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SINGULAR POINTS OF ANALYTIC FUNCTIONS EXPANDED IN SERIES OF FABER POLYNOMIALS

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ABSTRACT. Let $a_n \geq 0$, $n = 0, 1, \dots$, be such that $\limsup_{n\to\infty}(a_n)^{1/n} = 1$. Then a theorem of Pringsheim states that the point $z = 1$ is a singular point for $f(z) = \sum_{n=0}^{\infty} a_n z^n$. It is the purpose of this note to extend Pringsheim's theorem by replacing the unit disk $|z| \leq 1$ by a compact simply connected set E (containing more than one point) and whose boundary $\mathrm{Br}\,(E)$ is an analytic Jordan curve, and by replacing the monomials z^n by the Faber polynomials for E.

1. Introduction. Let $a_n \geq 0$, $n = 0, 1, \dots$, be such that

$$\limsup_{n\to\infty}(a_n)^{1/n} = 1. \tag{1.1}$$

Then a theorem of Pringsheim [**8**] states that the point $z = 1$ is a singular point for

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

It is the purpose of this note to extend Pringsheim's theorem by replacing the unit disk $|z| \leq 1$ by a compact simple connected set E (containing more than one point) and whose boundary $\mathrm{Br}\,(E)$ is an analytic Jordan curve, and by replacing the monomials z^n by the Faber polynomials for E.

For the sake of notational simplicity we will assume that the capacity of E, $\mathrm{Cap}\,(E)$, is equal to 1. It will appear clearly, however, that our results hold for any positive value of $\mathrm{Cap}\,(E)$.

The function $w = \phi(z)$ which maps conformally the exterior of E, $\mathrm{Ext}\,(E)$ onto $|w| > 1$ and such that $\phi(\infty) = \infty$, has a Laurent expansion at infinity of the form

$$\phi(z) = z + a_0 + \frac{\alpha_{-1}}{z} + \cdots .$$

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(Recall that $\operatorname{Cap}(E) = 1$). The Faber polynomials for E, $\phi_n(z)$, consist of the terms with nonnegative powers of z in the Laurent expansion at infinity of $\phi(z)^n$.

The behavior of the mapping function $\phi(z)$ near the boundary $\operatorname{Br}(E)$ of E will play an important role in the sequel. It is known that in the case when $\operatorname{Br}(E)$ is an analytic Jordan curve, the inverse function $z = \psi(w)$ of $w = \phi(z)$ extends from $|w| > 1$ to $|w| > r_0$, for some $r_0 < 1$, in a conformal manner. We let $\psi(w)$ continue to denote this extension, and $\phi(z)$ continue to denote its inverse. For $r > r_0$, Γ_r is the level curve

$$\Gamma_r = \{z : |\phi(z)| = r\}.$$

With this notation $\Gamma_1 = \operatorname{Br}(E)$.

We are now in a position to state our main result.

Theorem 1. *Let E be compact and simply connected with $\operatorname{Br}(E)$ an analytic Jordan curve and $\operatorname{Cap}(E) = 1$. Let $a_n \geq 0$ satisfy (1.1), and let $z_0 \in \operatorname{Br}(E)$ be the unique point such that $\phi(z_0) = 1$. Then z_0 is a singular point for the function*

$$f(z) = \sum_{n=0}^{\infty} a_n \phi_n(z). \tag{1.2}$$

Example. Let E_2 be the ellipse with foci -1 and 1 and sum of semi-axes 2, together with its interior. The function $w = \phi(z) = (1/2)(z + \sqrt{z^2 - 1})$ maps $\operatorname{Ext}(E_2)$ conformally into $|w| > 1$. It follows that $\operatorname{Cap}(E_2) = 1$. The Faber polynomials for E_2 are $\phi_n(z) = (1/2^{n-1})T_n(z)$, $n \geq 1$, $\phi_0(z) = 1$, where the $T_n(z) = \cos(n \arccos(z))$ are the Chebyshev polynomials. Theorem 1 gives: For the function

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^n} T_n(z)$$

the point $z = 5/4$ is a singular point.

Remark. Clearly the above conclusion could have been found more directly from $T_n((1/2)(w + 1/w)) = (1/2)(w^n + 1/w^n)$, $w \neq 0$. The domain E_2 is used because it is one of the few sets for which the Faber polynomials are known explicitly.

II. Proof of Theorem 1. It is not evident from the outset that the series (1.2) converges anywhere. Indeed, it is known [**3**] that when $\mathrm{Br}\,(E)$ is a curve of bounded rotation, which is clearly the case here, $\|\phi_n(z)\| \leq M$, where $\|\cdot\|$ denotes the supremum norm on E. (The boundedness of the $\phi_n(z)$, in our setting, is also a consequence of Lemma 2.1 below). This and condition (1.1) do not guarantee the convergence of (1.2). However, we will show (Lemma 2.1) that in fact $\lim_{n\to\infty} \|\phi_n(z)\|_r^{1/n} = r$, $r > r_0$, where $\|\cdot\|_r$ denotes the supremum norm on Γ_r. Recalling that $r_0 < 1$, relations (1.1) and (2.1), in conjunction with the maximum principle, show that $f(z) = \sum_{n=0}^{\infty} a_n\phi_n(z)$ converges uniformly on the compact subsets of $\mathrm{Int}\,(E)$ so that $f(z)$ is analytic there.

Relation (2.1) also shows that $\sum_{n=0}^{\infty} a_n\phi_n(z)$, with a_n satisfying (1.1), cannot converge for $z \in \mathrm{Ext}\,(E)$ because such a $z \in \Gamma_r$ for some $r > 1$.

These are, of course, necessary conditions for points in $\mathrm{Br}\,(E)$ to be singular points for $f(z)$.

We first need preparatory results.

Lemma 2.1. *Let E, $\mathrm{Br}\,(E)$ be as in Theorem* 1. *Let $r_0 < 1$ be as described in Section* 1. *Then, with $\|\cdot\|_r$ as above,*

$$\lim_{n\to\infty} \|\phi_n(z)\|_r^{1/n} = r, \quad r > r_0 \tag{2.1}$$

and

$$\phi_n(\psi(w)) = w^n + h_n(w) \tag{2.2}$$

where $h_n(w)$ has the following property:

Given $\varepsilon > 0$ and K a compact set contained in $|w| > r_0$, there exists a constant M such that, for $n = 0, 1, \dots,$

$$\|h_n(w)\|_K \leq M(r_0 + \varepsilon)^n \tag{2.3}$$

where $\|\cdot\|_K$ denotes the supremum norm on K.

Equations (2.1) and (2.2) are well known for $r_0 = 1$ (in which case $\mathrm{Br}\,(E)$ need not satisfy smoothness conditions).

Proof. For the sake of completeness we first adapt to our setting the standard formulae relating the Faber polynomials $\phi_n(z)$ with the mapping function $\phi(z)$. Let $r > r_0$, $z \in \mathrm{Int}\,(\Gamma_r)$. Then, because $\phi(\zeta)^n - \phi_n(\zeta)$ has a zero at ∞ of order at least one,

$$\zeta \longmapsto \frac{\phi(\zeta)^n - \phi_n(\zeta)}{\zeta - z}$$

has a zero at ∞ of order at least two. Hence,

$$\frac{1}{2\pi i}\int_{\Gamma_r} \frac{\phi(\zeta)^n - \phi_n(\zeta)}{\zeta - z}\, d\zeta = 0$$

so that

$$\phi_n(z) = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{\phi(\zeta)^n}{\zeta - z}\, d\zeta. \tag{2.4}$$

(See also [**4**]). Let now $r_0 < r' < r$, $z \in \mathrm{Int}\,(\Gamma_r)$, $z \in \mathrm{Ext}\,(\Gamma_{r'})$. Then

$$\phi(z)^n = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{\phi(\zeta)^n}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\Gamma_{r'}} \frac{\phi(\zeta)^n}{\zeta - z}\, d\zeta$$

so that, in view of (2.4),

$$\phi_n(z) = \phi(z)^n + \frac{1}{2\pi i}\int_{\Gamma_{r'}} \frac{\phi(\zeta)^n}{\zeta - z}\, d\zeta. \tag{2.5}$$

(Again this is known for $1 < r' < r$. See [**1, 4**]). Relation (2.5) shows that

$$\|\phi_n(z)\|_r = r^n + O(r'^n)$$

from which (2.1) follows because $r' < r$. Now let $g_n(z)$ be the second function on the right of (2.5). Then

$$|g_n(z)| \le M_z r'^n$$

(where M_z depends on $\mathrm{dist}\,(z, \Gamma_{r'})$).

Let now C be a compact set in $\psi(|w| > r_0)$, let $\varepsilon > 0$ be given, and let $r' = r_0 + \delta$ where $\delta = \min(\varepsilon, \operatorname{dist}(\phi(C), |w| = r_0)/2)$. Note that $\delta > 0$ and that the level curve $\Gamma_{r'}$ does not intersect the compact set C. The above argument shows that

$$\|g_n(z)\|_C \leq M(r_0 + \delta)^n \leq M(r_0 + \varepsilon)^n. \tag{2.6}$$

Let now $h_n(w) = g_n(\psi(w))$. Then $\phi_n(\psi(w)) = w^n + h_n(w)$. Let K be a compact set in $|w| > r_0$ and consider $C = \psi(K)$. Relation (2.3) now follows from (2.6) because $\sup_{w \in K} |h_n(w)| = \sup_{z \in C} |g_n(z)|$.

The proof of Lemma 2.1 is complete. □

Lemma 2.2. *With* $h_n(w)$ *defined as in Lemma* 2.1 *and* (a_n) *satisfying* (1.1) (*or more generally* $\limsup_{n\to\infty} |a_n|^{1/n} = 1$), $\sum_{n=0}^{\infty} a_n h_n(w)$ *is analytic in* $|w| > r_0$.

Proof. Let $\varepsilon = (1 - r_0)/2$. Then $r_0 + \varepsilon < 1$ because $r_0 < 1$. With this value of ε, relations (1.1) and (2.3) show that the series $\sum_{n=0}^{\infty} a_n h_n(w)$ converges uniformly on the compact sets of $|w| > r_0$.

We now have built the necessary tools to prove Theorem 1. With $a_n \geq 0$ satisfying (1.1), we have

$$\sum_{n=0}^{\infty} a_n \phi_n(z) = \sum_{n=0}^{\infty} a_n w^n + \sum_{n=0}^{\infty} a_n h_n(w), \quad w = \phi(z).$$

Now by Lemma 2.2, $\sum_{n=0}^{\infty} a_n h_n(w)$ is analytic in $|w| > r_0$ whereas $\sum_{n=0}^{\infty} a_n w^n$ has $w = 1$ for a singular point in view of Pringsheim's theorem. If we recall that $r_0 < 1$, we see that $\sum_{n=0}^{\infty} a_n w^n + \sum_{n=0}^{\infty} a_n h_n(w)$ has a singular point at $w = 1$. It follows that $\sum_{n=0}^{\infty} a_n \phi_n(z)$ has a singular point at $z_0 = \psi(1)$.

The proof of Theorem 1 is complete. □

It is of interest to note the crucial role played by the analyticity of $\operatorname{Br}(E)$, which allows us to extend the mapping function. Without the possibility of this extension, the above argument does not hold.

III. Lacunary series of Faber polynomials.

Theorem 3.1. *Let E and* $\mathrm{Br}\,(E)$ *be as in Theorem* 1. *Let* (a_n) *be a sequence of complex numbers with the following properties*:

i) $\limsup_{n\to\infty} |a_n|^{1/n} = 1$.

ii) $a_n = 0$ *except when n belongs to a sequence* (n_k) *such that* $\lim_{n\to\infty}(n_k/k) = \infty$. *Then* $\mathrm{Br}\,(E)$ *is the natural boundary for*

$$f(z) = \sum_{n=0}^{\infty} a_n \phi_n(a).$$

The proof of Theorem 3.1 follows lines similar to those of Theorem 1, replacing Pringsheim's theorem by Fabry's gap theorem [**2**] and is therefore omitted.

Example. The function

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{2^n}} T_{2^n}(z)$$

has the ellipse

$$\frac{x^2}{(5/4)^2} + \frac{y^2}{(3/4)^2} = 1$$

for natural boundary.

See also Remark following the example of Part I.

It is well known that if Ω is a domain of the complex plane "most" functions analytic on Ω have $\mathrm{Br}\,(\Omega)$ for natural boundary. In the case when $\mathrm{Br}\,(\Omega)$ is an analytic Jordan curve, Theorem 3.1 provides a formula for such a function.

In [**5**] an example is given of a power series whose natural boundary is $|z| = 1$ and whose restriction to $|z| = 1$ is infinitely differentiable. We now show that the same situation prevails for series of Faber polynomials. We first need a preparatory result.

Lemma 3.2. *Let* Γ *be an analytic Jordan curve, and let* $k \geq 1$ *be an integer. Then there exists a constant* M *with the following property: If* $P_n(z)$ *is a polynomial of degree at most* n *and* $z_0 \in \Gamma$, *then*

$$|P_n^{(k)}(z_0)| \leq M n^k \|P_n(z)\|_\Gamma, \quad n = 1, 2, \dots .$$

Lemma 3.2 is a direct consequence of a theorem of Szegö [**6, 7**] if one remarks that the exterior angle at z_0 is π, Γ being analytic.

Lemma 3.3. *Let* E *be as in Theorem* 1, *let* $k \geq 0$ *be an integer, and let* (a_n) *be a sequence of complex numbers such that*

$$\sum_{n=0}^{\infty} n^k |a_n| < \infty. \tag{3.1}$$

Then the restriction of

$$f(z) = \sum_{n=0}^{\infty} a_n \phi_n(z)$$

to Br(E) *is* k*-times continuously differentiable.*

Proof. Recall that $\|\phi_n(z)\| \leq M$, Br(E) being analytic. Now Lemma 3.2 and (3.1) yield

$$\sum_{n=0}^{\infty} |a_n| \, \|\phi_n^{(k)}(z)\| < \infty$$

from which the conclusion follows. □

Theorem 3.1 and Lemma 3.3 yield

Proposition 3.4. *Let* E *and* Br(E) *be as in Theorem* 1. *Let* (a_n) *satisfy conditions* i) *and* ii) *of Theorem* 3.1 *and*

$$|a_n| = O\left(\frac{1}{n^k}\right), \quad k = 1, 2, \dots . \tag{3.2}$$

Then, in addition to having Br (E) *for a natural boundary, the function*

$$f(z) = \sum_{n=0}^{\infty} a_n \phi_n(z)$$

is infinitely differentiable on Br (E)*.*

Example. The function

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{2n} 2^{2^{n/2}}} T_{2^n}(z)$$

has the ellipse

$$\frac{x^2}{(5/4)^2} + \frac{y^2}{(3/4)^2} = 1$$

for natural boundary and is infinitely differentiable on this ellipse.

See also Remark following the example of Part I.

IV. Two open problems. As noticed above, the proofs of Theorem 1 (and of Theorem 3.1) do not hold without the condition of analyticity of Br (E). If we assume that Br (E) is of bounded rotation, so that $\|\phi_n(z)\| \le M$, the series (1.2) need not converge if only (1.1) is assumed. If, however, we replace (1.1) by

$$\sum_{n=0}^{\infty} a_n < \infty, \tag{4.1}$$

then clearly $f(z)$ defined by (1.2) is analytic in Int (E) (and continuous on E).

We recall that if Br (E) is a Jordan curve, which is the case if it is of bounded rotation, then the mapping function $w = \phi(z)$ extends to a homeomorphism between $\overline{\text{Ext}\,(E)}$ and $|w| \ge 1$. This is in the sense that $\phi(z_0) = 1$ must be understood in Conjecture 4.1 below.

We make the following

Conjecture 4.1. *Let E be compact and simply connected with* Br (E) *of bounded rotation and* Cap $(E) = 1$*. Let $a_n \ge 0$ satisfy* (1.1) *and*

(4.1), *and let* $z_0 \in \mathrm{Br}\,(E)$ *be such that* $\phi(z_0) = 1$. *Then* z_0 *is a singular point for the function* $f(z) = \sum_{n=0}^{\infty} a_n \phi_n(z)$.

Conjecture 4.2. *Let* E *and* $\mathrm{Br}\,(E)$ *be as in Conjecture* 4.1, *and let* a_n *satisfy conditions* i) *and* ii) *of Theorem* 3.1 *and* $\sum_{n=0}^{\infty} |a_n| < \infty$. *Then* $\mathrm{Br}\,(E)$ *is the natural boundary for* $f(z) = \sum_{n=0}^{\infty} a_n \phi_n(z)$.

However our efforts to prove these conjectures have been unsuccessful.

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