]).

Projection argument 1.3. *Let* W_γ, $\gamma \in \Gamma$, *be any indexed family of modules and* $0 \neq \xi \in E(\oplus_{\gamma\in\Gamma} W_\gamma)$. *Then there exist* $0 \neq r_0 \in R$, $\gamma \in \Gamma$ *and* $0 \neq w \in W_\gamma$ *such that* $0 \neq \xi r_0 R \cong wR < W_\gamma$ *with* $(\xi r_0)^\perp = w^\perp$.

2. Saturated classes. For a given, fixed R-module M, let $\sigma[M]$ be the full subcategory of $\mathcal{M}\mathrm{od}_R$ subgenerated by M [**38**, p. 118]. S. Page and Y. Zhou define in [**34**, p. 634] an M-natural class $\mathcal{K}$ to be a subclass $\mathcal{K} \subseteq \sigma[M]$ closed under (a) submodules, (b) isomorphic copies, (c) direct sums and M-injective hulls. (See also [**39**, p. 928]; and for $M = R$, [**35**, p. 2912].) An M-natural class is more general than a saturated class of modules below in 2.2. If M is a generator of $\mathcal{M}\mathrm{od}_R$, e.g., $M = R$ or $M = ER$, then the M-injective hull is just the usual injective hull. In this case, when $M = R$, the above definition of M-natural class becomes the same as the Definition 2.2 of a saturated class. Every one of S. Page's and Y. Zhou's many theorems proved for M-natural classes have immediate corollaries valid for saturated classes obtained by specializing M to $M = R$. Their work is in a quite different direction and does not overlap with this article because they ask and answer very different questions. Definition 2.2 below is completely identical to the definition of a 'saturated class' given in [**15**, Definition 3.2]. However, later there, and in [**13**, p. 329], and [**14**, p. 108] as well, in addition all modules were torsion-free, which in the present context is nevertheless still a specialized kind of saturated class, a so-called t.f. saturated class.

Definition 2.1. For any nonempty class of right R-modules Δ, define a *complementary class*

$$c(\Delta) = c\Delta = \{W \in \mathcal{M}\mathrm{od}_R \mid \forall\, 0 \neq V \leq W, V \notin \Delta\}.$$

Modules $W \in c\Delta$ are called Δ-*free*. Consequently,

$$c(c\Delta) = \{N \in \mathcal{M}\mathrm{od}_R \mid \forall\, 0 \neq W \leq N, \exists\, 0 \neq V \leq W, V \in \Delta\}.$$

Any module N possessing this latter defining property of $c(c\Delta)$ is said to be Δ-*dense*. Such modules N may be thought of as being locally in Δ.

For any module M and any class Δ, define $M_\Delta \leq M$ by $M_\Delta = \Sigma\{V \mid V \leq M, V \in \Delta\}$. Thus we also get $M_{c(\Delta)} \leq M$.

Definition 2.2. A nonempty class Δ of right R-modules is called a *saturated class* if Δ is closed under (a) submodules, (b) isomorphic copies, (c) arbitrary direct sums and (d) essential extensions. When (a) holds, (d) is equivalent to the condition that (d′) Δ is closed under injective hulls. The term 'saturated' refers to property (a) mainly, but also to (b). If Δ consists entirely of torsion-free, or torsion, modules it is called a t.f. saturated class or a torsion saturated class. Clearly, $\{(0)\}$ and $\mathcal{M}\text{od}_R$ are saturated classes. Since $Z_2E = EZ_2$, so are t.f. $\mathcal{M}\text{od}_R$ and all torsion R-modules tor$\mathcal{M}\text{od}_R$.

Definition 2.3. For any R, the class of all saturated classes of modules is denoted by $\Sigma(R)$. For $\alpha, \beta \in \Sigma(R)$, $\alpha \le \beta$ if and only if $\alpha \subseteq \beta$ defines a *partial order* on $\Sigma(R)$ with smallest and largest elements $\{(0)\} = 0 \in \Sigma(R)$ and $\mathcal{M}\text{od}_R = 1 \in \Sigma(R)$. For $\Delta_1, \Delta_2 \in \Sigma(R)$ whenever their greatest lower and least upper bounds exist in the poset $\Sigma(R)$, they are denoted as $\Delta_1 \vee \Delta_2$, $\Delta_1 \wedge \Delta_2 \in \Sigma(R)$, and similarly for infinite sups and infs.

Later it will be shown that the class $\Sigma(R)$ actually is a set.

Notation 2.4. The intersection of any set of saturated classes is again a saturated class. For any nonempty class of right R-modules Υ, $\langle \Upsilon \rangle$ denotes the saturated class generated by Υ, i.e., $\langle \Upsilon \rangle$ is the nonempty intersection of all saturated classes containing Υ. Again, it will be shown later that the class of all saturated classes is a set, and hence that above we are intersecting only a set of classes, and not a class of classes. When $\gamma = \{M\}$ is a singleton, abbreviate $\langle \{M\} \rangle = \langle M \rangle$.

Proposition 2.5. *Let Υ be any class of modules, and $\langle \Upsilon \rangle$ the unique saturated class generated by Υ. Let $\overline{\Upsilon}$ be the class of all modules isomorphic to a submodule of some element of Υ. Then the following hold, where J is a set.*

(i) $\langle \Upsilon \rangle = \{M_R \mid \exists N_j \in \Upsilon, j \in J \text{ such that } \exists M \hookrightarrow E(\oplus_{j \in J} N_j)\}$.

(ii) $\langle \Upsilon \rangle = \{M_R \mid \exists P_j \hookrightarrow N_j \in \Upsilon, j \in J; \exists \oplus_{j \in J} P_j \ll M\}$.

(iii) $\langle \Upsilon \rangle = \langle \overline{\Upsilon} \rangle = cc\overline{\Upsilon} = \{N_R \mid \forall\, 0 \neq W \le N, \exists\, 0 \neq V \le W, V \in \overline{\Upsilon}\}$.

Proof. (i) The right side of (i) satisfies 2.2(a), (b) and (c). The proof that it also satisfies Definition 2.2 (d) is not difficult and is omitted. Thus the right side of (i) is a saturated class which contains Υ and is necessary contained in $\langle \Upsilon \rangle$.

(ii) Define Y to be the right hand side of (ii). Then Y is closed under Definition 2.2 (b), (c) and (d) by definition. Since $\langle \Upsilon \rangle = \langle \overline{\Upsilon} \rangle$, and since Y remains the same if Υ is replaced by $\overline{\Upsilon}$, it suffices to prove (ii) for Υ replaced with $\overline{\Upsilon}$. To prove Definition 2.2 (a) for Y, it has to be shown that for any $V \leq M \in Y$, also $V \in Y$. Since $\oplus_{j \in J} P_j \ll M \in Y$, also $\oplus_{j \in J}(P_j \cap V) \ll V$. But $P_j \cap V < P_j \hookrightarrow N_j$ shows that $P_j \cap V \in \overline{\Upsilon}$. Thus $V \in Y$ and Y is a saturated class. Trivially, $\Upsilon \subseteq \overline{\Upsilon} \subseteq Y$ and hence $\langle \overline{\Upsilon} \rangle \subseteq Y$. Since every element of Y is obtained by applying Definition 2.2 (d′) first and 2.2 (a) second to modules belonging to $\overline{\Upsilon}$, necessarily $Y \subseteq \langle \overline{\Upsilon} \rangle$.

(iii) First some general comments. For any class Υ,

$$cc\Upsilon = \{N_R \mid \forall\, 0 \neq W \leq N, \exists\, 0 \neq V \leq W, V \in \Upsilon\}.$$

That is, $cc\Upsilon$ consists exactly of the Υ-dense modules. Consequently, hypothesis 2.2 (a) alone on Υ is enough to guarantee that $\Upsilon \subseteq cc\Upsilon$. Now suppose that Υ satisfies 2.2 (a) and (b). Clearly $cc\Upsilon$ is closed under essential extensions 2.2 (d) and 2.2 (d′). Use of the projection argument 1.3 shows that $cc\Upsilon$ is closed under submodules 2.2 (a). Lastly, 2.2 (b) on Υ always implies 2.2 (b) on $cc\Upsilon$. Thus $cc\Upsilon$ is a saturated class, and since $\Upsilon \subseteq cc\Upsilon$, also $\langle \Upsilon \rangle \subseteq cc\Upsilon$.

Since $\overline{\Upsilon}$ is closed under 2.2 (a) and (b), $\langle \Upsilon \rangle = \langle \overline{\Upsilon} \rangle \subseteq cc\overline{\Upsilon}$. In order to prove the converse, take any $N \in cc\overline{\Upsilon}$ as in the above formula. There exists by Zorn's lemma $\oplus_{j \in J} V_j \ll N$ with $V_j \in \overline{\Upsilon}$. Thus $N \ll E(\oplus_{j \in J} V_j) \in \langle \overline{\Upsilon} \rangle$, and hence $N \in \langle \overline{\Upsilon} \rangle$. Consequently, $cc\overline{\Upsilon} \subseteq \langle \overline{\Upsilon} \rangle$. Thus $\langle \Upsilon \rangle = \langle \overline{\Upsilon} \rangle = cc\overline{\Upsilon}$. The last equality in 2.5 (iii) follows from the fact noted above more generally that $cc\overline{\Upsilon}$ consists of all the $\overline{\Upsilon}$-dense modules.

For some one module M, the previous proposition is now specialized to $\Upsilon = \{M\}$, and hence to $\langle \Upsilon \rangle = \langle M \rangle$. □

Corollary 2.6. *For any module M, the saturated class $\langle M \rangle$ generated by M satisfies the following.*

(iv) $\langle M \rangle = \{N_R \mid$ *for all* $0 \neq W \leq N$, *there exists a* $0 \neq V \leq W$ *such that* $V \cong P \leq M$ *for some submodule* P *of* $M\}$.

(v) $\langle M \rangle = \{N_R \mid$ *there exists a set* $J, P_j \leq N, P_j \hookrightarrow M, j \in J$ *such that* $\sum_{j\in J} P_j = \oplus_{j\in J} P_j \ll N\}$.

Proof. (iv) If $\Upsilon = \{M\}$, then $\overline{\Upsilon}$ in 2.5 is the set of all submodules of M and 2.5 (iii) reduces to (iv).

(v) Define Y to be the set in (v), $Y = \{N\}$. In (v), for $N \in Y$ and for any $j \in J$, $P_j \in \langle M \rangle$, also $\oplus_{j\in J} P_j \in \langle M \rangle$, and hence $N \in \langle M \rangle$. Thus $Y \subseteq \langle M \rangle$. Conversely any $N \in \langle M \rangle$ by 2.6 (iv) is easily seen to contain an essential direct sum of submodules such that each direct summand is isomorphic to a submodule of M. Thus $N \in Y$, $\langle M \rangle \subseteq Y$. Therefore $\langle M \rangle = Y$. □

One of the more useful ways of defining a saturated class $\Delta \in \Sigma(R)$ is to find some easily describable cyclics which generate Δ.

Lemma 2.7. *Let* $X \subset \{E(R/L) \mid L < R\}$ *be a complete set without repetitions of isomorphy classes of injective hulls of cyclic* R*-modules. Define functions* $g : \mathcal{M}\mathrm{od}_R \to X$ *by* $g(M) = gM = \{E(R/m^{\perp}) \mid 0 \neq m \in M, E(R/m^{\perp}) \in X\}$ *and* $f : \Sigma(R) \to X$ *by* $f(\Delta) = f\Delta = \{E(R/L) \mid E(R/L) \in \Delta \cap X\}$. *Let* $M_1, M_2 \in \mathcal{M}\mathrm{od}_R$ *and* $\Delta_1, \Delta_2 \in \Sigma(R)$ *represent arbitrary elements. Then*

(i) $\langle M_1 \rangle = \langle M_2 \rangle \leftrightarrow gM_1 = gM_2$;

(ii) $\Delta_1 = \Delta_2 \Leftrightarrow f\Delta_1 = f\Delta_2$;

(iii) $|\Sigma(R)| = |\mathcal{P}(X)| \leq 2^{|\mathcal{P}(R)|}$;

(iv) *for all* $\Delta \in \Sigma(R)$, $\Delta = \langle \oplus_{E(R/L)\in f(\Delta)} R/L \rangle$.

Proof. First note that for any module M, if $T \subset M$ is any subset such that $\{E(R/x^{\perp}) \mid x \in T\} \subseteq X$ is a set of representatives of isomorphy classes of injective hulls of cyclic submodules of M without repetitions, then $gM = \{E(R/x^{\perp}) \mid x \in T\} \subseteq X$ and $|gM| = |T|$.

(i) It suffices to show that for any module M, $\langle M \rangle = \langle M_* \rangle$, where $M_* = \oplus\{E(xR) \mid x \in T\}$. Let $\oplus\{x_iR \mid i \in I\} \ll M$ be any essential direct sum of cyclics of M. Define $M_i = M_*$ for all i, and map $x_iR \to M_* = M_i \subset \oplus_{i\in I} M_i$. Then this extends to $\oplus_{i\in I} x_iR$ and from there to $M \hookrightarrow E(\oplus_I M_*)$. Hence $M \in \langle M_* \rangle$, and

$\langle M \rangle \subseteq \langle M_* \rangle$. For the converse, set $M_t = M$ for all $t \in T$, and map $tR \to M = M_t \subset \oplus_{t \in T} M_t$ by natural inclusions. This extends to $M_* \hookrightarrow E(\oplus_T M)$. Thus $M_* \in \langle M \rangle$ and $\langle M_* \rangle = \langle M \rangle$.

(ii) For any $\Delta \in \Sigma(R)$ and any $M \in \Delta$, $gM \subseteq f\Delta = \cup\{gN \mid N \in \Delta\}$. Let $f\Delta_1 = f\Delta_2$. Then for any $M \in \Delta_1$, $gM \subseteq f\Delta_2$. Then $M_* \in \Delta_2$ by the above proof of (i) and $M \in \langle M \rangle = \langle M_* \rangle \in \Delta_2$. Thus $\Delta_1 \subseteq \Delta_2$ and by symmetry $\Delta_2 \subseteq \Delta_1$, $\Delta_2 = \Delta_1$. If $\Delta_1 = \Delta_2$, then trivially $f\Delta_1 = f\Delta_2$.

(iii) By (ii), $f : \Sigma(R) \to \mathcal{P}(X)$ is one-to-one. It is easy to see that f is onto. Thus $|\Sigma(R)| = |\mathcal{P}(X)|$. Each element $E(R/L)$ of X is determined by at least one subset $L \subset R$ of R. Hence $|X| \leq |\mathcal{P}(R)|$ and $|\mathcal{P}(X)| \leq |\mathcal{P}(\mathcal{P}(R)) = 2^{|\mathcal{P}(R)|}$.

(iv) Set $N = \oplus\{R/L \mid E(R/L) \in f(\Delta)\}$. Since for each $E(R/L) \in f(\Delta)$, $R/L \ll E(R/L) \in \Delta$, $N \in \Delta$, and $\langle N \rangle \subseteq \Delta$. Conversely, for any $M \in \Delta$, let T and M_* be as at the beginning. Then there exists a dense embedding $\oplus_{x \in T} R/x^{\perp} \cong \oplus_{x \in T} xR \ll M_*$. As above, $\langle M_* \rangle = \langle M \rangle$. Since $\oplus_{x \in T} R/x^{\perp} \leq N$, $\langle M \rangle = \langle \oplus_{x \in T} R/x^{\perp} \rangle \subseteq \langle N \rangle$. Thus $M \in \langle N \rangle$ and $\Delta \subseteq \langle N \rangle$. Hence $\Delta = \langle N \rangle$. □

If below Δ were a t.f. saturated class, then it and $c\Delta$ would have several additional useful properties and more structure. It would be interesting to generalize (1) (iv) below in several ways.

Lemma 2.8. *Let Δ be a saturated class and M any R-module. Then*

(1) (i) *$c\Delta$ is a saturated class*;

(ii) $c(c\Delta) = \Delta$;

(iii) $\Delta \cap c\Delta = \{(0)\}$;

(iv) *Δ is closed under homomorphic images whose kernels are complement submodules.*

(2) (i) *Let $N \leq M$ be any maximal submodule of M with $N \in \Delta$. Then for all $D \leq M$, $N \cap D = 0 \Rightarrow D \in c(\Delta)$.*

(ii) *there exist complements $N, C \leq M$ with $N \oplus C \ll M$ and $N \in \Delta$, $C \in c(\Delta)$. For any such N, $C \leq M$,*

(iii) $\widehat{M} = \widehat{N} \oplus \widehat{C}$, $\widehat{N} \in \Delta$,$\widehat{C} \in c(\Delta)$.

Proof. (1) (i) The class

$$c(\Delta) = \{N_R \mid \forall\, 0 \neq V \leq N, V \notin \Delta\}$$

is closed under 2.2 (a), (b) and (d), and moreover by 1.3, also under 2.2 (c). (ii) As in the proof of 2.5 (iii) and its notation, we have $\Delta \subseteq c(c\Delta)$, $\Delta = \overline{\Delta} = \langle\overline{\Delta}\rangle = cc\overline{\Delta} = cc\Delta$. Conclusion (iii) is clear. (iv) Let $M \in \Delta$, $f : M \twoheadrightarrow M/K$ with $C \in \Delta$ by 2.2 (a), while $M/K \in \Delta$ by use of 2.2 (b) followed by 2.2 (d).

(2) (i) By maximality of N, in view of 2.2 (c), for any $0 \neq V \leq D$, $V \notin \Delta$. Hence $D \in c(\Delta)$.

(ii) By Zorn's lemma, there exists a maximal direct sum $\oplus_{i\in I} N_i \leq M$ of submodules $N_i \in \Delta$. If then $N \leq M$ is any complement submodule with $\oplus_{i\in I} N_i \ll N$, then also $N \in \Delta$. Now take $C \leq M$ to be any complement submodule of M maximal with respect to $N \cap C = 0$ and hence $N \oplus C \ll M$. By (2) (i), $C \in c(\Delta)$.

(iii) Thus $\widehat{M} = \widehat{N} \oplus \widehat{C}$ with $\widehat{N} \in \Delta$, $\widehat{C} \in c(\Delta)$, by 2.2 (d) or (d′). □

Notation 2.9. Above, for a fixed N, C need not be unique, not even up to isomorphism; and to begin with, certainly N need not be unique. Any such choice of complement submodules N and C of M will be denoted by $N = M_{(\Delta)}$ and $C = M_{(c\Delta)}$, even when possibly M is torsion-free.

In the torsion-free case when $ZM = 0$, $M_\Delta \equiv \Sigma\{V \mid V \leq M, V \in \Delta\} \leq M$ is unique, M_Δ is a complement submodule of M, and $M_\Delta \in \Delta$ ([**14**, Main Corollary, 3.17] or [**13**, Theorem 2.2]). Thus $M_\Delta \oplus M_{c\Delta}$, $M_\Delta \in \Delta$, $M_{c\Delta} \in c\Delta$ are three unique submodules of M. Note that with $ZM = 0$ if $f\Delta$ is defined as $f\Delta = \{V \in \Delta \mid ZV = 0\}$, then $M_\Delta = M_{f\Delta}$. Here most of the time we will reserve the notation "M_Δ" for the case when $M_{(\Delta)} = M_\Delta$ is unique.

Define $\widehat{M}_{(\Delta)}$ and $EM_{(\Delta)}$ by $\widehat{M}_{(\Delta)} = EM_{(\Delta)} \equiv (EM)_{(\Delta)} = E(M_{(\Delta)})$. Thus $\widehat{M} = \widehat{M}_{(\Delta)} \oplus \widehat{M}_{(c\Delta)}$.

Later the next result will be substantially generalized to any pairwise disjoint set of saturated classes which generate $\mathcal{M}\mathrm{od}_R$.

Direct sum and uniqueness 2.10. For any saturated class Δ and any module M, suppose that $N_1 \oplus C_1 \ll M$, $N_2 \oplus C_2 \ll M$ with $N_1, N_2 \in \Delta$ and $C_1, C_2 \in c\Delta$. Then

(i) $\widehat{N}_1 \oplus \widehat{C}_2 = \widehat{N}_2 \oplus \widehat{C}_1 = \widehat{M}$;

(ii) $\widehat{N}_1 \cong \widehat{N}_2$ and $\widehat{C}_1 \cong \widehat{C}_2$.

(iii) There exists $D_1 \ll N_1$, $D_2 \ll N_2$ and an isomorphism $f : D_1 \to D_2$.

(iv) for all $M_{(\Delta)} \oplus M_{(c\Delta)} \ll M$, in the decomposition $\widehat{M} = \widehat{M}_{(\Delta)} \oplus \widehat{M}_{(c\Delta)}$, the submodules $\widehat{M}_\Delta$ and $\widehat{M}_{c\Delta}$ of $\widehat{M}$ are unique up to superspectivity.

(v) $[M/M_{(\Delta)}]_{(\Delta)} = 0$.

Proof. (i) and (ii). Suppose that $M = N_1 \oplus C_1 = N_2 \oplus C_2$ are all injective. Then $N_1 \cap C_2 = 0$ and $N_2 \cap C_1 = 0$ by 2.8 (2)(i). Let $N_1 \oplus C_2 \oplus D = M$ for some $D < M$. It can be proved from 2.8 (2) (i) that $N_1 \in \Delta$ and $C_2 \in c\Delta$ are maximal submodules of M belonging to these respective classes. Use of 2.8 (2) (i) on N_1 shows that $D \in c(\Delta)$, while 2.8 (2) (i) applied to C_2 shows that $D \in \Delta$. Thus $0 = D \in \Delta \cap c\Delta = \{(0)\}$. Hence $M = \widehat{N}_1 \oplus \widehat{C}_2$ and for any $\xi \in \widehat{N}_1$, $\xi = \xi_2 + \eta$, $\xi_2 \in N_2$, $\eta_2 \in C_2$. Then $f(\xi) = \xi$ defines a monic R-map $f : \widehat{N}_1 \to \widehat{N}_2$. Conversely, $\widehat{N}_2$ is isomorphic to a submodule $\widehat{N}_1$. By [**5**], $\widehat{N}_1 \cong \widehat{N}_2$.

(iii) Now $N_i \oplus C_i \ll M \ll \widehat{M} = \widehat{N}_i \oplus \widehat{C}_i$, and let $f : \widehat{N}_1 \to \widehat{N}_2$ be an isomorphism. Since f is an isomorphism and $N_1 \ll \widehat{N}_1$, also $fN_1 \ll \widehat{N}_2$, and $(fN_1) \cap N_2 \ll \widehat{N}_2$. Since the inverse image of a large submodule is large, $D_1 = N_1 \cap f^{-1}[(fN_1) \cap N_2] \ll N_1$ and $fD_1 \subseteq N_2$. But since $f : \widehat{N}_1 \to \widehat{N}_2$ is an isomorphism it maps the large submodule $D_1 \ll \widehat{N}_1$ onto a large submodule $D_2 = fD_1 \ll \widehat{N}_2$. Then denote also by f the restriction and corestriction $f : D_1 \to D_2$.

(iv) Let $\widehat{M} = \widehat{N}_1 \oplus \widehat{C}_1 = \widehat{N}_2 \oplus \widehat{C}_2$ with $\widehat{N}_i \in \Delta$ and $\widehat{C}_i \in c\Delta$ arbitrary. From the proof of (i), $\widehat{M} = \widehat{N}_1 \oplus \widehat{C}_1 \Leftrightarrow \widehat{M} = \widehat{N}_2 \oplus \widehat{C}_1$. Thus $\widehat{N}_1$ and $\widehat{N}_2$ are superspective.

(v) If $K < M$ is a complement and $(M/K)_\Delta = V/K$, $K \leq V \leq M$, let $K \oplus W \ll V$. Then $(K \oplus W)/K \ll (M/K)_\Delta$ and $W \in \Delta$. In particular, if $K = M_{(\Delta)}$ then $M_{(\Delta)} \oplus W \in \Delta$ and $W = 0$. Consequently, $[M/M_{(\Delta)}]_{(\Delta)} = 0$. $\square$

Recall that in the poset $\Sigma(R)$, by definition, "$\leq$" is set inclusion.

Proposition 2.11. *For any subset* $\Gamma = \{\gamma(i) \mid i \in I\} \subseteq \Sigma(R)$, *define* $\alpha = \{M \in \mathcal{M}\mathrm{od}_R \mid$ *there exist* $M_i \leq M, M_i \in \gamma(i), i \in I; \oplus_{i\in I} M_i \ll M\}$ *and* $\beta = \{N \in \mathcal{M}\mathrm{od}_R \mid$ *for all* $0 \neq V \leq N$, *for all* $i \in I, V \notin \gamma(i)\}$. *Then*

(i) *there exists* $\inf \Gamma = \wedge \Gamma = \wedge_{i\in I}\gamma(i) = \cap_{i\in I}\gamma(i) \in \Sigma(R)$;

(ii) *there exist* $\sup \Gamma = \vee \Gamma = \vee_{i\in I}\gamma(i) = \alpha \in \Sigma(R)$;

(iii) $\alpha \wedge \beta = 0$ *and* $\alpha \vee \beta = 1 \in \Sigma(R)$.

Proof. (i) Since $\cap_{i\in I}\gamma(i) \subseteq \gamma(i)$ for all i, the saturated class $\cap_{i\in I}\gamma(i)$ is a lower bound of all the $\gamma(i)$. For any other lower bound $\delta \in \Sigma(R)$, $\delta \subset \gamma(i)$ for all $i \in I$, and hence $\delta \subseteq \cap_{i\in I}\gamma(i)$. Thus $\cap_{i\in I}\gamma(i)$ is the greatest lower bound of the $\gamma(i)$.

(ii) By its definition, the class α is closed under 2.2 (b) and (d). It is easily seen to be closed under 2.2 (c). For any $0 \neq N < M \in \alpha$, it is asserted that $N \in \alpha$. Let $\mathcal{S}$ be the set $\mathcal{S} = \{\{N_i\}_{i\in I} \mid N_i \leq N$, $N_i \in \gamma(i), i \in I; \sum_{i\in I} N_i = \oplus_{i\in I} N_i \leq N\}$. For $\{N_i'\}$, $\{N_i\} \in \mathcal{S}$, define $\{N_i'\} \leq \{N_i\}$ if for any $j \in I$, $N_j' \subseteq N_j$. Then $\{\{0\}_i \mid i \in I\} \in \mathcal{S} \neq \varnothing$, and S is inductive. By Zorn's lemma, let $\{N_i \mid i \in I\} \in \mathcal{S}$ be a maximal element so that $\oplus_{i\in I} N_i \leq N$ is a direct sum maximal in the above sense. By definition of α, there exists $\oplus_{i\in I} M_i \ll M \in \alpha$, $M_i \in \gamma(i)$. If $\oplus_{i\in I} N_i \ll N$, we are done. Otherwise there is a $0 \neq D \leq N$ with $(\oplus_{i\in I} N_i) \oplus D \leq N$. By the projection argument 1.3 there exists a $j \in I$ and submodules $0 \neq A \leq D \cap (\oplus_{i\in I} M_i)$, $0 \neq B \leq M_j$; with $A \cong B \in \gamma(j)$. Thus $(\oplus_{i\in I, i\neq j} N_i) \oplus (N_j \oplus A) \leq N$ contradicts the maximality of $\{N_i \mid i \in I\} \in \mathcal{S}$. Therefore $D = 0$ and $N \in \alpha$. Thus $\alpha \in \Sigma(R)$ is a saturated class. Since $\gamma(i) \subseteq \alpha$ for all i, α is an upper bound of the $\gamma(i)$. Suppose that $\gamma(i) \leq \delta$, $i \in I$ is any other upper bound. Since δ is closed under direct sums, and essential extensions, $\alpha \subseteq \delta$. Thus α is the least upper bound of $\{\gamma(i) \mid i \in I\}$.

(iii) By (i), $\alpha \wedge \beta = \alpha \cap \beta = \{(0)\} = 0$. By (ii) applied to $\{\alpha, \beta\}$ with $|I| = 2$, $\alpha \vee \beta = \{M_R \mid$ there exists an $A \in \alpha, B \in \beta, A \oplus B \ll M\}$.

We now show that any arbitrary $M \in \mathcal{M}\mathrm{od}_R$ belongs to $\alpha \vee \beta$. Let $A \leq M$ be a submodule that is maximal with respect to $A \in \alpha$. By 2.8 (2) (i) such an A exists, alternatively a Zorn's lemma argument of the type used in the proof of 2.8 (2) (ii) shows that A exists. (Then $A \leq M$ is a complement.) Let $A \oplus B \ll M$ for some $B \leq M$. If $B = 0$, then $B \in \beta$. Otherwise, for any $0 \neq V \leq B$, $V \notin \alpha$. For if $V \in \alpha$, then $A \oplus V \in \alpha$ would violate the maximality of A. Thus $B \in \beta$. By 2.11 (ii), $M \in \alpha \vee \beta = \mathcal{M}\mathrm{od}_R = 1 \in \Sigma(R)$. □

Corollary 2.12. *For any saturated class $\Delta \in \Sigma(R)$, the element $c\Delta \in \Sigma(R)$ is unique with respect to $\Delta \wedge c\Delta = 0 \in \Sigma(R)$ and $\Delta \vee c\Delta = \mathcal{M}\mathrm{od}_R = 1 \in \Sigma(R)$.*

Proof. Since "$<$" and "$\leq$" in the poset $S(R)$ are defined by set inclusion, and since $\Delta \cap c\Delta = \{(0)\}$, $\Delta \wedge c\Delta = 0$. For any module M, $M_{(\Delta)} \in \Delta \subseteq \Delta \vee c\Delta$, and also $M_{(c\Delta)} \in c\Delta \subseteq \Delta \vee c\Delta$. Consequently, $M_{(\Delta)} \oplus M_{(c\Delta)} \ll M \in \Delta \vee c\Delta = \mathcal{M}\mathrm{od}_R$. Now suppose that for some $\Upsilon \in \Sigma(R)$, $\Delta \wedge \Upsilon = 0$ and $\Delta \vee \Upsilon = 1$. Then $\Delta \cap \Upsilon = \{(0)\} = 0 \in \Sigma(R)$. For any $N \in \Upsilon$, and any $0 \neq V \leq N$, $V \notin \Delta$ because $V \in \Upsilon$. Thus $\Upsilon \subseteq \{N_R \mid \text{ for all } 0 \neq V \leq N, V \notin \Delta\} = c\Delta$. For any $M \in c\Delta$, $M \in \Delta \vee \Upsilon$. By 2.11 (ii) with $\Gamma = \{\Delta, \Upsilon\}$ we get that $M_1 \oplus M_2 \ll M$ with $M_1 \in \Delta$ and $M_2 \in \Upsilon$. Since $M \in c\Delta$, also $M_1 \in c\Delta$ and hence $M_1 = 0$. Hence $M_2 \ll M \in \Upsilon$. Thus $c\Delta \subseteq \Upsilon$ and $c\Delta = \Upsilon$. □

The proof that the distributive law holds in $\Sigma(R)$ will hinge on the next lemma.

Lemma 2.13. *For any saturated class Δ and any chain $\mathcal{C} \subset \Delta$ which is a set, also $\cup\mathcal{C} \in \Delta$.*

Proof. Set $B = \cup\{A \mid A \in \Delta\}$. Form $B_{(\Delta)} \oplus B_{(c\Delta)} \ll B$. Then $B_{(c\Delta)} = \cup\{A \cap B_{(c\Delta)} \mid A \in \mathcal{C}\}$. For every $A \in \mathcal{C} \subset \Delta$, $A \cap B_{(c\Delta)} \in \Delta$, and $A \cap B_{(c\Delta)} \leq B_{(c\Delta)} \in c\Delta$. Thus $A \cap B_{(c\Delta)} \in \Delta \cap c\Delta = \{(0)\}$. Hence $B_{c\Delta} = 0$ and $B \in \Delta$.

Lemma 2.14. *For any $\alpha, \beta, \gamma \in \Sigma(R)$,*

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma).$$

Proof. Always $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \leq \alpha \wedge (\beta \vee \gamma)$. It has to be shown that for any $M \in \alpha \wedge (\beta \vee \gamma)$, also $M \in (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$. Form the set $\mathcal{S} = \{(P, Q) \mid P + Q = P \oplus Q \leq M, P \in \alpha \wedge \beta, Q \in \alpha \wedge \gamma\}$. Partially order $\mathcal{S}$ by $(P, Q) \leq (P', Q') \in \mathcal{S}$ whenever $P \subseteq P'$ and $Q \subseteq Q'$. By 2.13, $\mathcal{S}$ is inductive. Also, $(\{0\}, \{0\}) \in \mathcal{S} \neq \varnothing$. Thus, by Zorn's lemma, there exists a maximal element $(P, Q) \in \mathcal{S}$ where $P \in \alpha \cap \beta$ and $Q \in \alpha \cap \gamma$. Take any $V \leq M$ with $P \oplus Q \oplus V \ll M$. Since $V \leq M \in \alpha$, and $M \in \beta \vee \gamma$, also $V \in \beta \vee \gamma$. By 2.11 (ii) there exist $B \in \beta$ and $C \in \gamma$ such that $B \oplus C \ll V$. Then $B \in \alpha \wedge \beta$, $C \in \alpha \wedge \gamma$ and $(P, Q) \leq (P \oplus B, Q \oplus C) \in \mathcal{S}$. By the maximality of (P, Q), $B = 0$, $C = 0$, hence $V = 0$, and $P \oplus Q \ll M$. Thus $P \in (\alpha \wedge \beta) \subseteq (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$, $Q \in \alpha \wedge \gamma \subseteq (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$, and consequently, $M \in (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ again by 2.11 (ii). Thus $\alpha \wedge (\beta \vee \gamma) \leq (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ and hence they are equal. □

Theorem 2.15. *For a ring R with unity, let $\Sigma(R)$ be the class of all saturated classes (Definition 2.2) of R-modules partially ordered by class inclusion. Then*

(1) *$\Sigma(R)$ is a set, $|\Sigma(R)| \leq 2^{|(\mathcal{P}(R)|}$.*

(2) *$\Sigma(R)$ is a complete Boolean lattice with largest element $1 = \mathcal{M}\mathrm{od}_R \in \Sigma(R)$, smallest element $0 = \{(0)\} \in \Sigma(R)$ under the following lattice operations for any $\Gamma \subseteq \Sigma(R)$ and $\Delta \in \Sigma(R)$.*

(i) $\wedge\Gamma = \inf \Gamma = \cap\Gamma$;

(ii) *$\vee\Gamma = \sup \Gamma$ is the unique saturated class $\langle\Gamma\rangle \in \Sigma(R)$ generated by Γ; alternatively,*

$$\vee\Gamma = \sup \Gamma = \{M \in \mathcal{M}\mathrm{od}_R \mid \exists \oplus_{\gamma \in \Gamma} M^\gamma \ll M, M^\gamma \in \gamma \in \Gamma\},$$

(iii) *$\Delta \wedge c(\Delta) = 0$, $\Delta \vee c(\Delta) = 1 \in \Sigma(R)$ where the complement $c(\Delta)$ consists of all R-modules which are Δ-free, i.e., contain no nonzero submodules belonging to Δ.*

Proof. (1). By 2.7 (iii). (2) (i), (ii). By 2.11 (i), (ii). (2) (iii). By 2.12. □

Remark 2.16. For any α, β, or $\Delta \in \Sigma(R)$, $c(\alpha)$, $c\beta$ or $c\Delta$ were defined in 2.1. By 2.15 (iii) and 2.11, $c(\alpha)$, $c\beta$ and $c\Delta$ are also the unique complements of α, β, Δ in the Boolean lattices $\Sigma(R)$.

Corollary 1 to Theorem I, 2.17. *For any ring R, and any* $\Delta \in \Sigma(R)$*, define*

$$t\Delta = \{A \in \Delta \mid Z_2 A = A\}, \qquad f\Delta = \{B \in \Delta \mid ZB = 0\};$$
$$\Sigma_t(R) = \{\Delta \in \Sigma(R) \mid \forall M \in \Delta, Z_2 M = M\},$$
$$\Sigma_f(R) = \{\Delta \in \Sigma(R) \mid \forall M \in \Delta, ZM = 0\}.$$

Then

(3) (i) $t\Delta, f\Delta \in \Sigma(R)$;

(ii) $t\Delta \wedge f\Delta = 0$, $f\Delta \vee f\Delta = \Delta$;

(iii) $\Sigma(R) = \Sigma_t(R) \oplus \Sigma_f(R)$ *is a lattice direct sum of complete and convex sublattices*; *alternatively and equivalently a ring direct sum of convex and complete ring ideals.*

Proof. (3) (i). It is easy to check that the operations 2.2 (a), (b), (c), (d) and (d′) preserve t.f. modules and torsion modules (but not singular ones). (3) (ii). Since $t\Delta, f\Delta \subseteq \Delta$, also $t\Delta \vee f\Delta \subseteq \Delta$. Let $M \in \Delta$. Take any $C \leq M$ with $Z_2 M \oplus C \ll M$. Then $Z_2 M \in t\Delta$, $C \in f\Delta$ and $M \in t\Delta \vee f\Delta$ by 2.15, 2 (ii). Thus $\Delta = t\Delta \vee f\Delta$.

(3) (iii). Since both $\Sigma_t(R)$ and $\Sigma_f(R)$ are closed under arbitrary suprema and infima (2.15 (2), (i), (ii)), and since their intersection is zero, they are convex, orthogonal sublattices with $\Sigma_t(R) \vee \Sigma(R) = \{\alpha \vee \beta \mid \alpha \in \Sigma_t(R), \beta \in \Sigma_f(R)\} = \Sigma_t(R) \oplus \Sigma_f(R) \subseteq \Sigma(R)$. The latter inclusion is an equality by (3) (ii). □

In the next corollary some lattice theoretic concepts are related to module theoretic properties. It merely scratches the surface; there still remains a lot to be done in explaining the algebraic significance of lattice order theoretic properties of $\Sigma(R)$. Note that for $\Delta \in \Sigma(R)$ and for a module M, $\langle M_{(\Delta)} \rangle$ is independent of the choice of $M_{(\Delta)} \leq M$.

Corollary 2 to Theorem I, 2.18. *For any module M and any saturated class Δ the following hold.*

(i) $\Delta \wedge \langle M \rangle = \langle M_{(\Delta)} \rangle$;

(ii) $\Delta \leq \langle M \rangle \Leftrightarrow \Delta = \langle M_{(\Delta)} \rangle$.

(iii) *For $\alpha, \beta \in \Sigma(R)$,*

$$\alpha \wedge \beta = 0 \Longleftrightarrow \forall M \in \mathcal{M}\mathrm{od}_R,$$

for all choice of $M_{(\alpha)}, M_{(\beta)} \leq M$, $M_{(\alpha)} \cap M_{(\beta)} = 0$.

Proof. (i) Note that (i) follows easily from (ii), and that "(ii) $\Leftarrow$" is trivial.

(ii) $\Rightarrow$. Since $M_{(\Delta)} \in \Delta$, also $\langle M_{(\Delta)} \rangle \subseteq \Delta$. Conversely, let $N \in \Delta \subseteq \langle M \rangle$. Hence $\oplus_{j \in J} P_j \ll N$ where $P_j \cong P'_j \leq M$, $j \in J$. In order to show that $N \in \langle M_{(\Delta)} \rangle$, it suffices to show that for each fixed j separately, for some $A \ll P_j$, A is isomorphic to a submodule of $M_{(\Delta)}$. Expand $P'_j \subseteq Q$ to a maximal Δ-submodule $Q \leq M$. By 2.10 (iii), there is a $D \ll Q$ and a monomorphism $f : D \to M_{(\Delta)}$. Let $A = D \cap P'_j \ll P'_j$. Then $A \cong f(A) \leq M_{(\Delta)}$. Thus $\langle M_{(\Delta)} \rangle = \Delta$.

(iii). Conclusion (iii) can be proved similarly, and is omitted. □

The proof of the next corollary uses heavily the fact that $\Sigma(R)$ is a complete lattice.

Corollary 3 to Theorem I, 2.19. *For any family of modules $\{N_i \mid i \in I\}$,*

$$\Big\langle \bigoplus_{i \in I} N_i \Big\rangle = \bigvee_{i \in I} \langle N_i \rangle \in \Sigma(R).$$

Proof. Set $M = \oplus_{i \in I} N_i$. Since $N_i < M$, $\langle N_i \rangle \leq \langle M \rangle$. Since $\Sigma(R)$ is complete by 2.11, it follows that $\vee_{i \in I} \langle N_i \rangle \leq \langle M \rangle$.

It suffices to show that $M \in \vee_{i \in I} \langle N_i \rangle$, because in that case by 2.6 also $\langle M \rangle \subseteq \vee_{i \in I} \langle N_i \rangle$. In 2.11 (ii), take $M_i = N_i \leq M$ and take $\Gamma = \{\langle N_i \rangle \mid i \in I\}$. Then $\oplus_{i \in I} M_i \ll M$ holds, in fact the two are equal. Hence $M \in \vee \Gamma$. □

3. Direct sum decompositions. Recall that for a t.f. module M and a t.f. saturated class $\Delta \in \Sigma_f(R) \subset \Sigma(R)$, the complement submodule $M_\Delta = \Sigma\{V \mid V \leq M, V \in \Delta\} \leq M$ is the unique f.i. maximal Δ submodule of M with $M_\Delta \in \Delta$. In the absence of the t.f. hypotheses, it is the failure of the later uniqueness property which makes direct sum decompositions much more complicated, difficult and challenging.

Theorem II, 3.1. *Let $\Gamma = \{\alpha, \beta, \gamma, \dots\} \subset \Sigma(R)$ be any finite or infinite pairwise disjoint set ($\alpha \wedge \beta = 0$ if $\alpha \neq \beta \in \Gamma$) whose supremum is $\sup \Gamma = \vee\Gamma = 1 \in \Sigma(R)$. For any module M, let $\{M_{(\alpha)} \mid \alpha \in \Gamma\}$ be any choice of maximal α submodules, i.e., each $M_{(\alpha)} \leq M$ is maximal with respect to $M_{(\alpha)} \in \alpha$. Let $\Omega \subseteq \Gamma$ be any subset, and let $K \leq M$ be any complement submodule such that $\Sigma\{M_{(\gamma)} \mid \gamma \in \Omega\} \ll K$. Note that such a K always exists. Then*

(1) (i) $\sum_{\gamma\in\Gamma} M_{(\gamma)} = \oplus_{\gamma\in\Gamma} M_{(\gamma)} \ll M$

(ii) *$K \in \vee\Omega$ is a maximal $\sup\Omega$ submodule of M. Hence, if $\oplus_{\gamma\in\Gamma\setminus\Omega} M_\gamma \ll L \leq M$ is any complement submodule, then we may take $K \equiv M_{(\vee\Omega)}$, $L \equiv M_{(c(\vee\Omega))}$, $M_{(\vee\Omega)} \oplus M_{(c(\vee\Omega))} \ll M$ where $\oplus_{\gamma\in\Omega} M_\gamma \ll M_{(\vee\Omega)} \in \vee\Omega$, $\oplus_{\gamma\in\Gamma\setminus\Omega} M_\gamma \ll M_{(c(\vee\Omega))} \in c(\vee\Omega)$.*

(iii) *In particular, if $\Gamma = \{\alpha\}$, then there exists a $\oplus_{\gamma\in\Gamma,\gamma\neq\alpha} M_\gamma \ll M_{(c(\alpha))} \in c(\alpha)$, $M_{(\alpha)} \oplus M_{(c(\alpha))} \ll M$.*

(2) (i) *Superspectivity* (*see* [**32**, p. 12]). *Suppose that $N^\alpha \leq M$, $N^\alpha \in \alpha \in \Gamma$ is any other choice of maximal α submodules, and $\oplus_{\alpha\in\Omega} N^\alpha \ll N^\Omega \in \vee\Omega$ is any complement of M (and hence a maximal $\vee\Omega$ submodule of M). Then $E(\oplus_{\gamma\in\Gamma\setminus\Omega} N^\gamma) \cong E(M_{(c(\vee\Omega))})$.*

$$\left(\bigoplus_{\gamma\in\Omega} N^\gamma\right) \oplus \left(\bigoplus_{\gamma\in\Gamma\setminus\Omega} M_{(\gamma)}\right) \ll N^\Omega \oplus M_{(c(\vee\Omega))} \ll M$$

$$E(N^\Omega) \cong E(M_{(\vee\Omega)}); \qquad E\left(\bigoplus_{\gamma\in\Gamma\setminus\Omega} N^\gamma\right) \cong EM_{(c(\vee\Omega))}.$$

(ii) *In particular, if $\Omega = \{\alpha\}$, and if $\oplus_{\gamma\in\Gamma,\gamma\neq\alpha} N^\gamma \ll N^{c\alpha} \leq M$ is any complement, and hence $N^{c\alpha} \in c\alpha$ is $c\alpha$ maximal, then*

$$E(M_{(\alpha)}) \cong E(N^\alpha); \qquad E(M_{(c\alpha)}) \cong E(N^{c\alpha}).$$

Proof. (1) (i). Suppose by induction that the sum of any n distinct M_γs is direct, but that $M_{(\alpha(0))} \cap (M_{(\alpha(1))} \oplus \cdots \oplus M_{(\alpha(n))}) \neq 0$ for $\alpha(0), \alpha(1), \ldots, \alpha(n) \in \Gamma$. Then by the projection argument, a nonzero submodule $0 \neq V \leq M_{(\alpha(0))}$ is isomorphic to a submodule of some $M_{(\alpha(i))}$, $V \cong U \leq M_{\alpha(i)}$. But then U or $V \in \alpha(0) \cap \alpha(i) = \alpha(0) \wedge \alpha(i) = \{(0)\}$, a contradiction.

Let $(\oplus_{\gamma\in\Gamma} M_{(\gamma)}) \oplus P \ll M$. Then $P \in \mathcal{M}\mathrm{od}_R = \vee\Gamma = 1 \in \Sigma(R)$. By 2.11 (ii), there exist $P_\gamma \leq P$, $P_\gamma \in \gamma \in \Gamma$, $\oplus_{\gamma\in\Gamma} P_\gamma \ll P$. If $P \neq 0$, then some $P_\gamma \neq 0$. But then $M_{(\gamma)} \oplus P_\gamma \in \gamma$ contradicts the maximality of $M_{(\gamma)}$.

(1) (ii) and (iii). By 2.11 (ii), $K \in \vee\Omega$. If K were not a maximal $\vee\Gamma$ submodule of M, then $K \subsetneqq C \leq M$ for some $C \in \vee\Omega$. Since K is a complement, $K \not< \!\!\!\!\not{\ } C$, and $K \oplus D \leq C$ for some $0 \neq D \in \vee\Omega$. In view of 2.11 (ii) again, $D \in \vee\Omega$ implies that $\oplus_{w\in\Omega} D_w \ll D$ for some $D_w \in w \in \Omega$. For some $\gamma \in \Omega$, $D_\gamma \neq 0$. But then $M_{(\gamma)} \neq M_{(\gamma)} \oplus D_\gamma \in \gamma$ contradicts the maximality of $M_{(\gamma)}$. Hence K is a maximal $\vee\Omega$ submodule of M.

In any complete Boolean lattice such as $\Sigma(R)$ for any subsets $\Omega \subseteq \Gamma$, if $\vee\Gamma = 1$ then $\vee(\Gamma\backslash\Omega) = c(\Omega)$. The previous argument applied to $\Gamma\backslash\Omega$ and L, in place of Ω and K, shows that $L \leq M$ is a maximal $c(\vee\Omega)$ submodule of M. The rest of (1) (ii) and (iii) is clear.

(2) (i) and (ii). Since $N^\Omega \in \vee\Omega$ is a maximal $\vee\Omega$ while $M_{(c(\vee\Omega))} \in c(\vee\Omega)$ is a maximal $c(\vee\Omega)$ submodule of M, it follows from 2.10 that $N^\Omega \oplus M_{(c(\vee\Omega))} \ll M$. The remainder can now either be proved directly, or deduced from 2.10. □

Corollary 1 to Theorem II, 3.2. *In the above theorem, define $M_{(t\alpha)} = Z_2 M_{(\alpha)}$, and let $M_{(f\alpha)} \leq M_{(\alpha)}$ be any complement submodule such that $M_{(t\alpha)} \oplus M_{(f\alpha)} \ll M_{(\alpha)}$. Then let $t\alpha, f\alpha \in \Sigma(R)$ be defined as the unique elements such that $\alpha = t\alpha \vee f\alpha$, $t\alpha \wedge f\alpha = 0$, $t\alpha$ is torsion, $f\alpha$ is torsion-free. Then $\{t\gamma, f\gamma \mid \gamma \in \Gamma\}$ and $\{M_{(t\alpha)}, M_{(f\alpha)} \mid \alpha \in \Gamma\}$ satisfy the hypotheses of the last theorem, and* 2.19 (1) (ii) *becomes $\oplus_{\gamma\in\Gamma} M_{(t\alpha)} \oplus (\oplus_{\gamma\in\Gamma} M_{f\gamma}) \ll M$, i.e.,*

(3) (i) *$M_{(t\alpha)} \in t\alpha$ is a maximal $t\alpha$ submodule of M and $M_{(f\alpha)} \in f\alpha$ is a maximal $f\alpha$ submodule of M; in particular, both are complements in M;*

(ii) $Z_2M \oplus (\oplus_{\alpha\in\Gamma} M_{(f\alpha)}) \ll M$

(iii) $Z_2(\oplus_{\gamma\in\Gamma} M_{(\gamma)}) = \oplus_{\gamma\in\Gamma} M_{(t\gamma)} \ll Z_2M$.

Proof. By definition, $t\alpha = \varepsilon_1 \wedge \alpha = \varepsilon_1\alpha$ and $f\alpha = \varepsilon_2 \wedge \alpha = \varepsilon_2\alpha$ where $1 = \varepsilon_1 \vee \varepsilon_2 = \varepsilon_1 + \varepsilon_2 \in \Sigma_t(R) \oplus \Sigma_f(R) = \Sigma(R)$, $\varepsilon_1 \in \Sigma_t(R) = \Sigma(R)\varepsilon_1$, $\varepsilon_2 \in \Sigma_f(R) = \Sigma(R)\varepsilon_2$. Moreover, $\varepsilon_1 = \{M \in \mathcal{M}\mathrm{od}_R \mid Z_2M = M\}$, while $\varepsilon_2 = \{M \mid ZM = 0\}$. Consequently, $Z_2M_{(\alpha)} \in \varepsilon_1 \cap \alpha = \varepsilon_1 \wedge \alpha = t\alpha$; and since $Z_2M_{(f\alpha)} = 0$, $M_{(f\alpha)} \in \varepsilon_2 \cap \alpha = \varepsilon_2 \wedge \alpha = f\alpha$.

By construction $M_{(t\alpha)}$ is a maximal $t\alpha$ submodule of $M_{(\alpha)}$ and $M_{(f\alpha)}$ a maximal $f\alpha$ submodule of $M_{(\alpha)}$. Hence, both are complements in $M_{(\alpha)}$, which in turn is a complement in M. Since, in general, a complement of a complement is a complement, both $M_{(t\alpha)}$ and $M_{(f\alpha)}$ are complement submodules also in M.

In order to see that $M_{(t\alpha)} = Z_2M_{(\alpha)}$ and $M_{(f\alpha)}$ are maximal $t\alpha$ and $f\alpha$ submodules of M (and not merely of $M_{(\alpha)}$), let $M_{(t\alpha)} \oplus C \le N_1$, $0 \ne Z_2C = C \in t\alpha \subset \alpha$, and $M_{(f\alpha)} \oplus D \oplus Z_2N_2 \ll N_2$, $0 \ne D \in f\alpha$, $Z_2D = 0$, where $N_1, N_2 \le M$ both are maximal α submodules of M. By 2.10 (ii), $\widehat{M}_{(\alpha)} \cong \widehat{N}_1 \cong \widehat{N}_2$. Hence $\widehat{M}_{(t\alpha)} \oplus \widehat{C} \le Z_2\widehat{N}_1 \cong Z_2\widehat{M}_\alpha = \widehat{M}_{(t\alpha)}$ and $\widehat{M}_{(f\alpha)} \oplus \widehat{D} \oplus Z_2\widehat{N}_2 \cong \widehat{M}_{(f\alpha)} \oplus \widehat{M}_{(t\alpha)}$.

The identity map $\widehat{M}_{(t\alpha)} \to \widehat{M}_{(t\alpha)}$ extends to an isomorphism $g : \widehat{N}_1 \to \widehat{M}_{(t\alpha)} \oplus \widehat{M}_{(f\alpha)} = \widehat{M}_{(\alpha)}$. Then $gC \cap \widehat{M}_{(t\alpha)} = 0$, and $C \cong (gC + \widehat{M}_{(t\alpha)})/\widehat{M}_{(t\alpha)}$ is torsion-free, a contradiction.

In the second case, the identity map $\widehat{M}_{(f\alpha)} \to \widehat{M}_{(f\alpha)}$ extends to another isomorphism $h : \widehat{N}_2 \to \widehat{M}_{(f\alpha)} \oplus \widehat{M}_{(t\alpha)} = \widehat{M}_{(\alpha)}$. Now $hD \cap \widehat{M}_{(f\alpha)} = 0$, and $D \cong (hD + \widehat{M}_{(f\alpha)}/\widehat{M}_{f\alpha}$ is torsion, again a contradiction. □

Corollary 2 to Theorem II, 3.3. *In the last theorem and corollary, let $C \le M$ be any complement submodule with $Z_2M \oplus C \ll M$. Let $C_{f\alpha} \in f\alpha$ be the unique maximal $f\alpha$ submodule $C_{f\alpha} = \Sigma\{V \mid V \le C, V \in f\alpha\} \le C$. Choose in the last theorem the $M_{(\alpha)}$ such that $C_{f\alpha} \subseteq M_{(\alpha)}$ for all $\alpha \in \Gamma$. Then the last Corollary* 3.2 *holds with $M_{(f\gamma)} = C_{f\gamma}$, i.e.,*

(4) (i) $\oplus_{\gamma\in\Gamma}(Z_2M_{(\gamma)} \oplus C_{f\gamma}) \ll \oplus_{\gamma\in\Gamma} M_{(\gamma)} \ll M$;

(ii) $Z_2M_{(\gamma)} \oplus C_{f\gamma} \ll M_{(\gamma)}$.

$$\bigoplus_{\gamma\in\Gamma} Z_2M_{(\gamma)} \ll Z_2M, \qquad \bigoplus_{\gamma\in\Gamma} C_{f\gamma} \ll C;$$

all submodules in (4) *are complements in* M.

Proof. In order to apply the last corollary, it has to be shown that $C_{f\alpha} \leq M_{(\alpha)}$ is a complement. However, $C_{f\alpha} \leq C$ is a complement, and $C \leq M$ is a complement. Hence $C_{f\alpha} \leq M$ is a complement, and finally $C_{f\alpha} \leq M_\alpha$ is also. The rest of the proof is omitted. □

4. Types. The t.f. universal saturated classes in [**14**] and [**15**] generalized and extended type I, II and III modules. Here all of the latter, and more, will be extended to the torsion, or more appropriately actually the general mixed case. To carry this out, a new method of defining saturated classes different from the one in the last section is needed. Some more theory is developed, which perhaps later could be applied to several classes of modules which are close to being injective, but not quite injective, of the kind one finds in [**32**]. Among other things some specific examples of saturated classes will be given in this section.

Definition 4.1. Let $\mathcal{C}$ be a nonempty class of modules closed under isomorphic copies. A module M is *essentially* $\mathcal{C}$*-dense* if for any $0 \neq N \leq M$, $\widehat{N}$ contains a nonzero member of $\mathcal{C}$. Define a class of modules ess-$\mathcal{C}$ by

$$\text{ess-}\mathcal{C} = \{N \mid \exists\, K \in \mathcal{C}, N \ll K\}.$$

Thus the terms "essentially $\mathcal{C}$-dense" and "ess $\mathcal{C}$-dense" are synonyms.

Recall that, for a nonempty class of modules $\mathcal{D}$, a module M is $\mathcal{D}$*-free* if M contains no nonzero member of $\mathcal{D}$. The module M is *locally* $\mathcal{D}$*-free* if every nonzero submodule of M contains a nonzero $\mathcal{D}$-free submodule, i.e., for any $0 \neq A \leq M$, there exists a $0 \neq B \leq A$ such that for every $0 \neq C \leq B$, $C \notin \mathcal{D}$.

In applications to types I, II and III, the class $\mathcal{C}$ will consist entirely of injective modules of some specified kind. Relatively recently various classes of noninjective modules which in some sense are close to being injective have received attention. [**32, 33, 34, 35, 39, 26, 27, 29** and **30**]. In order to later possibly apply the ideas in this article to these or similar classes of noninjective modules, here in the next proposition the theory is developed for not necessarily injective modules.

Sometimes it is difficult, if not impossible, to show that some given class of modules is closed under submodules, or even essential submodules. The first part of the next theorem has been specifically designed for these classes, while the second part contrasts this with the additional information and simplifications which closure under submodules provides.

Theorem III, 4.2. *Let $\mathcal{C}$ be any nonempty class of modules closed under isomorphic copies. Denote the* ess-$\mathcal{C}$ *dense R-modules by $\mathcal{D}$, i.e., $\mathcal{D} = \{M \in \mathcal{M}\mathrm{od}_R \mid$ for all $0 \neq N \leq M,$ there exists $0 \neq K \leq \widehat{N}$ with $K \in \mathcal{C}\}$. Then*

(i) *$\mathcal{D}$ is a saturated class*; *and*

(ii) *$c\mathcal{D} = \{M \in \mathcal{M}\mathrm{od}_R \mid$ for all $0 \neq A \leq M,$ there exists $0 \neq B \leq A$ such that for all $0 \neq C \leq \widehat{B}, C \notin \mathcal{C}\}$*; *i.e., $c\mathcal{D}$ are precisely the locally* ess-*$\mathcal{C}$-free modules.*

Now for (iii) *and* (iv) *assume in addition to the previous hypotheses that $\mathcal{C}$ is also closed under injective hulls and essential submodules. Then*

(iii) *$\mathcal{D}$ are the $\mathcal{C}$-dense modules, while*

(iv) *$c\mathcal{D}$ are the locally $\mathcal{C}$-free modules.*

(v) *If $\mathcal{C}$ is also closed under submodules, then $\mathcal{D} = \langle \mathcal{C} \rangle$ and hence $c\mathcal{D}$ are the $\mathcal{C}$-free modules.*

Proof. (i) Note that it is practically built into the definition of $\mathcal{D}$ that it is closed under isomorphic copies, injective hulls, as well as submodules. Let $M_i \in \mathcal{D}$, $i \in I$, and $0 \neq N \leq \oplus_{i \in I} M_i$. Then for some $i \in I$, and some $0 \neq A \leq N$, $A \cong B \leq M_i$. Thus, since $M_i \in \mathcal{D}$, there exists $0 \neq D \leq \widehat{B}$ with $D \in \mathcal{C}$. There is a $0 \neq K \leq \widehat{A}$, $K \cong D$. Hence $0 \neq K \leq \widehat{A} \leq \widehat{N}$ with $K \in \mathcal{C}$. Consequently, $\oplus_{i \in I} M_i \in \mathcal{D}$ and $\mathcal{D} \in \Sigma(R)$.

(ii) Let X be defined as the righthand side of the equation in (ii). Always, $c\mathcal{D} = \{M \mid \text{ for all } 0 \neq A \leq M, A \notin \mathcal{D}\}$. But $A \notin \mathcal{D}$ if and only if there exists $0 \neq B \leq A$, where $\widehat{B}$ contains no nonzero submodule of $\mathcal{C}$. Thus $c\mathcal{D} = X$.

(iii) Rewrite the definition of $\mathcal{D}$ as

$$\mathcal{D} = \{M \mid \forall\, 0 \neq N \leq M, \exists\, 0 \neq P \leq N, \widehat{P} \in \mathcal{C}\}.$$

But, by the additional hypotheses on $\mathcal{C}$, $\widehat{P} \in \mathcal{C}$ and $P \ll \widehat{P}$ implies that $P \in \mathcal{C}$, and hence that $\mathcal{D}$ are precisely the $\mathcal{C}$-dense modules.

(iv) For any nonempty $\mathcal{C}$, by definition, the locally $\mathcal{C}$-free modules are the class Y, where

$$Y = \{M \mid \forall\, 0 \neq A \leq M, \exists\, 0 \neq B \leq A, \forall\, 0 \neq C \leq B, C \notin \mathcal{C}\}.$$

In view of our assumptions on $\mathcal{C}$, again $C \notin \mathcal{C}$ if and only if $\widehat{C} \notin \mathcal{C}$. Hence Y translates into $Y = c\mathcal{D}$ as in (ii). Since in (v) $\mathcal{C}$ is closed under submodules the locally $\mathcal{C}$-free modules coincide with the $\mathcal{C}$-free ones. $\square$

Below the definition of the terms "square and square-free" are the same as those in [**32**, Definition 2.34]. However, here "square dense" modules are "square full" in the sense of [**32**, p. 35]. The latter is not used here.

Definition 4.3. For any module M, a submodule $N \leq M$ is an (*idempotent*) *square* if $N = P \oplus P$ for some $P \leq N$ (with $P \cong P \oplus P$). The module M is (*idempotent*) *square-free* if M contains no nonzero (idempotent) squares.

According to our already defined and established terminology, M is (*idempotent*) *square dense* if every nonzero submodule $0 \neq N \leq M$ contains a nonzero (idempotent) square $0 \neq P \oplus P \leq N$ ($0 \neq P \cong P \oplus P \leq N$). Clearly, M is (idempotent) square dense if and only if $\widehat{M}$ is likewise.

Surprisingly, square-free as opposed to idempotent square-free modules behave quite differently in (1) and (3) below.

Consequences 4.4. (1) *Every square-free submodule of M is contained in a maximal square-free submodule of M. In particular, every module contains a maximal square-free submodule.*

Proof. Note that $(0) < M$ is a square-free submodule. Let $D_0 \leq \cdots \leq D_i \leq \cdots \leq M$ be any ascending chain of square-free submodules of M, and suppose that $0 \neq P \oplus Q \leq \cup_i D_i$ with $f : P \to Q$ and isomorphism. For any $0 \neq x \in P$, set $y = fx \in Q$. Then $x, y \in D_i$ for some i, and hence $xR + yR = xR \oplus yR \leq D_i \leq M$ with $xR \cong yR$ a contradiction. □

Consequences 4.4. (2) *For any $A \ll B$, A is (idempotent) square dense if and only if B is (idempotent) square dense.*

Proof. The property of a module M of being dense with respect to any given class of modules whatsoever is inherited both by essential submodules as well as essential extensions. □

Consequences 4.4. (3) (i) *M is square dense if and only if every direct summand of $\widehat{M}$ is a square.*

(ii) *If M is idempotent square dense then $\widehat{M}$ is an idempotent square.*

Proof. (3) (i) $\Rightarrow$. For $\widehat{M} = N \oplus N'$, let $V = \oplus_{i \in I}(V_{1i} \oplus V_{2i}) \leq N$ with $V_{1i} \cong V_{2i}$, $i \in I$ an ordinal indexed direct sum maximal in the partial order under which $W = \oplus_{j \in J}(W_{1j} \oplus W_{2j}) \preceq V$ whenever $J \subseteq I$ and $W_{1j} = V_{1j}$, $W_{2j} = V_{2j}$ for all $j \in J$. Then $V \oplus C \ll N$ for some $C \leq N$. But by the square denseness of $M \cap C$ and the maximality of V, $C = 0$. Hence $N = V_1 \oplus V_2$, where $V_1 = E(\oplus_{i \in i} V_{1i}) \cong V_2 = E(\oplus_{i \in I} V_{2i})$.

(3) (ii) $\Rightarrow$. In this case, in addition $V_{1i} \cong V_{2i} \cong V_{1i} \oplus V_{2i}$, and hence $V_1 \cong V_2 \cong V_1 \oplus V_2$. □

Consequences 4.4. (4) *In particular, M is square dense if and only if $\widehat{M}$ is square dense if and only if for all $0 \neq N \leq \widehat{M} = N \oplus N'$, N contains a nonzero injective square.*

Definition 4.5. For an arbitrary injective module M, here N will stand for any arbitrary possible nonzero direct summand of M. Then M is *directly finite* if $N \ncong N \oplus N$ for all summands N of M; M is *directly infinite* if not directly finite. This means that there exists a direct summand $0 \neq N \oplus N \cong N \leq M = N \oplus N'$; alternatively $0 \neq P \cong P \oplus P \leq M = P \oplus P \oplus M''$. The module M is *abelian* if no nonzero direct summand N of M is a square, i.e., $M = P \oplus P \oplus V$ is possible only if $P = 0$.

Next, M is type I if every N contains a nonzero injective abelian submodule. It is type III if every N is directly infinite. Lastly, M is type II if every summand N is not abelian, and every N contains a nonzero directly finite direct summand.

The module M is defined to be *purely infinite* if it contains no direct summand N which is fully invariant and directly finite.

A completely arbitrary module M will be said to have any of the above in 3.3 listed properties or to belong to any one of the above seven classes (directly finite, directly infinite, abelian, I, II, III or purely infinite) if and only if $\widehat{M}$ does.

The next remark shows that our present definition of "abelian" generalizes the well-established old concept for t.f. injective modules. Conclusion (iv) below is proved in [**14**, p. 113, 4.2]. Although stated there for a t.f. continuous module, its proof there nowhere uses the "continuous" hypothesis.

Remark 4.6. For any module M, (i), (ii), (iii) and (iv) are equivalent.

(i) M is abelian;

(ii) M is square free;

(iii) $\widehat{M}$ is square free;

(iv) there does not exist an $x, y \in M\backslash\{0\}$, $xR \cong yR$, $xR \cap yR = 0$.

(v) If M is t.f., then (i)–(iv) are equivalent to the condition that all nonzero isomorphic direct summands of $\widehat{M}$ are equal.

Remark 4.7. In [**32**, Definition 1.24] a module is defined to be "directly finite" if it is not isomorphic to a proper summand of itself. It is proved in [**23**, 3.1(c)] that for t.f. injective modules the definition given here is equivalent to the [**32**] definition. There a module is defined to be "purely infinite" if it is a square isomorphic to itself [**32**, Definition 1.3.2.]. Again, it is proved in [**23**, Theorem 6.2] that the two definitions of "purely infinite" coincide on t.f. injective modules.

Corollary 1 to Theorem III, 4.8. *For a ring R,*

(i) *locally directly finite, type I, type II, type $III \in \Sigma(R)$ are saturated classes.*

(ii) *For any t.f. module M, M is type III if and only if $\widehat{M}$ is type III in the usual sense* (*as in* [**23**]). *Analogous statements hold for I, II and locally directly finite.*

Proof. (i) Note that the square-free modules $\mathcal{SF}$ are the same as the abelian ones, and the $\mathcal{SF} \subset \mathcal{DF}$, where $\mathcal{DF}$ are the directly finite ones. These two classes are closed under everything except direct sums, i.e., they are closed under isomorphic copies, injective hulls and submodules. Hence, by 3.2 (iii), type I, that is, the $\mathcal{SF}$-dense, and the $\mathcal{DF}$-dense or the locally directly finite classes of modules are saturated classes.

Let now $\mathcal{C}$ in 3.2 be the class of injective idempotent squares. Then type III are the ess-$\mathcal{C}$ dense modules, which are a saturated class by 3.2 (i).

Lastly type $II = c$ (type I) $\cap$ (locally directly finite) $\in \Sigma(R)$ because it is the meet of two elements of $\Sigma(R)$.

(ii) In the torsion-free case, the definitions of I, II, III and directly finite are equivalent to standard definitions. The proof of this is omitted but is spelled out in more detail in [**13**, Definition 3.3]. ◻

Consequences 4.9. *Let the lattice operations in $\Sigma(R)$ be as in* 2.11, *and let M be any module.*

(a) type $II \vee$ type III =*square-dense modules.*

(b) type $I \vee$ type II =*locally directly finite modules.*

(c) $M \in$ type III *if and only if for all* $0 \neq N \leq M$, $\widehat{N} \cong \widehat{N} \oplus \widehat{N}$.

(d) $M \in$ type $II \vee$ type III *if and only if for all* $0 \neq N \leq M$, $\widehat{N} \cong P \oplus P$, *some* $P < \widehat{N}$.

(e) *The directly infinite and the purely infinite modules are closed under direct sums, essential extensions, but not under submodules. Hence, neither one is a saturated class.*

Consequences 4.10. *For any ring* R, *let* I, II, $III \in \Sigma(R)$ *denote type* I, II *and* III *modules. Let* M *be any* R*-module. Then*

(i) $I \wedge II = 0$, $I \wedge III = 0$, $II \wedge III = 0$; $I \vee II \vee III = 1 \in \Sigma(R)$.

(ii) *There exists an* $M_I \oplus M_{II} \oplus M_{III} \ll M$, $M_I \in I$, $M_{II} \in II$, $M_{III} \in III$.

(iii) *In particular,* $\widehat{M} = \widehat{M}_I \oplus \widehat{M}_{II} \oplus \widehat{M}_{III}$ *is unique up to superspectivity.*

Proof. (i) Locally $\mathcal{DF}$-dense modules by 4.9(b) is the element $I \vee II \in \Sigma(R)$. By 4.6 $I =$ locally ess-$\mathcal{SF}$-dense modules, while by Definition 4.5, $II \subseteq$ locally $\mathcal{SF}$-free; hence, $I \wedge II = 0$.

Note that $\mathcal{DF} = \text{ess} - \mathcal{DF}$. By 4.5, $I, II \subseteq$ locally $\mathcal{DF}$-modules, while III is $\mathcal{DF}$-free; thus $III \wedge I = 0$, $III \wedge II = 0$. Moreover, $III = c(I \vee II)$. Consequently, $(I \vee II) \vee III = 1$.

(ii) and (iii) follow by (3.1). □

Definition 4.11. A module W is *atomic* if $W \neq 0$ and for any $0 \neq V < W$, $\langle W \rangle = \langle V \rangle$, i.e., if and only if $\langle W \rangle$ is an atom in the poset $\Sigma(R)$. More generally, a module A is *molecular* if every nonzero submodule contains an atomic one, i.e., if and only if $A \hookrightarrow E(\oplus_{i \in I} W_i)$, where all the W_i, $i \in I$ are atomic.

A module D is *discrete* if it contains an essential direct sum of uniform modules; C is *continuous* if it contains no uniform submodules; B is *bottomless* if it contains no atomic submodules.

As in previous articles on t.f. modules, we will continue using "A, B, C, D" as subscripts and superscripts for entities associated with these classes, as well as "CA" for the continuous molecular modules.

Notation 4.12. We denote these classes of R modules by $A(R), B(R), C(R), D(R)$; write $(CA)(R) = C(R) \cap A(R)$ for the continuous molecular modules.

Abbreviate the classes of all type I, II or III R-modules as $I(R)$, $II(R)$, $III(R)$, also $III_t(R)$, $III_f(R)$, etc., as before. Thus, $III(R) = III_t(R) \vee III_f(R) \in \Sigma(R)$ where $III_f(R) \in \Sigma_f(R)$ are the expected previous type III modules including also noninjective ones. Since the symbol "Type III" with capital "T" has long been traditionally used for operator algebras and t.f. injective modules, we are not extending its definition, i.e., Type $III \neq III$; if anything, Type $III = III_f$.

When the ring is understood and fixed, above the "R" is omitted, e.g., $M_I \oplus M_{II} \oplus M_{III}$ instead of $M_{I(R)} \oplus M_{II(R)} \oplus M_{III(R)}$, or as in the previous sentence. When modules over several rings R, S with identity are used, we write $A(S), B(S), \ldots, III(S)$ for classes of S-modules. Let $A, B, C, D, CA, I, II, III, A_t, A_f, \ldots, III_t, III_f$ denote functions from the class of all rings with identity to classes of modules, e.g., the values of B at R and III at S are $B(R)$ and $III(S)$.

In the next corollary, the "R" has been omitted in $A(R)$, etc. Also its proof is easier than the I, II, III case because the classes of A–D are naturally defined in terms of the partial order structure of $\Sigma(R)$.

Corollary to Theorem 4.13. *For any ring R and any R-module M, the following hold*

(i) A, B, C, D *and* $CA \in \Sigma(R)$ *are saturated classes.*

(ii) $D \subseteq A$, $B \subseteq C$.

(iii) $C \vee D = 1$, $C \wedge D = 0$.

(iv) $A \vee B = 1$, $A \wedge B = \varnothing$.

(v) $CA \vee B \vee D = 1$, $D \wedge CA = 0$, $D \wedge B = 0$, $CA \wedge B = 0$;

(vi) $C = CA \vee B$, $A = D \vee CA$.

(vii) *There exists an* $M_{CA} \oplus M_B \ll M_C$, $M_C \oplus M_D \ll M$; $M_{CA} \oplus M_B \oplus M_D \ll M$.

(viii) $\widehat{M} = \widehat{M}_C \oplus \widehat{M}_D = \widehat{M}_{CA} \oplus \widehat{M}_B \oplus \widehat{M}_D$ *uniquely up to superspectivity.*

Proof. (i) By definition, $D = \langle\{U \in \mathcal{M}\text{od}_R \mid U \text{ is uniform}\}\rangle$ while $C = c(D)$. Hence (iii) automatically follows. Similarly, $A = \langle\{W_R \mid W \text{ is atomic}\}\rangle$ and $B = c(A)$. Thus (iv) follows.

(v) Since $D, CA \subseteq A$, also $D \vee CA \subset A$. But $CA \subseteq C$, and hence $D \wedge CA \subseteq D \wedge C = 0$. Since $\Sigma(R)$ is a distributive lattice, $A = A \wedge (C \vee D) = (A \wedge C) \vee (A \wedge D) = CA \vee D$. Thus $1 = A \vee B = CA \vee D \vee B$. The rest is clear, i.e., $D, CA \subseteq A$, $A \wedge B = 0$. So $D \wedge B = CA \wedge B = 0$.

(vi) Again, by distributivity, $C = C \wedge (A \vee B) = (C \wedge A) \vee (C \wedge B) = CA \vee B$ because $B \subseteq C$.

(vii) and (viii) follow from 3.1. □

5. Functors. Some facts about singular submodules are developed for later use. Categories $\mathcal{A}^* \subset \mathcal{A}$ and $\mathcal{B}$ are defined so that $\Sigma, \Sigma_t : \mathcal{A} \to \mathcal{A}$ and $\Sigma_f : \mathcal{A}^* \to \mathcal{B}$ become functors. Universal saturated classes are defined and their connection with direct sums and products of functors explored.

Module categories 5.1. Here the symbol for any category is used to refer to the disjoint union of its objects and its morphisms. As before, $\mathcal{M}\text{od}_R$ denotes the category of unital right R-modules. Denote by t.f. $\mathcal{M}\text{od}_R$ the subcategory of torsion-free modules, where the morphisms have closed kernels. Similarly, tor$\mathcal{M}\text{od}_R$ denote the full subcategory of $\mathcal{M}\text{od}_R$ consisting of all torsion modules.

Ring categories 5.2. The category of all rings $R, S, \ldots$ with identity and (a) identity preserving ring homomorphisms (b) which are onto is denoted by $\mathcal{A}$. Then $\mathcal{A}^* \subset \mathcal{A}$ is the subcategory having the same objects as $\mathcal{A}$, but where $\mathcal{A}^*$ contains only those morphisms (c) $\varphi \in \mathcal{A}$ whose kernels $\varphi^{-1}0$ are closed right ideals.

Lattice category 5.3. Any function $f : L_1 \to L_2$ of complete lattices L_1 and L_2 will be said to be *complete* if f preserves arbitrary infima and suprema. Let $\mathcal{B}$ denote the category whose objects are complete Boolean lattices $L_1, L_2, \ldots$ with smallest and largest elements 0 and 1. The morphisms $f : L_1 \to L_2 \in \mathcal{B}$ are zero preserving (a) $f0 = 0$ lattice homomorphisms, which are (b) one-to-one, and with (c) convex images $fL_1 \subseteq L_2$. Note that (c) is equivalent to $fL_1 = \{y \in L_2 \mid y \leq f1\}$, and that (c) implies that fL_1 is complete, that is, it is closed under arbitrary infs and sups. Moreover, (b) and (c) imply that all $f \in \mathcal{B}$ are complete. The degenerate Boolean lattice $\{0\} \in \mathcal{B}$ with $0 = 1$ is also allowed and used.

Remarks 5.4. (1) It would be tempting to say that, under the contravariant functor, that properties 5.2 (a), (b) and (c) of $\varphi \in \mathcal{A}^*$ are carried over into the dual properties 5.3 (a), (b) and (c) of $\Sigma(\phi)$. This will be the case for the first two (a) and (b), but surprisingly, not the last one (c).

(2) The limited infinite distributive law for complete Boolean lattices ([**4**, Theorem 16] or [**24**, Lemma 10]) is available and is used here.

Next, some useful basic facts about Boolean lattices are stated in a form in which they will illuminate subsequent results.

Facts 5.5. For any set X, recall that the Boolean lattices of all subsets of X is written as $\mathcal{P}(X) = \langle \mathcal{P}(X), \cap, \cup, \setminus, X, \phi \rangle$. Let $L_1 = \langle L_1, \vee, \wedge, ', 1, 0 \rangle$ and $L_2 = \langle L_2, \vee, \wedge, ^c, 1, 0 \rangle$ be Boolean lattices, where the complement of $b \in L_2$ is $b^c \in L_2$, $b \wedge b^c = 0$, $b \vee b^c = 1 \in L_2$. Assume that $L_2 = A \oplus B$ is a direct product of sublattices $A, B \subset L_2$. Write $1 = e \vee e^c = e + e^c$, $e \in A$, $1 - e = e^c \in B$. Then $A = eL_2$, $B = (1-e)L_2$ and $L_2 = eL_2 \oplus (1-e)L_2$. Note that any convex subset $0 \in A \subset L_2$ with a largest element is necessarily of the above form, and conversely, $A = \{y \in L_2 \mid y \leq e\}$. For $a \in A$, define $a^* = a^c \wedge e \in A$. Then

(1) $\langle A, \vee, \wedge, ^*, e, 0 \rangle$ is a complete Boolean lattice. Let $\langle A, [+], \cdot, e, 0 \rangle$ be its associated Boolean ring [**4**, Theorem 19]. Then the ring structure on A as an ideal of the Boolean ring L_2 coincides with $\langle A, [+], \cdot, e, 0 \rangle$.

(2) For any Boolean lattices L_1, L_2 (complete or not), a zero preserving lattice homomorphism $f : L_1 \to L_2$ is the same as a ring homomorphism of the associated Boolean rings L_1, L_2 which in general need not preserve identities [**24**, Theorem 9].

(3) For a lattice homomorphism $f : L_1 \to L_2$ of complete Boolean lattices L_1, L_2 with $f1 = 1 \in L_2$, the following are equivalent: (i) f

preserves arbitrary sups; (ii) f preserves arbitrary infs; (iii) f is complete.

(4) Next, assume that $L_1, L_2 \in \mathcal{B}$ and $f \in \mathcal{B}$, $f : L_1 \to L_2$. Define $e = f1 \in L_2$, $A = eL_2$ and $B = (1-e)L_2$, $1 - e = e^c \in L_2$. Then $L = A \oplus B$ satisfies all of the above including (1). Let f_A be the corestriction of f to its image $f_A : L_1 \to A$. Then both

(i) f_A and f_A^{-1} are identity and complement preserving (order preserving) lattice isomorphisms of Boolean lattices L_1 and $A = \langle A, \vee, \wedge, {}^*, e, 0\rangle$.

(ii) f preserves arbitrary infima and suprema.

Notation 5.6. An arbitrary identity preserving ring homomorphism of $\mathcal{A}$ will be denoted as $\phi : R \to S$ with kernel $\phi^{-1}0 = I \triangleleft R$. For simplicity it may be assumed without loss of generality throughout here that $\phi : R \to R/I = S$ is the natural projection. Right singular submodules and right injective hulls with respect to the ring S are denoted by Z^S, Z_2^S and E_S. In this section N will be a right S-module (notation: N_S, or $N = N_S$, or $N \in \mathcal{M}\text{od}_S$). The induced R-module on N is denoted by N_ϕ where $n \cdot r \cong n(\phi r) = n(r + I)$, $n \in N$, $r \in R$. Since $(ZN)_\phi$, $(EN)_\phi$ are meaningless, define ZN_ϕ and EN_ϕ to be $ZN_\phi = Z(N_\phi)$ and $EN_\phi = E(N_\phi)$.

The set of large submodules of a module M over a ring R is denoted by $\mathcal{L}_R(M)$, and for N, likewise $\mathcal{L}_S(N)$ over the ring S. Large right submodules or right ideals with respect to rings other than R are denoted by "$\leq_e$," for example large right S-submodules.

Observation 5.7. Any right S-modules N, N' satisfy

(i) $\mathrm{Hom}_S(N, N') = \mathrm{Hom}_R(N_\phi, N'_\phi)$.

(ii) The lattice of S-submodules of N is exactly the same as the lattice of R-submodules of N_ϕ. The following properties in this lattice coincide:

$$\text{large}\, S - \text{submodule} = \text{large}\, R - \text{submodule};$$
$$\text{right}\, S - \text{complement} = \text{right}\, R - \text{complement}.$$

Moreover,

$$S\text{–quotient module of } N = R\text{–quotient module of } N_\phi.$$

(iii) By (ii), $N_\phi \ll (E_S N)_\phi$. Hence there exists an embedding $N_\phi \ll (E_S N)_\phi \ll EN_\phi$. It follows from (i) that

(iv) $E_S N = \{x \in EN_\phi \mid xI = 0\}$; furthermore, (i) implies that

(v) $(E_S N)_\phi$ is a quasi-injective R-module. Hence, EN_ϕ contains one unique S-injective hull of N as in (iii).

(vi) ϕ induces a covariant functor $\mathcal{M}\mathrm{od}_S \to \mathcal{M}\mathrm{od}_R$ which maps $N \to N_\phi$ and which is the identity on morphisms. The image of this induced functor is the full subcategory $\{V \in \mathcal{M}\mathrm{od}_R \mid VI = 0\}$, which as a category is isomorphic to $\mathcal{M}\mathrm{od}_S$ via the functor induced by ϕ. This explains why above 5.7 (i) and (ii) hold. Let $\phi^* : \Sigma(S) \to \Sigma(R)$ by $\phi^*(\Delta^S) = \langle\{N_\phi \mid N \in \Delta^S\}\rangle$ for $\Delta^S \in \Sigma(S)$.

The next lemma explores how singular submodules are mapped under the above functor $\mathcal{M}\mathrm{od}_S \to \mathcal{M}\mathrm{od}_R$.

Lemma 5.8. *For $\phi : R \to S = R/I$ with $I = \ker\phi$, let N, N_ϕ, Z^S and Z_2^S be as above. Then*

(i) $Z^S N \subseteq Z(N_\varphi)$;

(ii) $Z_2^S N \subseteq Z_2(N_\phi)$.

(iii) *If $I < R$ is a right complement, then for any $N = N_S$*

(a) $Z^S N = Z(N_\varphi)$; *hence*

(b) $Z_2^S N = Z_2(N_\varphi)$; *and in particular*

(c) $Z^S(R/I) = Z(R/I)$, *and* $Z_2^S(R/I) = Z_2(R/I)$.

Proof. (i) For any $n \in N$, since $nI = 0$, $\mathrm{ann}_S n = \{r + I \in S \mid nr = 0\} = n^\perp/I$. If $n^\perp <\not\leq R$, then for some $0 \neq B \leq R$, $n^\perp \oplus B \leq R$. But then $(n^\perp/I) \oplus [(B+I)/I] \leq R/I$ shows that also $n \notin Z^S N$. Thus $Z^S N \subseteq Z(N_\varphi)$.

(ii) Any quotient or submodule of N with respect to S is simultaneously also an R-module, and the S and R homomorphisms of these modules coincide. Let π be the natural projection $\pi : N/Z^S N \to N/ZN_\varphi$ included by (i). Since Z^S is a subfunctor of the identity functor on $\mathcal{M}\mathrm{od}_S$, it restricts and corestricts to give a map $\pi : Z^S[N/Z^S N] \to Z^S[N/ZN_\varphi] \subset Z[N/ZN_\varphi]$. The last inclusion follows by (i).

(iii) (a) It suffices to show that for any $n \in Z(N_\varphi)$, $\mathrm{Ann}_S n = n^\perp/I$ is a large right S ideal of R/I. Since $I < R$ is a right R-complement, and $n^{\perp} \ll R$ we know that $n^{\perp}/I \ll R/I$ as a right R-module. Since R/I is also an S-module, and since the essential R and essential S-submodules of R/I coincide, also $n^{\perp}/I \leq_e S$ is large as a right S-ideal. (iii) (b) and (c) follow from (a).

Lemma 5.9. *For $S = R/I$ with $I < R$ a right complement, let $K = I + ZR$ and as before $\overline{K}$ is defined as the complement closure of K. Let $\varphi : R \to S = R/I \in \mathcal{A}^*$ be the natural projection. Then*

(i) *$K \ll \overline{K} \triangleleft R$ and $Z_2^S(S) = \overline{K}/I$.*

(ii) *$\varphi^*\Sigma_f(S) = \{\Delta \mid$ for all $M \in \Delta$, there exists an index set J and there exists an $M \hookrightarrow E(\oplus_J R/\overline{K})\} \subseteq \Sigma_f(R)$.*

Proof. (i) This is proved in [**11**, Proposition C].

(ii) Note that $\langle R/\overline{K}\rangle \in \Sigma_f(R)$ because for $\varphi \in \mathcal{A}^*$, $Z_2^S(S) = Z_2(R/I)$ and hence $Z(R/\overline{K}) = 0$. Write $1 = t + f = t \vee f \in \Sigma_t(S) \oplus \Sigma_f(S)$, $t \wedge f = 0$, and $\Sigma_f(S) = f\Sigma(S)$. Since, for any $N = N_S$, $Z^S N = 0$ implies that $ZN_{\varphi} = 0$, $\varphi^*\Sigma_f(S) \subseteq \Sigma_f(R)$. For any $\Delta^S \in \Sigma_f(S)$, $\Delta^S \leq f$; hence $\varphi^*\Delta^S \leq \varphi^* f = \varphi^*(\langle S/Z_2(S)\rangle^S) = \langle (S/Z_2(S))_{\varphi}\rangle = \langle R/\overline{K}\rangle$ by (i). Therefore, for any $N \in \Delta^S$, $N_{\varphi} \hookrightarrow E(\oplus_{\Gamma} R/\overline{K})$ for some Γ. Since $(R/\overline{K})I = 0$ and for any J, $\oplus_J R/\overline{K} \ll E(\oplus_J R/\overline{K})$, by 5.10 (i) $\varphi^* f$ is exactly the set in (ii). □

Lemma 5.10. *For any $\Delta^S \in \Sigma(S)$, define $\Delta_{\varphi}^S = \{N_{\varphi} \mid N \in \Delta^S\} \subset \mathcal{M}\mathrm{od}_R$, and then let $\langle \Delta_{\varphi}^S\rangle \in \Sigma(R)$ be the saturated class of R-modules generated by Δ_{φ}^S. For $I \triangleleft R$ with $S = R/I$ and any R-module M, define $\mathrm{Ann}_M I = \{m \in M \mid mI = 0\} \leq M$. Then*

(i) $\langle \Delta_{\varphi}^S\rangle = \{M \in \mathcal{M}\mathrm{od}_R \mid$ there exists an $N \in \Delta_{\varphi}^S, N_{\phi} \ll M\}$;

(ii) $\langle \Delta_{\varphi}^S\rangle = \{M \mid \mathrm{Ann}_M I \ll M, \mathrm{Ann}_M I \in \Delta^S\}$.

Proof. (i) Since Δ_{φ}^S to start with is already closed under submodules, isomorphic copies, and direct sums, 2.5 (ii) translates into conclusion (i).

(ii) By 2.5 (i), $\langle \Delta_{\varphi}^S\rangle = \{M \mid$ there exists an $N \in \Delta^S$ and $M \hookrightarrow EN_{\varphi}\}$. Thus for $M \in \langle \Delta_{\varphi}^S\rangle$ and N as above, $M \cap N_{\varphi} \ll \mathrm{Ann}_M I \ll M$.

From 5.6 (ii) it follows that $M \cap N_\varphi$ is a large S-submodule of the S-module $\mathrm{Ann}_M I$. Since $M \cap N \in \Delta^S$, also $\mathrm{Ann}_M I \in \Delta^S$. □

Definition and Proposition 5.11. *For any $\phi \in \mathcal{A}$, define a function $\phi^* : \Sigma(S) \to \Sigma(R)$ by $\phi^*\Delta^S = \langle \Delta^S_\varphi \rangle$ for $\Delta^S \in \Sigma(S)$. Then*

(i) *ϕ^* is one-to-one.*

(ii) *$\phi^*\Delta_1^S \leq \phi^*\Delta_2^S \in \Sigma(R) \leftrightarrow \Delta_1^S \leq \Delta_2^S \in \Sigma(S)$ for any $\Delta_1^S, \Delta_2^S \in \Sigma(S)$.*

(iii) *$\phi^*(\Sigma(S))$ is convex and upper directed in $\Sigma(R)$.*

Proof. (i) Suppose that $\phi^*\Delta_1^S = \phi^*\Delta_2^S$ for some $\Delta_1^S, \Delta_2^S \in \Sigma(R)$. It suffices to show that for any $P \in \Delta_1^S$, also $P \in \Delta_2^S$. The definition of ϕ^* implies that, first, $P_\phi \in \phi^*\Delta_1^S = \phi^*\Delta_2^S$, and second, that there exists some $Q \in \Delta_2^S$ with $Q_\phi \ll P_\phi$. By 5.7 (ii), likewise $Q_S \leq_e P_S$, and by 2.2 (d), since $Q \in \Delta_2^S$, also $P \in \Delta_2^S$. Thus, ϕ^* is one-to-one.

(ii) From $\Delta_1^S \subseteq \Delta_2^S \in \Sigma(S)$ it immediately follows that $\phi^*\Delta_1^S \subseteq \phi^*\Delta_2^S$. Thus $\Delta_1^S \leq \Delta_2^S \Rightarrow \varphi^*\Delta_1^S \leq \phi^*\Delta_2^S$. Conversely, assume that $\phi^*\Delta_1^S \subseteq \phi^*\Delta_2^S \in \Sigma(R)$ for some $\Delta_1^S, \Delta_2^S \in \Sigma(S)$, and take any $P \in \Delta_1^S$. Then $P_\phi \in \phi^*\Delta_1^S \subseteq \phi^*\Delta_2^S$. Hence there exists a $Q \in \Delta_2^S$ with $Q_\phi \ll P_\phi$. Again, by 5.7 (ii), $Q_S \leq_e P_S$, and by 2.2 (d), also $P \in \Delta_2^S$. Thus $\Delta_1^S \subseteq \Delta_2^S$. Hence $\Delta_1^S \leq \Delta_2^S \Leftrightarrow \phi^*\Delta_1^S \leq \phi^*\Delta_2^S$.

(iii) Suppose that $\Delta \in \Sigma(R)$ with $\Delta \leq \phi^*\Delta_1^S \in \Sigma(R)$ for some Δ_1^S in $\Sigma(S)$. Define a subclass $\Delta_2^S \subseteq \Delta_1^S$ by $\Delta_2^S = \{P \in \Delta_1^S \mid P_\phi \in \Delta\}$. In order to prove that $\Delta_2^S \in \Sigma(S)$, we omit the proof of 2.2 (a), (b) and (c) but show (d). So suppose that Q is an S-module such that $P \leq_e Q$ for some $P \in \Delta_2^S$. Since $P \in \Delta_1^S$ and $P \leq_e Q$, also $Q \in \Delta_1^S$. Now since $P_\phi \in \Delta$ and $P_\phi \ll Q_\phi$, also $Q_\phi \in \Delta$. But then $Q \in \Delta_2^S \in \Sigma(S)$.

Next it will be proved that $\phi^*\Delta_2^S = \Delta$. By definition of ϕ^*,

$$\begin{aligned}\phi^*\Delta_2^S &= \{M \in \mathcal{M}\mathrm{od}_R \mid \exists\, N \in \Delta_2^S, N_\phi \ll M\}\\ &= \{M \mid \exists\, N \in \mathcal{M}\mathrm{od}_S, N_\phi \in \Delta, N \in \Delta_1^S, N_\phi \ll M\} \subseteq \Delta.\end{aligned}$$

The last inclusion holds because Δ is closed under essential extensions. To show conversely that $\Delta \subseteq \phi^*\Delta_2^S$, take any $M \in \Delta$. Since $\Delta \subseteq \phi^*\Delta_1^S$, $M \in \phi^*\Delta_1^S$. The latter means that $W_\phi \ll M$ for some $W \in \Delta_1^S$. But $W_\phi \ll M \in \Delta$ implies that also $W_\phi \in \Delta$. Hence $W \in \Delta_2^S$. But then

$M \in \phi^* \Delta_2^S$, and $\Delta \subseteq \phi^* \Delta_2^S$. Therefore $\phi^* \Delta_2^S = \Delta$, and $\phi^*(\Sigma(S))$ is a convex subset of $\Sigma(R)$, which is easily seen to be upper directed.

Corollary 5.12. *For any $\phi \in \mathcal{A}$, $\phi^* : \Sigma(S) \to \Sigma(R)$ belongs to $\mathcal{B}$, and in particular is a complete lattice homomorphism.*

Proof. Let ϕ_A^* be the corestriction of ϕ^* to its image $A \equiv \phi^* \Sigma(S) = \{y \in \Sigma(R) \mid y \leq e = \phi^* 1\}$, which is a complete lattice by 5.11 (iii) and the completeness of $\Sigma(R)$. But then $\phi_A^* : \Sigma(S) \to A$ is a bijective map of two lattices $\Sigma(S)$ and A such that both ϕ_A^* and ϕ^* preserve order. Consequently, ϕ_A^* and hence automatically ϕ^* are lattice homomorphisms. (See [**4**, Lemma 1.2].)

Theorem IV, 5.13. *Let $\mathcal{A}^* \subset \mathcal{A}$ and $\mathcal{B}$ be the ring and lattice categories of* 5.2 *and* 5.3. *Let $\varphi : R \to S \in \mathcal{A}$ be a surjective ring homomorphism with kernel $\varphi = I \triangleleft R$. Let $\Sigma(S) = \Sigma_t(S) \oplus \Sigma_f(S)$ be as defined in* 2.2 *and* 2.17. *Define $\Sigma(\varphi) = \varphi^*$ as in* 5.11. *For $K = I + ZR < R$, let $\overline{K}$ be its complement closure, i.e., $K \ll \overline{K} \triangleleft R$* (*see* 1.1). *Then*

(i) *$\Sigma : \mathcal{A} \to \mathcal{B}$ is a contravariant functor. In particular, $\varphi^* : \Sigma(S) \to \Sigma(R)$ is a zero preserving monic lattice homomorphism* (*equivalently ring homomorphism of associated Boolean rings*) *whose image $\varphi^* \Sigma(S)$ is convex* (*and hence complete*) *in $\Sigma(R)$. Consequently, φ^* preserves arbitrary infima and suprema. Moreover,*

(ii) *$\varphi^* \Sigma_t(S) \subseteq \Sigma_t(R)$, and $\Sigma_t \leq \Sigma$ is a subfunctor.*

In (iii) *and* (iv) *below in addition assume that $\varphi \in \mathcal{A}^*$. Then the following hold.*

(iii) *$\varphi^* \Sigma_f(S) \subseteq \Sigma_f(R)$, $\Sigma_f \leq \Sigma$ is a subfunctor, and $\Sigma = \Sigma_t \oplus \Sigma_f$ is a direct sum of functors.*

(iv) *$\varphi^* \Sigma_f(S) = \{\Delta \in \Sigma(R) \mid$ for all $M_R \in \Delta$, there exists an index set J and there exists some $M \hookrightarrow E(\oplus_J R/\overline{K})\}$.*

Corollary to Theorem IV, 5.14. *For an arbitrary $\varphi \in \mathcal{A}$ as above, $\Sigma(R) = A \oplus B$ is a direct sum of unique convex sublattices A and B where $A = \varphi^* \Sigma(S)$. Let $\langle A, \vee, \wedge, {}^*, e, 0 \rangle$ be the induced Boolean lattice* (*see* 5.5(1)). *Then the corestriction $\varphi^* : \Sigma(S) \to A$ is a complete* *isomorphism of Boolean lattices* (*equivalently isomorphism of associated Boolean rings*). *In particular,* $\varphi^*0 = 0$*,* $\varphi^*(1_S) = 1_A$*, and* φ^* *preserves complements.*

6. Universal classes and direct sums of functors. It was shown that a pairwise disjoint set Γ of elements of the lattice $\Sigma(R)$ whose join is $\vee\Gamma = 1 \in \Sigma(R)$ give direct sum decompositions of R-modules. In general, Γ and the direct sum decomposition is only available for the ring R. Section 6 will answer the following question. Just what is it about certain sets of saturated classes like types I, II, III, or C, D, or C, A, B, D which over any ring R or S always give direct sum decompositions of injective modules over the ring in question, e.g., $M = \widehat{M} = M_I \oplus M_{II} \oplus M_{III}$, or $M = C \oplus D$, or $M = M_{CA} \oplus M_B \oplus M_D$. In order to explain what is really going on, or what is so-to-speak behind this phenomenon, first universal saturated classes are defined and studied. Then it is shown, among other things, that the above direct sum phenomenon over any rings holds if and only if Σ is a subdirect product of subfunctors, e.g., $\Sigma = \Sigma_I \oplus \Sigma_{II} \oplus \Sigma_{III}$ or $\Sigma = \Sigma_C \oplus \Sigma_D$ or $\Sigma = \Sigma_{CA} \oplus \Sigma_D \oplus \Sigma_B$. Anticipating further developments, our direct sum theorem in Section 3 was done for possibly infinite sets Γ, and here we also develop the direct sum decomposition of Σ as a direct product of either finite or infinite number of subfunctors.

For $\varphi : R \to S = R/I \in \mathcal{A}$ as before, or $\mathcal{A}^*$, let $\varphi^\# : \mathcal{M}\mathrm{od}_S \to \mathcal{M}\mathrm{od}_R$ be the induced functor, where $\varphi^\#(N_S) = N_\varphi$ and $\varphi^\#$ is the identity on morphisms. If $N = N_I \oplus N_{II} \oplus N_{III}$ is a type I, II, III decomposition of N_S over S, then it turns out that it is also simultaneously the corresponding decomposition over R, i.e., $N_\varphi = (N_I)_\varphi \oplus (N_{II})_\varphi \oplus (N_{III})_\varphi$. However, the functor $\varphi^\#$ does not usually preserve injective modules. Clearly, if $R \to S$ then for a universal saturated class Δ, the class $\Delta(R)$ and $\Delta(S)$ over R and S have to be connected or interrelated in some way, which is formulated and explained in general in this section, so that $\{I, II, III\}$ or $\{C, D\}$ or $\{CA, D, B\}$ are merely special cases of a very general and pervasive phenomenon.

Note that in (ii) below, upon taking injective hulls, the condition (ii) could equivalently be formulated for direct sum decompositions of modules.

Observation 6.1. *For* $\varphi : R \to S \in \mathcal{A}$*, let* $\Delta(S) \in \Sigma(S)$ *and* $\Delta(R) \in \Sigma(R)$ *with* $\varphi^*\Delta(S) \subseteq \Delta(R)$*. Then the following conditions are all equivalent*:

(i) *For all* $P \in c\Delta(S) \Rightarrow P_\varphi \notin \Delta(R)$.

(ii) *For all* $N \in \mathcal{M}\mathrm{od}_S$*, for all* $N_1 \oplus N_2 \leq_e N$, $N_1 \in \Delta(S)$, $N_2 \in c\Delta(S) \Rightarrow N_{1\varphi} \oplus N_{2\varphi} \ll N_\varphi$ *with* $N_{1\varphi} \in \Delta(R)$ *and* $N_{2\varphi} \in c\Delta(R)$.

(iii) $\varphi^*c\Delta(S) \subseteq c\Delta(R)$.

Proof. Trivially, (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i). (i) $\Rightarrow$ (iii). If (iii) fails, then there is a $V \in c\Delta(S)$ with $V_\varphi \notin c\Delta(R)$. By definition of the latter, there exists a $0 \neq P_R \leq V_\varphi$ with $P_R \in \Delta(R)$. Since $P \subseteq V \in c\Delta(S)$ viewing P as an S-module, we conclude that $P \in c\Delta(S)$. But then, by (i), $P = P_\varphi \notin \Delta(R)$, a contradiction. □

The next definition will correct an oversight, in [**14**], where the condition 6.2 (b) below should have been included as part of [**14**, Definition 3.13].

Definition 6.2. For any subcategory $\mathcal{C} \subseteq \mathcal{A}$, a $\mathcal{C}$-*universal saturated class* is a function Δ which assigns to every ring R with identity a saturated class of modules $\Delta(R) \in \mathcal{C}$ satisfying the following additional coherence conditions relative to $\mathcal{C}$. For any (surjective) ring homomorphism $\varphi : R \to S$ in the category $\mathcal{C}$ and any modules $N \in \Delta(S)$, $P \in c\Delta(S)$, the induced R-modules satisfy (a) $N_\varphi \in \Delta(R)$ and (b) $P_\varphi \in c\Delta(R)$. An equivalent lattice theoretic definition is that Δ is a $\mathcal{C}$-universal saturated class if for every $\varphi \in \mathcal{C}$ as above,

(a) $\varphi^*\Delta(S) \subseteq \Delta(R)$ and (b) $\varphi^*c\Delta(S) \subseteq c\Delta(R)$.

An $\mathcal{A}$-universal saturated class will be called simply a *universal saturated class*.

For any function Δ:ring $\to$ saturated classes with $\Delta(R) \in \Sigma(R)$ for all R, define three more associated functions $c\Delta = c(\Delta)$, Δ_t and Δ_f by $(c\Delta)(R) = c(\Delta(R))$, $\Delta_t(R) = t(\Delta(R))$ and $\Delta_f(R) = f(\Delta(R))$ as in 2.17. For two or more such functions Δ_1, Δ_2, define $\Delta_1 \vee \Delta_2$ by $(\Delta_1 \vee \Delta_2)(R) = \Delta_1(R) \vee \Delta_2(R)$, and similarly for $\Delta_1 \wedge \Delta_2$. Note that, in general, $(c\Delta)_t \neq c(\Delta_t)$.

If Δ is a $\mathcal{C}$-universal saturated class then so also is $c\Delta$. The class of all unital modules, which assigns $\mathcal{M}\mathrm{od}_R$ to R, and its complementary class $\{(0)\}$ are trivially universal saturated classes.

Lemma 6.3. *If Δ is a universal saturated class, then Δ_t and Δ_f are $\mathcal{A}^*$-universal saturated classes.*

Proof. (i) For $\varphi : R \to S$ in $\mathcal{A}$, for any $N \in \Delta_t(S)$, $N = Z_2^S N \subseteq Z_2 N_\varphi$ by 5.8. Hence, also $N_\varphi = Z_2 N_\varphi$, and consequently, $\varphi^*[t(\Delta(S))] \subseteq t(\Delta(R))$ which by 6.2 says that $\varphi^* \Delta_t(S) \subseteq \Delta_t(R)$. Here the additional restriction that $\varphi \in \mathcal{A}^*$ is needed for the first time to conclude for $P \in \Delta_f(S)$ that also $ZP_\varphi = 0$ and hence that $P_\varphi \in \Delta_f(R)$, and $\varphi^*(\Delta_f(S)) \subseteq \Delta_f(R)$. Similarly, 6.2(b) follows.

Note that the previous lemma implies that $c(\Delta_t), c(\Delta_f), (c\Delta)_t$ and $(c\Delta)_f$ are four more $\mathcal{A}^*$-universal saturated classes.

Notation 6.4. Let $\mathcal{C} \subseteq \mathcal{A}$ be any subcategory (and, in particular, $\mathcal{C}$ may be $\mathcal{C} = \mathcal{A}$). Even when Δ is a $\mathcal{C}$-universal saturated class, if M is an R-module and if either R is fixed, or understood from context, we abbreviate $\Delta(R) \equiv \Delta$ and define "M_Δ" by $M_\Delta = M_{(\Delta(R))}$ as in Section 2; similarly, $M_c\Delta = M_{c(\Delta)} = M_{((c\Delta)(R))}$, $M\Delta_t = M_{(\Delta_t(R))}$, etc.

Below, we use $c\Delta = \Delta^c$ and $\Delta(S)^c = c(\Delta(S))$ interchangeably.

Consequence 6.5. *Let $\mathcal{C} \subseteq \mathcal{A}$ be any subcategory, and in particular $\mathcal{C}$ may be $\mathcal{C} = \mathcal{A}$. If Δ_1 and Δ_2 are $\mathcal{C}$-universally saturated classes, then so are all their associated Boolean combinations*:

(i) $\Delta_1 \vee \Delta_2$, $\Delta_1 \wedge \Delta_2$ *and* $\Delta_1 + \Delta_2 = [\Delta_1 \vee \Delta_2] \wedge [\Delta_1 \wedge \Delta_2]^c = [\Delta_1 \wedge c\Delta_2] \vee [\Delta_2 \wedge c\Delta_1]$.

(ii) *$\mathcal{C}$-universal saturated classes are closed under arbitrary infima and suprema.*

Proof. (i) The proof for $\Delta_1 \vee \Delta_2$ and $\Delta_1 \wedge \Delta_2$ is omitted. For any $\varphi : R \to S$ in $\mathcal{C}$, the map $\varphi^* : \Sigma(S) \to \Sigma(R)$ preserves arbitrary sups and infs. By 6.2,

$$\begin{aligned}\phi^*[\Delta_1(S)+\Delta_2(S)] &= [\phi^*\Delta_1(S)\vee\phi^*\Delta_2(S)]\wedge\phi^*\{[\Delta_1(S)\wedge\Delta_2(S)]^c\}\\ &\le [\Delta_1(R)\vee\Delta_2(R)]\wedge\phi^*\{c\Delta_1(S)\vee c\Delta_2(S)\}\\ &= [\Delta_1(R)\vee\Delta_2(R)]\wedge[\phi^*c\Delta_1(S)\vee\phi^*c\Delta_2(S)]\\ &\le [\Delta_1(R)\vee\Delta_2(R)]\wedge[c\Delta_1(R)\vee c\Delta_2(R)]\\ &= [\Delta_1(R)\vee\Delta_2(R)]\wedge[\Delta_1(R)\wedge\Delta_2(R)]^c\\ &= \Delta_1(R)+\Delta_2(R).\end{aligned}$$

De Morgan's law gives $(\Delta_1+\Delta_2)^c = ([\Delta_1\vee\Delta_2]\wedge[\Delta_1\wedge\Delta_2]^c)^c = [\Delta_1\vee\Delta_2]^c\vee[\Delta_1\wedge\Delta_2] = [\Delta_1^c\wedge\Delta_2^c]\vee[\Delta_1\wedge\Delta_2]$. Then

$$\begin{aligned}\varphi^*[c(\Delta_1+\Delta_2)(S)] &= \varphi^*[c\Delta_1(S)\wedge c\Delta_2(S)]\vee\varphi^*[\Delta_1(S)\wedge\Delta_2(S)]\\ &= \{\varphi^*[c\Delta_1(S)]\wedge\varphi^*[c\Delta_2(S)]\}\\ &\quad\vee\{\varphi^*\Delta_1(S)\wedge\varphi^*\Delta_2(S)\}\\ &\le \{c\Delta_1(R)\wedge c\Delta_2(R)\}\vee\{\Delta_1(R)\wedge\Delta_2(R)\}\\ &= c(\Delta_1+\Delta_2)(R)\end{aligned}$$

as required. Hence $\Delta_1+\Delta_2$ is a $\mathcal{C}$-universal saturated class.

(ii) For any family $\{\Delta_\alpha\}_\alpha$ of $\mathcal{C}$-universal saturated classes, since φ^* is complete $\varphi^*[(\vee_\alpha\Delta_a)(S)] = \vee_\alpha\varphi^*(\Delta_\alpha(S)) \le \vee_\alpha\varphi^*(\Delta_\alpha(R)) = \varphi^*[(\vee_\alpha\Delta_\alpha)(R)]$. For the complementary class, use of infinite De Morgan's rule gives

$$\begin{aligned}\varphi^*\left\{c\left[\bigvee_\alpha\Delta_\alpha(S)\right]\right\} &= \varphi^*\left\{\bigwedge_\alpha[c\Delta_\alpha(S)]\right\}\\ &= \bigwedge_\alpha\{\varphi^*[c\Delta_\alpha(S)]\}\\ &\le \bigwedge_\alpha[c\Delta_\alpha(R)]\\ &= c\left[\bigwedge_\alpha\Delta_\alpha(R)\right],\end{aligned}$$

and hence $\varphi^*c\vee_\alpha\Delta_\alpha(S)\le c\vee_\alpha\Delta_a(R)$. Thus $\vee_\alpha\Delta_\alpha$ is a $\mathcal{C}$-universal saturated class. The proof for $\wedge\Delta_\alpha$ is also is similar, and is omitted. □

Definitions and notation 6.6. For any $\delta \in \Sigma(R)$, define $\Sigma_\delta(R) = \{x \in \Sigma(R) \mid x \leq \delta\} = [0, \delta]$. Then $\Sigma(R) = \Sigma_\delta(R) \oplus \Sigma_{c\delta}(R)$ is a lattice direct sum of convex sublattices if $\Sigma(R)$ is a Boolean lattice. For any function Δ of rings to saturated classes such that $\Delta(R) \in \Sigma(R)$ for all R, define a function Σ_Δ : rings $\to \mathcal{B}$ by $\Sigma_\Delta(R) = \Sigma_{\Delta(R)}(R) = [0, \Delta(R)]$.

Suppose that $\mathcal{C} \subseteq \mathcal{A}$ is a subcategory, that $\Sigma_i : \mathcal{C} \to \mathcal{B}$, $i \in I$, is a class of subfunctors $\Sigma_i \leq \Sigma$ of Σ restricted to $\mathcal{C}$, where I can be finite, a set, or a class. Assume that, for any R, (i) $\{i \in I \mid \Sigma_i(R) \neq 0\}$ is always a set, and that (ii) $\Sigma(R) \cong \prod_{i \in I} \Sigma_i(R)$ are isomorphic as Boolean lattices, where the operations in the latter are coordinatewise. For any i, view $\Sigma_i(R) \subset \Sigma(R)$ as a (convex) sublattice. Then let $1 \to (e_i)_{i \in I}$, where e_i is the identity element in $e_i \in \Sigma_i(R)$. Then for every $y \in \Sigma(R)$, $y = \sup\{y \wedge e_i \mid i \in I\} \to (y \wedge e_i)_{i \in I}$. Sometimes for the sake of simplicity of notation we will replace "$\cong$" by "$=$" and identify $\Sigma(R) = \prod_{i \in I} \Sigma_i(R)$ under this isomorphism. Lastly, assume that (iii) for any $\varphi : R \to S \in \mathcal{C}$, $\varphi^* \Sigma_i(S) \subseteq \Sigma_i(R)$ for all i.

It is a straightforward consequence of the completeness of φ^* that φ^* acts coordinatewise, i.e., for any $x = (x_i)_{i \in I} \in \Sigma(S)$, $\varphi^* x = (\varphi^* x_i)_{i \in I} \in \Sigma(R)$.

If (i), (ii) and (iii) hold, then Σ is said to be a *product of the subfunctors* $\Sigma_i \leq \Sigma$, $i \in I$, written as $\Sigma \cong \prod_{i \in I} \Sigma_i$ or $\Sigma = \prod_{i \in I} \Sigma_i$. In the case when $I = \{1, \ldots, n\}$ is finite, Σ is said to be a direct sum of subfunctors, written as $\Sigma = \Sigma_1 \oplus \cdots \oplus \Sigma_n$. In all these cases, strictly speaking, it is Σ restricted to $\mathcal{C}$.

Theorem V, 6.7. *Suppose that Δ_i, $i \in I$, is a class of functions mapping rings to saturated classes as in the last definition.*

(1) *Then for a fixed ring R the following three conditions* (a), (b) *and* (c) *are all equivalent.*

(a) *For all $M \in \mathcal{M}\mathrm{od}_R$, for all $M_i \in \Delta_i(R)$, if $i \neq j$, then $M_i \cap M_j = 0$; and there exist $M_i \in \Delta_i(R)$ such that $\oplus_{i \in I} M_i \ll M$.*

(b) $\sup\{\Delta_i(R) \mid i \in I\} = 1 \in \Sigma(R)$ *and* $\Delta_i(R) \wedge \Delta_j(R) = 0$ *for* $i \neq j \in I$.

(c) $\Sigma(R) \cong \prod_{i \in I} \Sigma_{\Delta_i}(R)$ *is a lattice direct product of convex* (*and hence complete*) *sublattices.*

(2) *Now assume, in addition, that Δ_i, $i \in I$, satisfy* (1) (a), (b) *and*

(c) *for all rings* $R \in \mathcal{C} \subseteq \mathcal{A}$ *and for some subcategory* $\mathcal{C}$. *Let* R, S *and* $\varphi : R \to S \in \mathcal{C}$ *represent any completely arbitrary element. Then the following two conditions* (d) *and* (e) *are equivalent.*

(d) *For any* $i \in I$ *and any* $N \in \Delta_i(S)$, $N_\varphi \in \Delta_i(R)$.

(e) $\Sigma = \prod_{i\in I} \Sigma_{\Delta_i} : \mathcal{C} \to \mathcal{B}$ *is a direct product of subfunctors* $\Sigma_{\Delta_i} \leq \Sigma$, *i.e., equivalently* $\varphi^*\Sigma_{\Delta_i}(S) \subseteq \Sigma_{\Delta_i}(R)$ *for all* $i \in I$.

Proof. First note that, for any R, $\{\Delta_i(R) \neq 0 \mid i \in I\}$ is a set, and hence so also is $\{i \mid \Sigma_{\Delta_i}(R) \neq 0\}$. (1) That (c) $\Leftrightarrow$ (b) is standard lattice theory, while (b) $\Rightarrow$ (a) follows from 3.1 (applied with $\Gamma = \{\Delta_i(R) \neq 0 \mid i \in I\}$. (1) (a) $\Rightarrow$ (b). If $\Delta_i(R) \wedge \Delta_j(R) \neq 0$, then there exists a $0 \neq M \in \Delta_i(R) \cap \Delta_j(R)$. Define $M_i = M$ and $M_j = M$. Then $0 \neq M = M_i \cap M_j$ violates (a). Hence $\Delta_i(R) \wedge \Delta_j(R) = 0$.

In order to show that $\vee_{i\in I}\Delta_i(R) = \mathcal{M}\mathrm{od}_R = 1 \in \Sigma(R)$, let $M \in \mathcal{M}\mathrm{od}_R$ be arbitrary. Let $\oplus_{i\in I} M_i \ll M$ be as in (a). Then $\langle M_i\rangle \leq \Delta_i(R)$ all i, and

$$M \in \langle M\rangle = \Big\langle \bigoplus_{i\in I} M_i \Big\rangle = \bigvee_{i\in I} \Delta_i(R) = 1.$$

(2) (d) $\Rightarrow$ (e). By 1(c), $\Sigma(R) = \Pi\Sigma_{\Delta_i}(R)$ for any R. Let $\Delta^S \in \Sigma_{\Delta_i}(S)$. Hence $\Delta^S \subseteq \Delta_i(S)$. Then by definition of φ^*, $\varphi^*\Delta^S = \langle\{N_\varphi \mid N \in \Delta^S\}\rangle$. By hypothesis (d), it now follows that $N_\varphi \in \Delta_i(R)$. Since all the generators of $\varphi^*\Delta^S$ belong to $\Delta_i(R)$, we have $\varphi^*\Delta^S \leq \Delta_i(R)$ or $\varphi^*\Delta^S \in [0, \Delta_i(R)] = \Sigma_{\Delta_i}(R)$. Hence $\varphi^*\Sigma_{\Delta_i}(S) \subseteq \Sigma_{\Delta_i}(R)$. Since $\varphi^* : \prod_{i\in I} \Sigma_{\Delta_i}(S) \to \prod_{i\in I} \Sigma_{\Delta_i}(R)$ by hypothesis (1) (a), (b), (c) and since $\varphi^* \in \mathcal{B}$ is complete, and since the Boolean lattice operations in each component are coordinatewise, $\Sigma = \prod_{i\in I} \Sigma_{\Delta_i}$ is a direct product of subfunctors.

(2) (e) $\Rightarrow$ (d). For $i \in I$ and $N \in \Delta_i(S)$ as in (2) (d), $N_\varphi \in \varphi^*(\langle N\rangle^S) \in \varphi^*\Sigma_{\Delta_i}(S) \subseteq \Sigma_{\Delta_i}(R) = [0, \Delta_i(R)]$ or $N_\varphi \in \varphi^*(\langle N\rangle^S) \subseteq \Delta_i(R)$, and $N_\varphi \in \Delta_i(R)$.

If I is finite, $I = \{1, \ldots, n\}$, then everywhere in the last theorem the product can be replaced by the finite direct sum "$\oplus_{i=1}^n$." If we start out with some one universal saturated class Δ in which we are interested in, we can still apply the last theorem as is described in the next corollary. □

Corollary to Theorem V, 6.8. *Let $\mathcal{C} \subseteq \mathcal{A}$ be a subcategory S, a ring $(1 \in S)$, and $N = N_S$ are S-module. Then there exist*

(i) *two complement submodules $N_i \le N$, with $N_1 \oplus N_2 \le_e N$, $E_S(N) = E_S N_1 \oplus E_S N_2$, $N_1 \in \Delta(S)$, $N_2 \in c\Delta(S)$.*

(ii) *For any $\varphi : R \to S \in \mathcal{C}$, $(E_S N)_\varphi = (E_S N_1)_\varphi \oplus (E_S N_2)_\varphi \ll EN_\varphi$ with $(E_S N_1)_\varphi \in \Delta(R)$ and $(E_S N_2)_\varphi \in c\Delta(R)$.*

(iii) *$\Sigma = \Sigma_\Delta \oplus \Sigma_{c\Delta} : \mathcal{C} \to \mathcal{B}$ is a direct sum of subfunctors Σ_Δ,$\Sigma_{c\Delta}$ of Σ restricted to $\mathcal{C}$, i.e., $\varphi^* : \Sigma(S) = \Sigma_\Delta(S) \oplus \Sigma_{c\Delta}(S) \to \Sigma(R) = \Sigma_\Delta(R) \oplus \Sigma_{c\Delta}(R)$ where these are lattice direct sums of convex sublattices and $\varphi^* \Sigma_\Delta(S) \subseteq \Sigma_\Delta(R)$, $\varphi^* \Sigma_{c\Delta}(S) \subseteq \Sigma_{c\Delta}(R)$.*

Proposition 6.9. *The functions A, B, C, D, CA defined in 4.12 are universal saturated classes.*

Proof. For any R, S and any $\varphi : R \to S \in \mathcal{A}$, as before, let $\varphi^{\#}$ be the induced functor $\varphi^{\#} : \mathcal{M}\mathrm{od}_S \to \mathcal{M}\mathrm{od}_R$, and let $N \in \mathcal{M}\mathrm{od}_S$. Also let $\varphi^* : \Sigma(S) \to \Sigma(R) = A \oplus G$ where the corestriction $\varphi^* : \Sigma(S) \to A$ is a complete isomorphism. An element $q \in \Sigma(S)$ is an atom, is a supremum of atoms, or does not dominate any atoms if and only if $\varphi^*(q)$ satisfies the same.

The functor $\varphi^{\#}$ preserves submodules, essential submodule (complements), direct summands, essential direct sums and intersections of submodules. Thus $N \in D(S)$ (or $N \in C(S)$) if and only if $N_\varphi \in D(R)$ (or $N_\varphi \in C(R)$). Since $\langle N_\varphi \rangle \in A \cong \Sigma(S)$, A inherits the order theoretic properties of $\Sigma(S)$. Thus N_S is atomic if and only if N_φ is. Next, $N_S \in A(S)$ if and only if $\langle N_S \rangle^S \in \Sigma(S)$ is a supremum of atoms of $\Sigma(S)$ which holds if and only if $\langle N_R \rangle \in \Sigma(R)$ is likewise. Thus, $N \in A(S)$, $B(S)$ or $CA(S)$ if $N_\varphi \in A(R), B(R)$ or $CA(R)$. □

Remarks 6.10. (1) All of the above could also be proved non-categorically by standard ring theoretic arguments. For example, if $N \in B(S)$, we would have to show that for any $0 \ne y \in N$, there exists $b \in R$ such that for any set J, there does *not* exist an embedding $(yR)_\varphi \hookrightarrow E(\oplus_J (ybR)_\varphi)$. This would show that N_φ contains no R-atomic submodules.

(2) The functor $\mathcal{M}\mathrm{od}_S \to \mathcal{M}\mathrm{od}_R$ induced by φ does not in general

preserve injective modules, and types I, II and III where defined via injectives.

Proposition 6.11. *The type I, II and III functions are universal saturated classes.*

The proof is omitted but is based on two principles. (i) For any S, an S-module N is one of the following if and only if $E_S N$ has the same corresponding property: square-free, locally square-free, i.e., ess-square-free dense, directly finite, directly infinite, locally directly finite, i.e., ess-directly finite dense, and locally directly infinite.

(ii) In general, for $\varphi : R \to S \in \mathcal{A}$, the induced functor $\varphi^{\#} : \mathcal{M}\mathrm{od}_S \to \mathcal{M}\mathrm{od}_R$, $\varphi^{\#}N = N_\varphi$, does not preserve injective modules. However, $N_\varphi \ll E_S(N)_\varphi \ll E(N_\varphi)$, and now (i) (for the ring R in place of S) can be used.

Remark 6.12. Let $N = E_S(N) \in \mathcal{M}\mathrm{od}_S$ and $N = N_I \oplus N_{II} \oplus N_{III}$ or $N = N_{CA} \oplus N_D \oplus N_B$ be as in Section 4 over the ring S. Then these are also the corresponding direct sum decompositions of N_φ over R; $N_I, \ldots, N_B$ need not be R-injective. However, $EN_\varphi = E(N_{I\varphi}) \oplus E(N_{II\varphi}) \oplus E(N_{III\varphi})$ and $EN_\varphi = E(N_{CA\varphi}) \oplus E(N_{D\varphi}) \oplus E(N_{B\varphi})$ are also the corresponding direct sum decompositions of the now R-injective module EN_φ.

Previously, type I, II and III decompositions of t.f. injective modules have been considered for a fixed ring. Aside from extending this to arbitrary modules, the next application puts their theory in a functorial context.

Application 1, 6.13. For $n = 3$, let $\Delta_1 = I$, $\Delta_2 = II$ and $\Delta_3 = III$ be type I, II and III modules. Then

(i) for all R, $II(R) \wedge III(R) = 0$, etc.; $I(R) \vee II(R) \vee III(R) = 1$.

(ii) For any R module M, there exist three complement submodules of M each being maximal in its class $M_I \in I(R)$, $M_{II} \in II(R)$, $M_{III} \in III(R)$ such that $M_I \oplus M_{II} \oplus M_{III} \ll M$.

(iii) $\widehat{M} = \widehat{M}_I \oplus \widehat{M}_{II} \oplus \widehat{M}_{III}$ is a type I, II, III direct sum decomposition which is unique up to superspectivity.

Now let $I \triangleleft R$, $\varphi : R \to S = R/I \in \mathcal{A}$, let $N = N_S$ be an S-module, and $N_I \oplus N_{II} \oplus N_{III} \leq_e N$ be (ii) applied to N and S (in place of M and R). Then

(iv) $N_{I\varphi} \oplus N_{II\varphi} \oplus N_{III\varphi} \ll N_\varphi$ is the type I, II, III decomposition of N_φ given by (ii).

(v) In particular, if N is S-injective and $N = N_I \oplus N_{II} \oplus N_{III}$, then

$$N_\varphi = N_{I\varphi} \oplus N_{II\varphi} \oplus N_{III\varphi}$$

and

$$EN_\varphi = EN_{I\varphi} \oplus EN_{II\varphi} \oplus EN_{III\varphi}$$

are type I, II, III decompositions.

(vi) For any R, $\Sigma(R) = \Sigma_I(R) \oplus \Sigma_{II}(R) \oplus \Sigma_{III}(R)$ is a lattice direct sum of convex sublattices. For any $\Delta \in \Sigma_{III}(R)$, Δ consists of type III modules, and similarly for I and II. If $1 = e_I + e_{II} + e_{III}$ is the corresponding decomposition of 1 into orthogonal elements, then $e_{III} \in \Sigma_{III}(R)$ and $\cup\Sigma_{III}(R) = e_{III}\Sigma(R)$ is the class of all type III modules, and similarly for I and II.

Application 2, 6.14. In 6.7 take $\Delta = A$ to be the molecular modules so that $c\Delta = B$ are the bottomless ones. Then, for any S-module N,

(i) there exists an $N_A \oplus N_B \leq_e N$, $N_A \in A(S)$, $N_B \in B(S)$. If $\varphi : R \to S \in \mathcal{A}$, then

(ii) $N_{A\varphi} \oplus N_{B\varphi} \ll N_\varphi$, $N_{A\varphi} \in A(R)$, $N_{B\varphi} \in B(R)$, that is, $EN_\varphi = (EN_{A\varphi}) \oplus (EN_{B\varphi})$ is the decomposition of EN_φ as a direct sum of a molecular module and a bottomless module, unique up to superspectivity.

(iii) $\Sigma(R) = \Sigma_A(R) \oplus \Sigma_B(R)$ is a lattice direct sum of convex sublattices such that $\Sigma_A(R) \cong \mathcal{P}(X)$, where $X \subseteq \Sigma(R)$ is the set of atoms of $\Sigma(R)$, while $\Sigma_{\mathcal{B}}(R)$ is a complete atomless Boolean lattice.

We next continue with the previous application by splitting A up into $A = (C \wedge A) \vee D$.

Application 3, 6.15. In the theorem, take $n = 3$ and $\Delta_1 = CA$, $\Delta_2 = B$, $\Delta_3 = D$. Then $CA \wedge B = 0$, $CA \wedge D = 0$, $B \wedge D = 0$ and $CA \vee B \vee D = 1$. Let M be any R-module. Then there exist three complement submodules of M maximal in their respective classes such that

(i) $M_{CA} \oplus M_B \oplus M_D \ll M$, $M_{CA} \in CA(R)$, $M_B \in B(R)$, $M_D \in D(R)$; M_D and M_{CA} each contain essential direct sums of atomic modules, M_D uniform ones and M_{CA} continuous atomic ones.

(ii) $C = CA \vee B$, $C \wedge D = 0$, $C \vee D = 1$. There exists a continuous submodule of M, $M_C \in C(R) = CA(R) \vee B(R)$ such that $M_{CA} \oplus M_B \ll M_C$ and $M_C \oplus M_D \ll M$ for M_D as in (i).

(iii) $\Sigma(R) = \Sigma_{CA}(R) \oplus \Sigma_D(R) \oplus \Sigma_B(R)$ is a lattice direct sum of convex sublattices; for $\Sigma_A(R)$ as in 6.9 (iii), $\Sigma_A(R) = \Sigma_{CA}(R) \oplus \Sigma_D(R) \cong \mathcal{P}(Y) \oplus \mathcal{P}(Z)$ where the set of atoms $X = Y \cup Z \subset \Sigma(R)$, $Y \cap Z = \varnothing$, is a disjoint union of continuous atoms Y and discrete atoms Z.

Application 4, 6.16. Let t be torsion modules, $ct = f$ the t.f. ones and $\varphi : R \to S = R/I \in \mathcal{A}^*$ with $I \triangleleft R$ a right complement. For any $N \in \mathcal{M}\mathrm{od}_S$,

(i) $Z_2^S N \oplus P \leq_e N$, $Z^S P = 0$. Then

(ii) $(Z_2^S N)_\varphi \oplus P_\varphi \ll N_\varphi$, $(Z_2^S N)_\varphi = Z_2 N_\varphi$, $ZP_\varphi = 0$.

(iii) $\Sigma(R) = \Sigma_t(R) \oplus \Sigma_f(R)$ is a lattice direct sum of convex sublattices. For any $g : T \to R \in \mathcal{A}$, $g^*\Sigma_t(T) \subseteq \Sigma_t(R)$; hence, $\Sigma_t \leq \Sigma : \mathcal{A} \to \mathcal{B}$ is a subfunctor.

(iv) $\Sigma = \Sigma_t \oplus \Sigma_f : \mathcal{A}^* \to \mathcal{B}$ is a direct sum of subfunctors; i.e., $\varphi^*\Sigma_f(S) \subseteq \Sigma_f(R)$ (as well as $\varphi^*\Sigma_t(S) \subseteq \Sigma_t(R)$ which holds as in (iii)).

Definition 6.17. The infinite Goldie dimension of a module W_R is the cardinal number $\mathrm{Gd}\, W = \sup\{|I| \mid \text{ there exists } 0 \neq W_i \leq W, \sum_{i\in I} W_i = \oplus_{i\in I} W_i \leq W\}$ [**18**, p. 297]. If $\mathrm{Gd}\, W$ is *not* inaccessible, then above $\mathrm{Gd}\, W = |I|$ is attained for some I. There do exist rings R and modules W where $\mathrm{Gd}\, W$ is not attained (with $\mathrm{Gd}\, W$ necessarily being inaccessible). For each R, the function "Gd" is additive in the sense that $\mathrm{Gd}\,(\oplus_{j\in J} M_j) = \sum_{j\in J} \mathrm{Gd}\, M_j$ [**18**, Theorem 3].

Definition 6.18. For any module M over any ring R, and any cardinal $\aleph$ with $\aleph = 1$ or $\aleph_0 \leq \aleph$, define M to be of *local Goldie dimension* $\aleph$ if for any $0 \neq V \leq M$, there exists $0 \neq W \leq V$ with $\operatorname{Gd} W = \aleph$.

For $\aleph = 1$, or for any infinite cardinal $\aleph \geq \aleph_0$, for any R define $\Delta_\aleph(R) = \{M \in \mathcal{M}\mathrm{od}_R \mid M \text{ is of local Goldie dimension } \aleph\}$ to be all such right R-modules having locally Goldie dimension $\aleph$. Note that $\Delta_1(R) = D(R)$ are our previous discrete modules.

Main application 6.19. *For $\aleph = 1, \aleph_0, \aleph_1, \dots$ and for any ring R, the following hold*:

(i) *$\Delta_\aleph(R)$ is a saturated class*; *there is a cardinal $\tau(R)$ such that $\Delta_\aleph(R) = 0$ and $\Sigma_{\Delta_\aleph}(R) = 0$ for all $\aleph \geq \tau(R)$.*

(ii) *$\Delta_\kappa(R) \cap \Delta_\aleph(R) = 0$ for $\kappa \neq \aleph$; $\vee_\aleph \Delta_\aleph(R) = \vee_{\aleph < \tau(R)} \Delta_\aleph(R) = 1$ for all $\aleph$.*

(iii) *$\Delta_\aleph$ are universal saturated classes.*

(iv) *For any R-module M, there exist* (*complement*) *submodules which are maximal with respect to $M_\aleph \in \Delta_\aleph(R)$ and $M_\aleph \leq M$ and with $\sum_\aleph M_\aleph = \oplus_\aleph M_\aleph \ll M$. Hence $EM = E(\oplus_\aleph M_\aleph)$. The $M_\aleph \leq M$ are unique up to superspectivity. If M is t.f. all $M_\aleph \leq M$ are unique.*

(v) *For any ring S and surjective ring homomorphism $\varphi : R \to S \in \mathcal{A}$ and any S-module N, if $\oplus_\aleph N_\aleph \leq_e N$ is as guaranteed by* (iv)*, then*

$$\bigoplus_\aleph N_{\aleph_\varphi} \ll N_\varphi \quad \text{and} \quad N_{\aleph_\varphi} \in \Delta_\aleph(R).$$

(vi) *$\Sigma = \prod_\aleph \Sigma_{\Delta_\aleph} : \mathcal{A} \to \mathcal{B}$ is a direct product of subfunctors $\Sigma_{\Delta_\aleph} \leq \Sigma$ of Σ; in particular, $\Sigma(R) \cong \prod_\aleph \Sigma_{\Delta_\aleph}(R)$ is a lattice direct product of convex and complete sublattices $\Sigma_{\Delta_\aleph}(R) \subset \Sigma(R)$ with componentwise Boolean operations. Moreover, $\varphi^* \Sigma_{\Delta_\aleph}(S) \subseteq \Sigma_{\Delta_\aleph}(R)$ for all $\aleph$.*

Proof. (i) By its very definition, $\Delta_\aleph(R)$ is closed under isomorphic copies and submodules. Since $\operatorname{Gd} W = \operatorname{Gd} V$ for $W \ll V$, it is closed under injective hulls. The projection argument 1.3 reduces this question for a direct sum to the corresponding property for each of the summands, thus showing that this class is also closed under direct sums. Hence $\Delta_\aleph(R)$ for all $\aleph = 1, \aleph_0, \dots$ is a saturated class.

If $\Delta_\aleph(R) \neq 0$, then there exists a cyclic module R/L with $L < R$ and $\mathrm{Gd}\, R/L = \aleph$. Let $\tau(R)$ be the smallest cardinal larger than $\mathrm{Gd}\, R/L$ for all $L < R$.

(ii) and (iv). Let $\kappa < \aleph$ where $\kappa = 2, \aleph_0, \ldots$, and suppose that $0 \neq \Delta_\kappa(R) \cap \Delta_\aleph(R)$. Since $0 \neq M \in \Delta_\kappa(R)$, there exists $0 \neq V \leq M$, $\mathrm{Gd}\, V = \kappa$. Since $0 \neq M \in \Delta_\aleph(R)$, also $0 \neq V \in \Delta_\aleph(R)$, and hence there exists $0 \neq W \leq V$ with $\aleph = \mathrm{Gd}\, W \leq \mathrm{Gd}\, V = \kappa$, a contradiction. Hence the $\Delta_\aleph$ are pairwise disjoint.

For any R-module M, there exist $M_\aleph \leq M$ with $M_\aleph \in \Delta_\aleph(R)$ being maximal. Then $\Sigma_\aleph M_\aleph = \oplus_\aleph M_\aleph \leq M$. If the latter is not essential, take any $0 \neq C \leq M$ with $(\oplus_\aleph(M_\aleph)) \oplus C \leq M$. If $0 \neq U \leq C$ is uniform, then $M_1 \oplus U \in \Delta_1(R)$ contradicts the maximality of M_1. Let $\kappa = \text{minimum } \{\mathrm{Gd}\, V \mid 0 \neq V \leq C\}$, and take any $0 \neq V \leq C$ for which $\mathrm{Gd}\, V = \kappa$. Then $V \in \Delta_\kappa(R)$, and now $M_\kappa \oplus V \in \Delta_\kappa(R)$ contradicts the maximality of M_κ. Hence $C = 0$, and $\oplus_\aleph M_\aleph = \oplus_{\aleph \leq \mathrm{Gd}\, M} M_\aleph \ll M$. Thus $M \in \vee_{\aleph \leq \tau(R)} \Delta_\aleph(R) = 1$.

(iii) For any $\varphi : R \to S \in \mathcal{A}$, let $N \in \Delta_\aleph(S)$. Since the S-submodules of N and the R-submodules of the induced module N_φ coincide, $\mathrm{Gd}\, N_S = \mathrm{Gd}\, N_\varphi$ and $N_\varphi \in \Delta_\aleph(R)$. Thus $\varphi^* \Delta_\aleph(S) \subseteq \Delta_\aleph(R)$. For any $\aleph$, $c\Delta_\aleph(R) = \{M \mid \text{ for all } 0 \neq W \leq M, \mathrm{Gd}\, W \neq \aleph\}$. In particular, $c\Delta_1(R) = C(R)$ are the continuous modules. Hence, also $\varphi^* c\Delta_\aleph(S) \subseteq c\Delta(R)$. Therefore, all the $\Delta_\aleph$ are universal saturated classes.

(v) and (vi). These follow from Theorem 6.7. □

7. Examples. Some examples of rings R are given such that $ZR \ll Z_2 R = R$ and consequently every R-module is torsion. In order later to construct from these rings torsion bottomless modules in these examples in addition $R \backslash ZR$ will be units and hence they are special kinds of local rings [**1**, Proposition 15.15].

Example 7.1. Let D be a commutative p.i.d. and $p \in D$ a prime. Let $R = D/(p^n)$ for some $n = 1, 2, \ldots$. Then $Z(R) = (p)/(p^n)$, $(R/Z(R))^\perp = (p)/(p^n) \ll R$, and $Z_2(R) = R$.

The next example is used for other purposes in [**37**].

Example 7.2. For a field F and indeterminates $x_1, x_2, \ldots$, let R be the polynomial ring $F[x_1, x_2, \ldots]$ subject to the relations $x_{i+1}^2 = x_i$, $x_1^2 = 0$. For some smallest i and odd k or $k = 0$, every element $r \in R$ is uniquely of the form $r = x_i^k u$ where $u \in R$ is a polynomial in x_i and nonzero constant term; u is a unit. Since $x_i^{2^i} = 0$, $r^\perp = x_i^{2^i - k} R$. Thus $r^\perp = 0$ if and only if $r = u$, and otherwise when $k \neq 0$, $r^\perp \ll R$. Hence the unique maximal ideal $Z(R) = \{x_i^k u \mid u \in F[x_i], u \text{ is a unit}; i, k = 1, 2, \ldots\} \ll R$ of R is nil. Since $r = x_i^k u = (x_{i+1}^k)(x_{i+1}^k u) \in (ZR)^2$, $(ZR)^2 = ZR$. (If $\operatorname{char} F = 0$, then actually $r = (x_{i+1}^k \sqrt{u})^2$ with $\sqrt{u} \in R \backslash ZR$.) Any two elements $r, s \in R$ can be written with the same i both in the form $r = x_i^k u$, $s = x_i^m v$ for units $u, v \in R \backslash ZR$. Not only is R uniform, but all the ideals of R form a chain because for cyclics, either $m \leq k$, $r = s x_i^{k-m} v^{-1} u$, $rR \subseteq sR$ or $sR \subset rR$.

Example 7.3. For a prime p and two noncommuting indeterminates x and y, let R be the free algebra $Z_{p^2}\{x, y\}$ subject to the relations $x^2 = xy = yx = y^2 = 0$. Every element $r \in R$ is uniquely of the form $r = k_0 + xk_1 + yk_2$, $k_i \in Z_{p^2}$. In this and similar examples we will use the fact that $Z_{p^2}\{x, y\}$, Z_{p^2} and R are $\mathbf{Z}$-bimodules in a canonical way. Then R is commutative, and it can be shown that its unique maximal ideal is $ZR = pZ_{p^2} + xR + yR \ll R$, $(ZR)^3 = 0$. Here $xR \cap yR = 0$ and R is not uniform.

The next construction gives a large and diverse class of bottomless torsion rings and modules and also type I torsion modules. We begin below by introducing some notation which later will also be useful in other examples.

Construction 7.4. Let T_i, $i \in I$, be an infinite family of rings with unique maximal ideal $P_i \triangleleft T_i$ which is its first singular submodule $Z^{T_i}(T_i) = P_i$, and $P_i \leq_e T_i$ is essential, where as before "$\leq_e$" denotes right essential submodules over rings other than R. Examples of such rings T_i are 7.1, 7.2 and 7.3. Set $T = \prod_{i \in I} T_i$, $S = \oplus_{i \in I} T_i$ and $R = T/S$. Note that R is canonically a T-bimodule. It is important to note that the elements of $T_i \backslash P_i$ are invertible [**1**, Proposition 15.15]. For $t = (t_i)_{i \in I} \in T$, the support of t is $\operatorname{supp} t = \{i \in I \mid t_i \neq 0\}$. For any subsets of T_i whatsoever such as $P_i \subseteq T_i$, view $\prod_{j \in Y} P_j \subset T$ canonically as consisting of all $t = (t_i)_{i \in I}$ with $\operatorname{supp} t \subseteq Y$ and values

$t_j \in P_j$ for all $j \in Y$.

Evidently R is continuous, and it is shown next that $ZR \ll R$. For any $i \in I$ and any $c_i \in T_i$, set $c_i^{-1}0 = \{t \in T_i \mid c_i t = 0\}$. For an arbitrary element $\overline{c} = c + S \in R$, $c = (c_i)_{i\in I} \in T$, let $Y = \{i \in I \mid c_i \in P_i \triangleleft T_i\}$. Note that $I \backslash \operatorname{supp} c \subseteq Y$, and if $c_i = 0$, then $c_i^{-1}0 = T_i$. Since, by hypothesis, $c_i \in T_i \backslash P_i$ are units in T_i, $c_i^{-1}0 = 0$. Hence if $\overline{c}\overline{d} = 0$, $\overline{d} = d + S \in R$, without loss of generality, $\operatorname{supp} d \subseteq Y$. But then $\{i \in Y \mid c_i d_i \neq 0\}$ is finite and $d \in \prod_{i\in Y} c_i^{-1}0 + S$. Consequently,

$$\overline{c}^{\perp} = \left(\prod_{i\in Y} c_i^{-1}0 + S\right)/S.$$

Next it is shown that, for the above, $\overline{c}, \overline{c}^{\perp} \ll R$ exactly when $Y = I$ and all $c_i \in P_i$. First of all, if $|I\backslash Y| \geq \aleph_0$ is infinite, let $k = (k_i)_{i\in I} \in T$ with infinite support $\operatorname{supp} k \subseteq I \backslash Y$. Since $\operatorname{supp} c \subseteq Y$, $cT \cap kT = 0$, $\overline{c}^{\perp} \neq \overline{c}^{\perp} \oplus \overline{k}R \leq R$, and $\overline{c} \notin ZR$. So, without loss of generality, let $Y = I$, and consequently all $c_i \in P_i$.

Let $\overline{0} \neq \overline{k} = \overline{(k_i)}_{i\in I} \in T$ be arbitrary. We will show that $\overline{c}^{\perp} \cap \overline{k}R \neq 0$. Since $Y = I$ and since $c_i \in Z^{T_i}(T_i) = P_i \leq_e T_i$, $c^{-1}0 \leq_e T_i$. Consequently, for any $i \in \operatorname{supp} k$, $k_i \neq 0$, and for some $t_i \in T_i$, $0 \neq k_i t_i \in c_i^{-1}0 \cap k_i T_i \neq 0$. Define $t \in T$ with $\operatorname{supp} t = \operatorname{supp} k$, where the components t_i of t are as above for $i \in \operatorname{supp} k$, and zero otherwise. Thus $c_i k_i t_i = 0$ for all i, while $k_i t_i \neq 0$ for all $i \in \operatorname{supp} k = \operatorname{supp} t$. Thus $ckt = 0$, $\overline{c}\overline{kt} = 0$, where $\overline{kt} \neq 0$. Hence, for any $0 \neq \overline{k} \in R$, $\overline{c}^{\perp} \cap \overline{k}R \neq 0$ and $\overline{c}^{\perp} \ll R$. Therefore,

$$ZR = \left(\prod_{i\in I} P_i + S\right)/S.$$

In order to see that $ZR \ll R$, it suffices to show that for any $0 \neq \overline{k} = \overline{(k_i)}_{i\in I} \in R$, $\overline{k}R \cap ZR \neq 0$. But by hypothesis $P_i \leq_e T_i$ for all i, and hence if $k_i \neq 0$, then $k_i T_i \cap P_i \neq 0$. For all $i \in \operatorname{supp} k$, select $t_i \in T_i$ with $0 \neq k_i t_i \in P_i$. Again, define $t \in T$ with $\operatorname{supp} t = \operatorname{supp} k$ by $t = (t_i)_{i\in I}$ for the above chosen t_i for $i \in \operatorname{supp} k$. For $i \notin \operatorname{supp} k$, set $t_i = 0$. Then $\operatorname{supp} kt = \operatorname{supp} k$, and $kt \in \prod_{i\in I} P_i$. Hence $0 \neq \overline{kt} \in ZR \ll R$.

Clearly $R_R = Z_2R$ is continuous, and the proof that it is bottomless is much the same as the proof in [**14**, p. 116]. It hinges on the following fact. For any R-map $f : \overline{a}R \to \overline{b}R$ with $\overline{b} = f\overline{a}$ and $\overline{a} = a + S$, $a \in T$, there exists a coset representative $b = (b_i) \in T$ whose support in I satisfies $\operatorname{supp} b = \{i \in I \mid b_i \neq 0\} \subseteq \operatorname{supp} a$. For a proof, see [**14**, p. 115].

Clearly, each of the rings in 7.1, 7.2 or 7.3 is torsion type I. The proof given in [**14**, p. 116] shows that, under the additional hypothesis that all T_i are type I rings, R_R is torsion type I and, as already noted, continuous bottomless.

The next example shows that torsion type III modules exist.

Example 7.5. For p a prime and $Z_{p^2} = \mathbf{Z}/p^2\mathbf{Z}$, let $R = Z_{p^2}\{x, y\}$ be the free algebra as in 7.3. It will be shown that $ZR = pR \ll R = Z_2R$, and that $R = Z_2R$ is a type III torsion module.

Since $(pR)^{\perp} = pR \ll R$ and $pR \subseteq ZR$, we conclude that $Z_2R = R$. A *term* τ of R is an element of the form

$$\tau = x^{\varepsilon(1)}y^{\eta(1)}x^{\varepsilon(2)}y^{\eta(2)}\cdots x^{\varepsilon(n)}y^{\eta(n)}$$

where $x^0 = y^0 = 1$, $1 \leq \varepsilon(2), \varepsilon(3), \dots, \varepsilon(n)$; $1 \leq \eta(1), \eta(2), \dots, \eta(n-1)$; but $\varepsilon(1), \eta(n) = 0, 1, 2, \dots$. A *monomial* of R is a constant times a term, i.e., $k\tau = \tau k$, $k \in Z_{p^2}$. Note that as a Z_{p^2}-module, the terms form a free basis, i.e., $R = \oplus\{Z_{p^2}\tau \mid \tau \text{ is a term}\}$ over Z_{p^2}. The degree of τ is $\deg \tau = \varepsilon(1) + \eta(1) + \cdots + \varepsilon(n) + \eta(n)$, and similarly we have $0 \leq \deg r$ for any $0 \neq r \in R$. Every element of r is uniquely of the form $r = r_1 + pr_2$, $r_i \in R$, where every monomial of r_1 and r_2 is not divisible by p, hence $p \nmid r_1$, $p \nmid r_2$. Moreover, the nonzero monomials appearing in r_1, r_2 are jointly all Z_{p^2} independent.

For any $r \in R$ as above, suppose that $rr' = 0$ or $(r_1 + pr_2)(r_1' + pr_2') = r_1r_1' + p(r_1r_2' + r_2r_1') = 0$. Then $r_1r_1' = 0$ and $r_1r_2' + r_2r_1' = 0$. Hence if $r_1 \neq 0$, then $r_1' = 0$. But then $r_1r_2' = 0$, and thus $r_2' = 0$. Therefore, $p \nmid r$ if and only if $r_1 \neq 0$ if and only if $r^{\perp} = 0$. Alternatively, for $r \neq 0$, $p \mid r$ if and only if $r_1 = 0$ if and only if $r^{\perp} = pR$. Thus ZR is exactly $ZR = pR$.

Take any $0 \neq v = v_1 + pv_2$ written as above with $p \nmid v$ and $v_1 \neq 0$. Take any countably infinite direct sum of cyclics $\oplus_{\alpha} x_{\alpha}R \ll R$. Since

$R \to vR$, $r \to vr$ is an isomorphism of R modules, it maps essential submodules to essential submodules $\oplus_\alpha v x_\alpha R \ll vR$.

Since left multiplication by pv is not an isomorphism, no such argument is available to conclude that $\oplus_a pvx_\alpha R \ll pvR$. However, in this case all we need is that there exists some infinite countable direct sum of cyclics $\oplus_\alpha px'_\alpha \ll pvR$ with $0 \neq x'_\alpha \in R \backslash pR$, i.e., with $p \nmid x'_\alpha$. Simply take any $\oplus_\alpha pvx_\alpha R \oplus D \ll pvR$. Take any finite or infinite $\oplus_\beta d_\beta R \ll D$, $0 \neq d_\beta \in D$. Since $pD \subseteq p^2 vR = 0$, all $pd_\beta = 0$, and each $d_\beta = pd_{\beta_2}$, $p \nmid d_{\beta_2}$. Thus $(\oplus_\alpha pvx_\alpha R) \oplus (\oplus_\beta pd_{\beta_2} R) \ll pvR$ as required.

Finally, consider any direct summand of $\widehat{R}_R$ of the form $\widehat{N} \leq \widehat{R}_R$, where $N = \widehat{N} \cap R$. Take any essential direct sum of cyclics $[\oplus_{i\in I} v_i R] \oplus [\oplus_{j\in J} pw_j R] \ll N$ where $p \nmid v_i$, $p \nmid w_j$. The previous argument shows that we can choose the v_i and w_j so that $|J| = \aleph_0$ is infinite, and either $I = \varnothing$, or also $|I| = \aleph_0$.

In either case we can partition the index sets into two equal-sized parts, $J = J_1 \cup J_2$, $J_1 \cap J_1 = \varnothing$, $|J| = |J_1| = |J_2| = \aleph_0$ and $I = I_1 \cup I_2$, $I_1 \cap I_2 = \varnothing$, with also $|I| = |I_1| = |I_2|$. Then

$$\widehat{N} = E\bigg(\bigoplus_{i\in I_1} v_i R\bigg) \oplus E\bigg(\bigoplus_{i\in I_2} v_i R\bigg) \oplus E\bigg(\bigoplus_{j\in J_1} pw_j R\bigg) \oplus E\bigg(\bigoplus_{j\in J_2} pw_j R\bigg),$$

$$\widehat{N} = E\bigg(\bigoplus_{i\in I} v_i R\bigg) \oplus E\bigg(\bigoplus_{j\subset J} pw_j R\bigg).$$

Since $v_i^\perp = 0$, $v_i R \cong R$ while $(pw_j)^\perp = pR$, $pw_j R \cong R/pR$ for all i and j, it now follows that $\widehat{N} \cong \widehat{N} \oplus \widehat{N}$. Hence $\widehat{R}_R = Z_2 \widehat{R}$ and $R_R = Z_2 R$ or $pR = ZR$ all are torsion type III modules.

The next example shows that for every cardinal $\aleph = \aleph_0, \ldots$ there exist rings R with $\Delta_\aleph(R) \neq 0$.

Example 7.6. For any field F and set X with $2 \leq |X| = \aleph$ the free algebra $R = F\{X\}$ on X has $\operatorname{Gd} R = \max\{\aleph_0, |X|\} = \aleph$. For any $0 \neq b \in R$, $bR \cong R$, and $\operatorname{Gd} bR = \aleph$. Hence, $R_R \in \Delta_\aleph(R)$. Actually more is true. For any $0 \neq L < R$, if $\oplus_{i\in I} b_i R \ll L$, then $|I| = \aleph$, and

$\operatorname{Gd} L = \operatorname{Gd}(\oplus_{i\in I} b_i R) = \sum_{i\in I} \operatorname{Gd} b_i R = \aleph \cdot \aleph = \aleph$ by [**18**, Theorem 3]. Note that, for $X = \{x\}$ a singleton, likewise $F\{\{x\}\} = F[x] \in \Delta_1(R)$. For $2 \leq |X|$, $F\{X\}$ is t.f. continuous, molecular, and type III [**13**, Example 4.4], i.e., $R = F\{X\} \in C_f \cap A_f = (CA)_f$ and $R \in III$.

A quotient of an example from [**18**, Example 9] is used in the next example.

Example 7.7. For any infinite cardinal $\aleph$ and any ring F with $|F| \leq \aleph$, set $F_i = F$ for all ordinals $i < \aleph$. Let $S = \prod_{i<\aleph}^{<\aleph} F_i$ be the $\aleph$-product of the $F_i{}'s$ consisting of all $t = (t_i)_{i<\aleph}$ whose support $\operatorname{supp} t = \{i < \aleph \mid t_i \neq 0\}$ has cardinality $|\operatorname{supp} t| < \aleph$. Let $P \subseteq \prod_{i<\aleph} F_i$ be the subring of all eventually constant vectors, and set $R = P/S$. Then $|P| = \aleph$. Let $0 \neq \overline{a}R \leq R$ be arbitrary, $\overline{a} = (a_i)_{i<\aleph} + S$, $(a_i)_{i<\aleph} \in P$. Partition $\operatorname{supp}(a_i)_{i<\aleph} = X = \cup_{i<\aleph} X_i$, $X_i \cap X_j = \varnothing$ for $i \neq j$, and all $|X_i| = \aleph$. Let $\chi_i \in P$ be the characteristic function of X_i. Then R is a P-module, and $\oplus_{i<\aleph}(a\chi_i)^- R = \oplus_{i<\aleph}\overline{a}\aleph_i R \ll \overline{a}R$ and $\aleph \leq \operatorname{Gd}\overline{a}R \leq |R| = \aleph$. Thus $R_R \in \Delta_\aleph(R)$. If $Z^F F = 0$, R is t.f. If $Z_2^F F = F$, R is torsion, e.g., for $F = \mathbf{Z}/p^2\mathbf{Z}$.

Example 7.8. Let $|X| \geq \aleph_0$ and $J \triangleleft \mathcal{P}(S)$ be the ideal $J = \{Y \subset X \mid |Y| < |X|\}$. Set $R = \mathcal{P}(X)/J$, and let $S = E(R)$ be the completion of R (see [**13**, Example 4.4]). Then $R, S_R \in B_f(R) \cap I_f(R)$ as well as $S_S \in B_f(S) \cap I_f(S)$.

If the GCH is assumed, then $\operatorname{Gd} R = \operatorname{Gd}(ER) = \operatorname{Gd} S_S = 2^{|X|}$ (see [**13**, p. 337]). For any $0 \neq yR \leq R$, $y = Y + J$, $Y \subset X$, $Y \notin J$, and $|Y| = |X|$. Again, use the GCH to obtain the existence of an almost disjoint family $\mathcal{F} = \{A, B, \dots \mid A, B \subset Y, |X| = |Y| = |A| = |B| = \dots, |A \cap B| < |X|\}$. Then $A \notin J$ for $A \in \mathcal{F}$, and $\oplus\{(A+J)R \mid A \in \mathcal{F}\} \ll yR$ shows that $2^{|X|} = |\mathcal{F}| \leq \operatorname{Gd} yR \leq |R| = 2^{|X|}$. Thus $\operatorname{Gd} yR = 2$, and $R, ER = S_R \in \Delta_\aleph(R)$ for $\aleph = 2^{|X|}$.

Note that this example is related to the previous one. In 7.7, take $F = Z_2$. Then $\prod_{i<\aleph} Z_2 \cong \mathcal{P}(X)$, $\prod_{i<\aleph}^{<\aleph} Z_2 \cong J$, and thus $\prod_{i<\aleph} Z_2 / \prod_{i<\aleph}^{<\aleph} Z_2 \cong \mathcal{P}(X)/J$ for $|X| = \aleph$.

Example 7.9. For a field F, let $F_i = F$, $i \in I$ with $\aleph \leq |I|$. For

$S = \prod_{i\in I}^{<\aleph} \subset T = \prod_{i\in I} F_i$ set $R = T/S$. For any $0 \neq V \leq R$, there exists $0 \neq \overline{a}R \leq V$ with $\overline{a} = (a_i)_{i\in I} + S$, $|\mathrm{supp}\,(a_i)_{i\in I}| = \aleph$. Let $P \triangleleft T$ be the ideal of all such $\overline{a}$. Then $P/S \ll R$ and $P/S \in \Delta_{\aleph}(R)$.

REFERENCES

1. F. Anderson and K. Fuller, *Rings and categories of modules*, Graduate Texts in Math. **13**, Springer, New York-Berlin, 1974.

2. A.H. Al-Huzali, S.K.Jain and S.R. López-Permouth, *Rings whose cyclics have finite Goldie dimension*, J. Algebra **153** (1992), 37–40.

3. S.K. Berberian, *Baer *-rings*, Grundlehren Math. Wiss. **195**, Springer, New York-Berlin, 1972.

4. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Pub. **25**, Providence, RI, 1967.

5. R.T. Bumby, *Modules which are isomorphic to submodules of each other*, Arch. Math. (Basel) **16** (1965), 184–185.

6. J. Dauns, *Chains of modules with completely reducible quotients*, Pacific J. Math. **17** (1966), 235–241.

7. ———, *Uniform modules and complements*, Houston J. Math. **6** (1980), 31–40.

8. ———, *Sums of uniform modules*, in *Advances in non-commutative ring theory*, Lecture Notes in Math. **951** (1982), 68–87.

9. ———, *Uniform dimensions and subdirect products*, Pacific J. Math. **136** (1987), 1–19.

10. ———, *Subdirect products of injectives*, Comm. Alg. **17** (1989), 179–196.

11. ———, *Torsion free modules*, Ann. Mat. Pura Appl. (4) **154** (1989), 49–81.

12. ———, *Semiprime modules and rings*, in *Non-commutative ring theory*, Lecture Notes in Math. **1448** (1990), 41–62.

13. ———, *Classes of modules*, Forum Math. **3** (1991), 327–338.

14. ———, *Torsion free types*, Fund. Math. **139** (1991), 99–117.

15. ———, *Direct sums and subdirect products*, in *Methods in module theory* (G. Abrams, J. Haefner, K.M. Rangaswamy, eds.), Marcel Dekker, New York, 1992, 39–65.

16. ———, *Module classifying functors*, Czechoslovak Math. J. **42** (1992), 741–756.

17. ———, *Functors and Σ-products*, in *Ring theory* (S.K. Jain and S. Tariq Rizvi, eds.), World Sci. Pub., Singapore, 1993, 149–171.

18. J. Dauns and L. Fuchs, *Infinite Goldie dimensions*, J. Algebra **115** (1988), 297–302.

19. ———, *κ-products of slender modules*, Acta Sci. Math. **56** (1992), 205–213.

20. K.R. Goodearl, *Prime ideals in regular self-injective rings*, Canad. J. Math. **25** (1973), 829–839.

21. ———, *Ring theory*, Marcel Dekker, New York, 1976.

22. ———, *Von Neumann regular rings*, Pitman, London, 1979.

23. K.R. Goodearl and A.K. Boyle, *Dimension theory for nonsingular injective modules*, Amer. Math. Soc. Memoir **177**, Providence, RI, 1976.

24. G. Grätzer, *General lattice theory*, Academic Press, New York, 1978.

25. P. Halmos, *Lectures on Boolean algebras*, Studies No. 1, D. Van Nostrand Co., Princeton, New Jersey, 1963.

26. S.K. Jain and S.R. López-Permouth, *A survey on the theory of weakly-injective modules*, in *Computational algebra*, Proceedings of the Mid-Atlantic Algebra Conference, George Mason University, Marcel Dekker, New York, 1994, 205–232.

27. S.K. Jain, S.R. López-Permouth and Surjeet Singh, *On a class of QI-rings*, Glasgow Math. J. **34** (1992), 75–81.

28. I. Kaplansky, *Rings of operators*, W.A. Benjamin, New York, 1968.

29. S.R. López-Permouth, *Rings characterized by their weakly-injective modules*, Glasgow Math. J. **34** (1992), 349–353.

30. S.R. López-Permouth and S.T. Rizvi, *On certain classes of QI rings*, in *Methods in module theory* (G. Abrams, J. Haufner, K.M. Rangaswamy, eds.), Marcel Dekker, 1992.

31. F.J. Murray and J. von Neumann, *On rings of operators*, Ann. of Math. **37** (1936), 116–229.

32. S.H. Mohamed and B.J. Müller, *Continuous and discrete modules*, London Math. Soc. Lecture Note Ser. **147** (1990).

33. S. Page and Y. Zhou, *When direct sums of singular injectives are injective*, in *Ring theory* (S.K. Jain and S. Tariq Rizvi, eds.), World Scientific Publishing Co., New Jersey, 1993, 276–286.

34. ———, *On direct sums of injective modules and chain conditions*, Canad. J. Math. **46** (1994), 634–647.

35. ———, *Direct sums of quasi-injective modules, injective covers, and natural classes*, Comm. Algebra **22** (1994), 2911–2923.

36. J. Rios Montes and G. Tapia Sánchez, *A general theory of types for nonsingular injective modules*, Comm. Algebra **20** (1992), 2337–2364.

37. H. Storrer, *On Goldman's primary decomposition*, in *Tulane ring year lecture notes*, Lecture Notes in Math. **246** (1972), 617–661.

38. R. Wisbauer, *Foundations of module and ring theory*, Gordon and Breech Science Publishers, Philadelphia, 1991.

39. Y. Zhou, *Direct sums of M-injective modules and module classes*, Comm. Algebra **23** (1995), 927–940.

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