A CONVERGENCE RESULT
FOR REGULAR BLASCHKE FRACTIONS

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1. Introduction. For a given α, $0 < |\alpha| < 1$, we shall call

$$(1.1) \qquad \zeta(z) = \frac{\bar{\alpha}}{|\alpha|} \frac{\alpha - z}{1 - \bar{\alpha}z}$$

a *Blaschke factor*. Although $\zeta(z)$ is meaningful on the whole Riemann sphere (analytic except for a pole of order 1 at $z = 1/\bar{\alpha}$) it is often assumed that $|z| < 1$, in which case also $|\zeta(z)| < 1$, or even more: The function $z \to \zeta(z)$ maps the open unit disk D one to one onto itself. In the literature Blaschke factors are sometimes defined slightly differently from (1.1). A consequence of the normalizing in (1.1) is that

$$(1.2) \qquad \zeta(0) = |\alpha| > 0,$$

which, in particular for *Blaschke products*,

$$(1.3) \qquad B_n(z) = \prod_{k=1}^{n} \zeta_k(z) = \prod_{k=1}^{n} \frac{\bar{\alpha}_k}{|\alpha_k|}\left(\frac{\alpha_k - z}{1 - \bar{\alpha}_k z}\right),$$

is of advantage.

Sometimes it is convenient to accept also $\alpha = 0$ or $|\alpha| = 1$ or both, although in both cases something gets lost.

For $\alpha = 0$ one sometimes makes the convention that

$$\zeta(z) = z.$$

We shall here, unless otherwise stated, follow this convention. This can be obtained by taking $\bar{\alpha}/|\alpha|$ to be -1 if $\alpha = 0$. What here gets lost is seen as follows. Take

$$\alpha = \rho e^{i\theta}.$$

We then have

(1.4)
$$\zeta(z) = \frac{\rho - ze^{-i\theta}}{1 - \rho e^{-i\theta} z}.$$

If we keep θ fixed and let $\rho \to 0$, we get as the limit value $-ze^{-i\theta}$. Hence, the convention mentioned above means to pick the particular value $\theta = \pi$, i.e., to take $\alpha = -\rho$ before letting $\rho \to 0$. But, with an arbitrarily fixed θ, we may get as a limit value any point on the circle with center at the origin and radius $|z|$.

For $|\alpha| = 1$, the natural way is to look at

$$\lim_{|\alpha| \to 1} \zeta(z).$$

From (1.4) it follows that the limit is 1, regardless of θ. In this case we lose the property of mapping D one to one onto D, since here D is mapped to the point 1.

The Blaschke factors and the Blaschke products are the essential building blocks in the Szegő theory for rational functions as introduced by Bultheel, et al. [**1, 2, 3**]. The study of series similar to power series but with powers of z replaced by Blaschke products, and continued fraction expansions, where z is replaced by Blaschke factors, was suggested by Olav Njåstad. The series will be related to Newton series, but the terms would, at least for $|z| < 1$, be easier to handle, since the factors will have absolute value at most 1. Similarly, continued fractions of the form

(1.5)
$$\mathop{\mathbf{K}}_{n=1}^{\infty} \frac{a_n \zeta_n(z)}{1}$$

are related to Thiele fractions. Since they reduce to a regular C-fraction when all $\alpha_k = 0$, continued fractions (1.5) are called regular Blaschke fractions (regular B-fractions).

Other types related to the classical ones may also be defined; for instance, a Blaschke-Thron fraction (BT-fraction)

(1.6)
$$\mathop{\mathbf{K}}_{n=1}^{\infty} \frac{F_n \zeta_n(z)}{1 + G_n \zeta_n(z)},$$

to mention but one example.

In the classical theory some continued fractions correspond to power series, for instance regular C-fractions to power series where the coefficients satisfy certain conditions. But correspondence is also a type of interpolation. Interpolation is a key word in the relation between the B-fraction (1.5) and the series

$$(1.7) \qquad \sum_{n=1}^{\infty} c_n B_n(z).$$

Expanding a function in a continued fraction (1.5) or a series (1.7) is in fact an interpolation problem. Questions about *when* this is possible and *how* it can be done are important but will not be discussed here. In the present paper the starting point is the regular B-fraction (1.5).

2. A convergence result. If we are trying to deal with an interpolation problem by using Blaschke-fractions, we may possibly end up with a regular B-fraction (1.5). It then becomes very important to know something about convergence, preferably uniform convergence in some domain, since in such a case we may conclude analyticity of the continued fraction value. The following trivial example will illustrate this. We assume that in the continued fraction (1.5) all a_n are of absolute value $\leq 1/4$. Since the Worpitzky disk $|w| \leq 1/4$ is a uniform convergence set, see, e.g., [6] and [7], the continued fraction

$$(2.1) \qquad \mathop{\mathbf{K}}_{n=1}^{\infty} \frac{a_n \zeta_n(z)}{1}, \quad z \in D, \qquad |a_n| \leq \frac{1}{4},$$

will converge in the unit disk D, regardless of the values of α_k, $0 \leq |\alpha_k| \leq 1$. And because of the uniformity and the fact that all approximants have absolute value $\leq 1/2$ and hence are analytic in D, the continued fraction (2.1) converges uniformly on any disk $|z| \leq r < 1$ to a function f, analytic in D and of absolute value $\leq 1/2$. Replacing a_1 by $2a_1$, we get a function $g = 2f$, analytic in D and mapping D into D. Furthermore, with $w_1 = 0$ and

$$(2.2) \qquad w_n = 2 \mathop{\mathbf{K}}_{k=1}^{n-1} \frac{a_k \zeta_k(\alpha_n)}{1},$$

it satisfies $g(\alpha_n) = w_n$ for $n = 1, 2, 3, \ldots$. This condition is of greatest interest if the numbers α_n are all distinct. If, for a given sequence of interpolating points, the values w_n are to be picked in advance, they will be subject to rather strong restrictions. So far these restrictions, to their full extent, are not known, and a discussion of this matter is outside the scope of the present paper.

The convergence question for (2.1) is substantially more complex if we move from the Worpitzky disk to a parabolic convergence set, although we also there will be helped by the fact that a bounded subset is a uniform convergence set. We shall be searching for sufficient conditions for (2.1) to converge to an analytic function. An interesting thing is that the case when the interpolation points α_n are near the unit circle, even $|\alpha_n| \to 1$ turns out to be the easiest to deal with, whereas in other interpolating problems it is easier to have all α_n bounded away from the unit circle.

Before stating and proving the theorem we shall study some properties of what conveniently are called Blaschke sets. These properties will, together with the uniformity, be crucial in the proof.

3. Properties of Blaschke sets. The goal of this section is to determine for a fixed $z \in D$ and a given a_n the set of possible values of the partial numerator

$$a_n \zeta_n(z)$$

of the continued fraction (2.1) when α_n is arbitrary in $D \backslash \{0\}$. For convenience, we shall also include 0 and any value on the unit circle, i.e., we shall let the set of α_n-values be the closed unit disk but without the specific convention for $\alpha_n = 0$. We shall also be interested in more restrictive cases, in particular cases where $|\alpha_n|$ is close to 1, and briefly touch upon the case when all α_n are real and negative.

In the discussion in the present section we will leave out a_n since the set of all values of $a_n \zeta_n(z)$ for fixed z is trivially obtained from the set of all $\zeta_n(z)$ by multiplication by a_n. We shall furthermore omit the subscripts. Hence we shall, for a fixed z in D, primarily find the set of all values of (1.4), i.e., the set

$$(3.1) \qquad H(|z|) = H(z) := \left\{ \frac{\rho - ze^{-i\theta}}{1 - \rho e^{-i\theta}z}; \ 0 \leq \rho \leq 1, \ 0 \leq \theta < 2\pi \right\},$$

but in fact we shall be more interested in certain subsets of it. That we can write $H(|z|)$ instead of $H(z)$ is a consequence of the fact that (3.1) depends only upon $|z|$ and not z itself. (This comes out in the computation, but is also easily seen in advance from the definition.) The inclusion of $\rho = 0$ will imply that the circle of radius $|z|$ will be included, and not only the point z. The impact is the same as if we had used only $0 < \rho \leq 1$ and subsequently had taken the closure.

The sets we shall be even more interested in are, for $0 \leq r \leq 1$,

$$(3.2) \qquad H(|z|, r) := \left\{ \frac{\rho - ze^{-i\theta}}{1 - \rho e^{-i\theta} z};\ r \leq \rho \leq 1,\ 0 \leq \theta < 2\pi \right\}.$$

We have in particular $H(|z|, 0) = H(|z|)$ and $H(|z|, 1) = \{1\}$.

The way to determine $H(|z|, r)$ for $0 < r < 1$ shall here be as follows. We fix ρ and let θ vary. This gives a circle, degenerate if $\rho = 1$. Next we take the union of the circles when ρ varies in the closed interval $[r, 1]$. The set thus obtained is $H(|z|, r)$. The process of taking the union will, as it is done here, make use of the envelope of the circles.

Step 1. For a fixed $\rho \in [0, 1)$, the set of all points

$$\frac{\rho - ze^{-i\theta}}{1 - \rho e^{-i\theta} z}, \quad 0 \leq \theta < 2\pi$$

is the image of the unit circle $|w| = 1$ under the linear fractional transformation (LFT)

$$\omega = \frac{\rho - zw}{1 - \rho zw}.$$

Standard computation and use of properties for LFT yields the circle

$$(3.3) \qquad \left| \omega - \frac{\rho(1 - |z|^2)}{1 - |z|^2 \rho^2} \right| = \frac{|z|(1 - \rho^2)}{1 - |z|^2 \rho^2}.$$

This circle is symmetric with respect to the real axis and intersects it in the points

$$(3.4) \qquad \omega_l = \frac{\rho - |z|}{1 - \rho |z|} \quad \text{and} \quad \omega_r = \frac{\rho + |z|}{1 + \rho |z|}.$$

(For $\rho = 1$ the circle is degenerate and consists of only the point $\omega = 1$.) An observation of use later is that ω_l and ω_r, for a fixed $0 < |z| < 1$, increases when ρ increases from 0 to 1: ω_l increases from $-|z|$ to 1, ω_r from $|z|$ to 1. Simultaneously, the radius of (3.3) decreases from $|z|$ to 0. We also see that, for a fixed ρ, $0 < \rho < 1$, ω_l increases when $|z|$ decreases.

Step 2. The union of the disks (3.3) is the set bounded by the half-circle

$$(3.5) \qquad\qquad |\omega| = |z|, \qquad \operatorname{Re}\omega \le 0$$

and the envelope of (3.3) in the right half-plane. Writing it by using the real coordinates, $\omega = x + iy$, we get (3.3) in the following form:

$$(3.3') \qquad \left(x - \frac{\rho(1 - |z|^2)}{1 - |z|^2\rho^2}\right)^2 + y^2 = \frac{|z|^2(1 - \rho^2)^2}{(1 - |z|^2\rho^2)^2}.$$

If we compute the envelope the traditional way, i.e., by eliminating between (3.3′) and its partial derivative with respect to ρ, we find the parametric representation

$$(3.6) \qquad \begin{aligned} x &= \frac{\rho(1 + |z|^2)}{1 + \rho^2|z|^2} \\[2mm] y &= \pm\frac{|z|(1 - \rho^2)}{1 + \rho^2|z|^2} \end{aligned} \qquad , \quad 0 \le \rho \le 1.$$

If we eliminate the parameter ρ, we find, by simple computation, the two circular arcs

$$(3.6') \qquad x^2 + \left(y \pm \frac{1}{2}\left(\frac{1}{|z|} - |z|\right)\right)^2 = \frac{1}{4}\left(\frac{1}{|z|} + |z|\right)^2.$$

They are centered at

$$\pm\frac{i}{2}\left(\frac{1}{|z|} - |z|\right),$$

have (both) radius

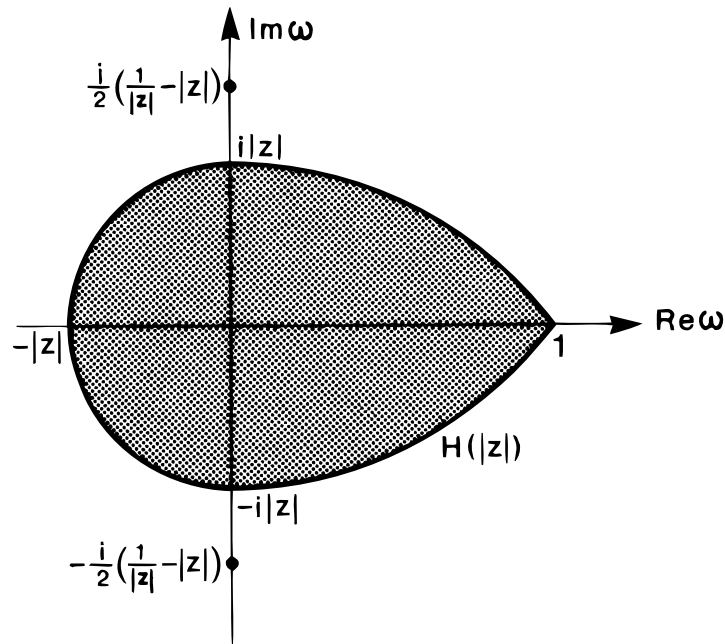$$\frac{1}{2}\left(\frac{1}{|z|} + |z|\right),$$

FIGURE 1.

and run from $\pm i|z|$ to 1.

It follows that the set $H(|z|)$ is the closed set, bounded by the half-circle (3.5) and the circular arcs (3.6′). We furthermore have that, for $0 < r < 1$, the set $H(|z|, r)$ is the closed set, bounded by the left arc from $\omega = \omega_0$ to $\omega = \bar{\omega}_0$ of the circle (3.3) with ρ replaced by r and the subarcs of (3.6′) in the right half-plane from ω_0 and $\bar{\omega}_0$ to $\omega = 1$, where ω_0 and $\bar{\omega}_0$ are the two points of tangency between the circle and the arcs.

Illustrations of $H(|z|)$ and $H(|z|, r)$ are given above and on the next page.

Combining the observations at the end of the description of Step 1 with the fact that distinct circles with centers on the imaginary axis and passing through the point 1 can intersect only in ± 1, one obtains the following.
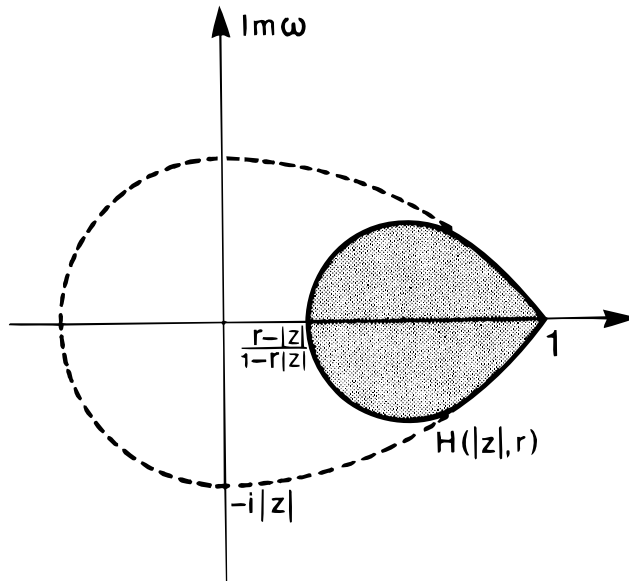
FIGURE 2.

Inclusion properties. For any fixed $|z|$, $0 \leq |z| < 1$, we have

$$(3.7) \qquad 0 \leq r_1 < r_2 \leq 1 \Longrightarrow H(|z|, r_2) \subseteq H(|z|, r_1).$$

For any fixed r, $0 \leq r < 1$, we have

$$0 \leq |z_1| < |z_2| < 1 \Longrightarrow H(|z_1|, r) \subseteq H(|z_2|, r).$$

These properties will turn out to be crucial in the convergence proof to come. We shall also make use of the following observation on

$$(3.8) \qquad a_n H(|z|, r), \quad 0 \leq |z| < 1, \quad 0 \leq r \leq 1,$$

readily seen from the description of the set: The set (3.8) is contained in a disk, centered at a_n and with radius

$$(3.9) \qquad |a_n| \frac{(1-r)(1+|z|)}{1-r|z|}.$$

Observe also that this set decreases (with respect to inclusion) when $|z|$ decreases and when r increases.

4. The main theorem. As mentioned in the introduction we shall here prove a convergence result for regular Blaschke fractions (1.5). The tool to be used is the uniform parabola theorem [**5, 6, 7**]. To each a_n, assumed to be in a parabolic (conditional) convergence region, there is, for a fixed z, associated a Blaschke set $a_n \cdot H(|z|)$ or $a_n \cdot H(|z|, r)$, and in order to be able to use the parabola theorem these sets have to be in the parabolic region. There are two ways of obtaining this (if we restrict the considerations to these two types of sets):

1) to restrict $|z|$ to "small" values.

2) To restrict $|\alpha_k|$ to values "near" 1.

Remarks to 1). When $|z| \to 0$, the set $a_n H(|z|)$ degenerates to the straight line segment from 0 to a_n, which is in the parabolic region if a_n is, since the parabolic region is star shaped with respect to the origin. But in most cases the restriction on $|z|$ would be too severe to be of interest. In the case of the simplest parabolic region, symmetric with respect to the real axis, we have to require $|a_n z| \le 1/4$ to make sure $a_n H(|z|)$ is in the parabolic region.

Remarks to 2). If a_n is an interior point of the parabolic region, such that a δ-neighborhood of a_n is in the parabolic region, it follows from the observation on the set $a_n \cdot H(|z|, r)$ that this set is contained in the parabolic region if

$$(4.1) \qquad |a_n| \frac{(1-r)(1+|z|)}{1 - r|z|} < \delta.$$

If we furthermore assume that all a_n are in a bounded part of the parabolic region, $|a_n| \le M$ for some $M > 0$, then all sets $a_n \cdot H(|z|, r)$ are in the parabolic region if

$$(4.1') \qquad \frac{(1-r)(1+|z|)}{1 - r|z|} < \frac{\delta}{M}.$$

We shall assume $\delta < M$, since that is essentially the only interesting case. We shall, for a fixed r, with

$$1 - \delta/M < r < 1$$

assume that, for all n,

$$|\alpha_n| > r.$$

We find that $(4.1')$ holds for all z with

$$(4.2) \qquad |z| \leq R = \frac{r - (1 - \delta/M)}{1 - r(1 - \delta/M)}.$$

Observe that when r increases from $1 - \delta/M$ to 1, R increases from 0 to 1.

Since every bounded subset of the parabolic convergence region (explicitly described below) is a uniform convergence region, we have the following result.

Theorem 1. *Let $\delta > 0$ and $M > 0$ be given, $\delta < M$, and let*

$$(4.3) \qquad \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{a_n \zeta_n(z)}{1}$$

be a continued fraction, where all a_n have absolute value $\leq M$, and where the δ-neighborhoods of all a_n are in the parabolic (conditional) convergence region given by

$$(4.4) \qquad |w| - \mathrm{Re}\left(we^{-2i\beta}\right) \leq \frac{1}{2}\cos^2\beta, \quad -\frac{\pi}{2} < \beta < \frac{\pi}{2}.$$

Let, furthermore, in

$$(4.5) \qquad \zeta_n(z) = \frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z},$$

$|\alpha_n| \geq r$ *for all n, where $1 - \delta/M < r < 1$. Then the continued fraction converges uniformly on*

$$(4.2) \qquad |z| \leq R = \frac{r - (1 - \delta/M)}{1 - r(1 - \delta/M)}$$

to a function, holomorphic in $|z| < R$.

The proof is already carried out if we keep in mind that all approximants are in a bounded part of the half-plane given by

$$(4.6) \qquad \mathrm{Re}\left(we^{-i\beta}\right) \geq -\frac{1}{2}\cos\beta$$

and hence are holomorphic. The uniform convergence on $|z| \leq R$ and hence on any compact subset of $|z| < R$ implies that the limit is holomorphic in $|z| < R$.

From (4.2) we see that we can have R-values arbitrarily close to 1 by taking r sufficiently close to 1. This gives rise to a corollary for continued fractions (4.3) where $|\alpha_n| \to 1$ as $n \to \infty$:

Corollary 2. *Let*

$$(4.3) \qquad \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{a_n \zeta_n(z)}{1}$$

be a continued fraction satisfying all the conditions in Theorem 1, except that the condition $|\alpha_n| \geq r$ is replaced by $|\alpha_n| \to 1$ when $n \to \infty$. Then, to each R, $0 < R < 1$, there is an N, such that for all $n \geq N$ the nth tail

$$\overset{\infty}{\underset{k=n+1}{\mathbf{K}}} \frac{a_k \zeta_k(z)}{1}$$

converges locally uniformly on $|z| < R$ to a holomorphic function. The continued fraction itself converges to a meromorphic function on the unit disk, uniformly on any compact subset of the unit disk minus the poles.

In the case dealt with in Corollary 2, the computation of the value of the continued fraction may be carried out by using the tail values of

$$(4.7) \qquad \overset{\infty}{\underset{n=1}{\mathbf{K}}} \frac{a_n}{1}$$

as modifying factors. The approximants are still rational functions, but using the tails of (4.7) represents an acceleration as compared to the ordinary approximants. This follows from results by Lisa Lorentzen [**4**]. She also has explicit estimates of the ratio of the two types of truncation errors.

5. A special case of a different kind. We shall briefly touch upon a case which is rather special but still may be of some interest. Instead of conditions on the absolute value of α_i we shall put conditions

on the argument. We shall restrict the discussion to the case when the argument of α is equal to π, i.e., $\alpha = -\rho$, $0 < \rho < 1$, and we shall also include the case $\rho = 0$. In this case we have

$$(5.1) \qquad \omega = \zeta(z) = \frac{\rho + z}{1 + \rho z}.$$

Keep z fixed in the open unit disk, and let ρ vary in the interval $[0, 1]$ (or in part of it). We shall here assume $\operatorname{Im} z > 0$ and $\operatorname{Re} z > 0$. Then, by simple calculation we find that ω varies along a circular arc of

$$(5.2) \qquad \left| \omega + \frac{1 - |z|^2}{2\operatorname{Im} z} i \right| = \frac{|1 - z^2|}{2\operatorname{Im} z}$$

in the first quadrant. (Other cases are equally easy to handle.)

Take, for instance, the case $a_n > 0$ for all n (Stieltjes case). Then $a_n \zeta_n(z)$ is inside the parabola

$$(5.3) \qquad |w| - \operatorname{Re}(w) = 1/2$$

if and only if

$$|z| - \operatorname{Re}(z) \le \frac{1}{2a_n},$$

and if all $a_n \le M$ the Blaschke-fraction converges when

$$|z| - \operatorname{Re}(z) \le \frac{1}{2M}.$$

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