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THE EXISTENCE OF SHAPE-PRESERVING OPERATORS WITH A GIVEN ACTION

B.L. CHALMERS AND M.P. PROPHET

ABSTRACT. We study the existence of shape-preserving projections and, more generally, the existence of shape-preserving operators with a given (fixed) action.

Introduction and preliminaries. Let X denote a Banach space and V an n-dimensional subspace of X. We will use the following notation. An n-tuple from X is to be considered a column vector while an n-tuple from X^* will be a row vector. Elements of $\mathbf{R}^n$ will be column vectors.

Let $S \subset X$ denote the set of all elements that possess a specified "shape." For example, S might denote the set of convex functions or the set of monotone functions in $C[0,1]$. The problems involved with preserving the "shape," i.e., leaving S invariant, while approximating elements of X by elements of V have been the object of much study, especially in the case of best approximation (see, for example, [**2, 4, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19**]). Best approximation operators that are invariant on S are, in general, nonlinear and their existence is usually not an issue. It is in the attempt to preserve a "shape" using *linear* operators that existence becomes problematic. As illustrated in the following example, small variations in the "action" of a linear operator on V may greatly influence the ability of that operator to leave S invariant.

Example 1.1. Let Π_2 denote the space of second-degree algebraic polynomials, considered as a subspace of $C[0,1]$. The second-degree Bernstein operator $B_2 : C[0,1] \to \Pi_2$ is a linear operator that preserves monotonicity. This is accomplished while *nearly fixing* Π_2 (B_2 fixes the lines and $B_2 t^2 = (t+t^2)/2$). However, no linear operator *fixing* Π_2 can preserve monotonicity. Indeed, if such an operator $P : C[0,1] \to \Pi_2$ did exist, we could rewrite it as $P = \sum_{i=1}^{3} u_i \otimes t^{i-1}$ where

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each u_i is a, real-valued, regular Borel measure. Then P would also preserve monotonicity from $(C^1[0,1], \|\cdot\|)$ onto Π_2, where $\|f\| = \max_{i=0,1}\{\|f^{(i)}\|_\infty\}$. But this is in contradiction to [**6**, Lemma 2.2], which shows that such an operator, $P = \sum_{i=1}^{3} u_i \otimes t^{i-1} : C^1 \to \Pi_2$, must have $u_2 = \delta_0'$, where δ_0' denotes derivative-evaluation at $t = 0$.

In the case S is a cone induced by a vector lattice, one usually refers to S as the *positive cone* and to an operator invariant on S as a *positive operator*, see, e.g., [**17**]. In the following, we will be interested in cones, and thus "shapes," derived in a different manner, using subsets of X^* to define S. We call linear operators invariant on S *shape-preserving* operators, and we will discuss the existence and characterization of these operators.

Denote by $\mathcal{B} = \mathcal{B}(X, V)$ the space of bounded linear operators from X to V. Given $P \in \mathcal{B}$, there exists $\mathbf{u} = (u_1, \dots, u_n) \in (X^*)^n$ and basis $\mathbf{v} = (v_1, \dots, v_n)^T \in (V)^n$ such that the representation $P = \mathbf{u} \otimes \mathbf{v} = \sum_{i=1}^{n} u_i \otimes v_i$ is valid, where $Pf = \sum_{i=1}^{n} \langle f, u_i \rangle v_i$.

Definition 1.1. For a given $n \times n$ nonsingular matrix A, $P \in \mathcal{B}$ is said to be an *A-action operator* if P can be written as $P = \sum_{i=1}^{n} u_i \otimes v_i$ such that $(\langle v_i, u_j \rangle) = A$, i.e., $P\mathbf{v} = A\mathbf{v}$.

Note that there is an entire equivalence class of matrices associated with a particular A-action operator. That is to say, if $P = \mathbf{u} \otimes \mathbf{v}$ is an A-action operator, then P is also an MAM^{-1}-action operator, for any nonsingular matrix M, since $P = \mathbf{u}M \otimes M^{-1}\mathbf{v}$ and $(\langle (M^{-1}\mathbf{v})_i, (\mathbf{u}M)_j \rangle) = MAM^{-1}$. In the following, it will frequently be advantageous for us to rewrite an operator's representation, as above. To this end we will resist *fixing* a particular nonsingular matrix A and instead simply refer to a given 'action' and use A to denote a representative from the equivalence class.

We will now consider the existence of A-action operators that preserve the "shape" of elements of X in the following sense, see [**1**] and [**10**] for related considerations. We will take the term *cone* to mean a convex set, closed under nonnegative scalar multiplication. A *pointed cone* is a cone that contains no lines.

Definition 1.2. Let S^* be a weak*-closed pointed cone in X^*. Then $f \in X$ is said to have *shape, in the sense of* S^*, if $\langle f, u\rangle \geq 0$ for all $u \in S^*$. Let S be the set of all elements of X with shape. Note that S is also a cone. Let $S_1^* = S^* \cap B(X^*)$, and let S_0^* denote the set of extreme points of S_1^* less zero. Note that S_1^* is the closed convex hull of $S_0^* \cup \{0\}$ by the Krein-Milman theorem. In order to emphasize the geometric flavor of our discussion, we will sometimes refer to S_0^* as "corners" of S_1^* and to $E(S^*) := \pi^{-1}(S_0^*)$ as the "edges" of the cone S^*, where $\pi(z) := z/\|z\|$. We will also say that S^* is generated by S_0^* or by $E(S^*)$ and write $S^* = \overline{\text{cone}}\,(S_0^*)$ or $S^* = \overline{\text{cone}}\,(E(S^*))$. Finally, we will sometimes refer to the edge of a cone as the ray generated by all positive scalar multiples of a particular nonzero element of the edge and sometimes identify such an element with the edge itself.

Note 1. $f \in X$ has shape, in the sense of S^*, if and only if $\langle f, u\rangle \geq 0$ for all $u \in S_0^*$ if and only if $\langle f, u\rangle \geq 0$ for all $u \in S_1^*$.

Assumptions. Unless otherwise noted, we assume that S^* is total over V, that is, we assume that $S^*_{|V}$ contains n independent elements (in Example 3.5 we examine a situation in which S^* is not total over V). Furthermore, we assume that $S \cap \sim (S^*)^{\perp} \neq \varnothing$ and that S contains at least n independent elements.

Lemma 1.1. *S and S^* are "dual" cones in the sense that, if $\langle f, u\rangle \geq 0$ for all $f \in S$, then $u \in S^*$.*

Proof. Suppose that $\langle f, u\rangle \geq 0$ for all $f \in S$ but $u \notin S^*$.

In the case where X is reflexive, we have an immediate contradiction since S^* being weakly closed and convex can be "separated" from u by a functional $f \in X^{**} = X$ such that $\langle w, f\rangle = \langle f, w\rangle \geq 0$ for all $w \in S^*$ and yet $\langle u, f\rangle = \langle f, u\rangle = -1$, i.e., f provides a "supporting hyperplane" for S^* separating S^* from u; but such an f is in S.

In the general case the "separating functional" f, in X^{**}, above is not necessarily in X and therefore not necessarily in S, and so the construction of a "separating" hyperplane must be modified as follows. Let $C = \overline{\text{co}}\,(S_0^*)$, where the closure is with respect to the weak* topology. Note that C is a convex, compact set, not containing the origin. Consider first the case that $u \notin -S^*$ (of course we still suppose that $u \notin S^*$). Then the entire subspace $[u]$ does not intersect C and thus, from the convexity and compactness of C, it follows that there exists an entire closed hyperplane H containing $[u]$ such that $H \cap C = \varnothing$, see [**8**]. Considering X^* with its weak*-topology, let $t \in (X^*/H)^*$ (t not identically zero), and let $q : X^* \to X^*/H$ be the natural map. Then $h = t \circ q$ is a (weak*) continuous linear functional (with kernel H) on X^* and thus $h \in X$ (a continuous linear functional on X^* with its weak*-topology must be in X); via scaling we may assume that $\min_{x \in C}\langle x, h\rangle = 1$. Finally we 'shift slightly' the hyperplane so that it strictly separates C from $[u]$. Indeed, let $g \in X$ be such that $\langle u, g\rangle = 1$. If $g \in C^{\perp}$, then $h - g$ strictly separates C and u; otherwise, let $1/c = \max_{x \in C}\langle x, g\rangle$ whence, for every $x \in C$, we have $\langle x, h - cg\rangle \geq 1 - \langle x, cg\rangle \geq 0$ and $\langle u, h - cg\rangle = \langle u, -cg\rangle = -c < 0$. In particular, we have shown that, if $u \notin S^* \cup -S^*$, then u cannot be nonnegative against S. Finally we consider the case $u \in -S^*$. Since $\langle f, u\rangle \geq 0$ for all $f \in S$, we see that u must vanish against S. Let $u_1 \in S^*$ be such that $\langle f, u_1\rangle > 0$ for some $f \in S$, see assumptions. Then the line segment $\lambda u + (1 - \lambda)u_1$, $\lambda \in [0, 1]$, does not pass through the origin. But every element on this line segment is nonnegative against S. Then, since both S^* and $-S^*$ are closed, there exists a (nonzero) element of the line segment that belongs to neither S^* nor $-S^*$. That is, there exists an element $\notin S^* \cup -S^*$ that is nonnegative against S, a contradiction to the above. We conclude that, in all cases, if $u \notin S^*$, then u cannot be nonnegative against S. $\square$

Example 1.2. Let $X = C[0, 1]$, and let $S_0^* = \{\delta_t : t \in [0, 1]\}$. Then S is the cone consisting of all nonnegative functions in $C[0, 1]$.

Definition 1.3. $P \in \mathcal{B}$ is said to be shape-preserving (in the sense of S^*) if, whenever f has shape, Pf has shape, i.e., $f \in S$ implies $Pf \in S$. Denote the set of all shape-preserving A-action operators (of $\mathcal{B}$) by $\mathcal{A}_{S^*}$.

From the above assumptions, the following lemma is immediate. We will say that a basis $v_1, \dots, v_n$ for V *has shape* if every basis element has shape.

Lemma 1.2. *If there does not exist a basis for V with shape, then $\mathcal{A}_{S^*} = \varnothing$ for all A.*

Proof. We prove the contrapositive. Let $P = \sum_{i=1}^n u_i \otimes v_i \in \mathcal{A}_{S^*}$ for some A and let $\mathbf{f} = (f_1, \dots, f_n)^T \in S^n$ be an n-tuple of independent elements. Then $P\mathbf{f} = \langle \mathbf{f}, \mathbf{u} \rangle \mathbf{v}$ is a basis that has shape. □

In the following we will therefore assume that V contains a basis with shape. As seen in [**6**], $\mathcal{A}_{S^*}$ may be empty for certain (standard) S^*, where $A = I_n$. In the following section we attempt to characterize when $\mathcal{A}_{S^*} \neq \varnothing$.

2. Characterization.

Lemma 2.1. *Let $P \in \mathcal{B}$. Then $PS \subset S \Leftrightarrow P^*S^* \subset S^*$.*

Proof. The proof is an immediate consequence of the duality equation $\langle Pf, u \rangle = \langle f, P^*u \rangle$ and Lemma 1.1. □

Note 2. If $P = \mathbf{u} \otimes \mathbf{v} = \sum_{i=1}^n u_i \otimes v_i$ preserves shape and has range V, then from Lemma 2.1 we see that, without loss (after a possible change of basis), we may assume that $u_i \in S^*$, $i = 1, \dots, n$. This fact already gives us much insight into the make-up of shape-preserving operators, i.e., the functionals of an operator preserving shape S^* must be, without loss, in S^* themselves.

Theorem 2.1 (Characterization). *$\mathcal{A}_{S^*} \neq \varnothing$ if and only if there exists $\mathbf{u} = (u_1, \dots, u_n) \in (S^*)^n$ such that $\mathbf{u}A\boldsymbol{\lambda}_u \in S^*$ for all $u \in S^*$, where A denotes an action matrix and $u_{|V} = \mathbf{u}_{|V}\boldsymbol{\lambda}_u$, where $\boldsymbol{\lambda}_u$ is a (column) vector of scalars.*

Proof. $\Rightarrow$. Suppose $P = \mathbf{u}' \otimes \mathbf{v}' = \sum_{i=1}^n u_i' \otimes v_i' \in \mathcal{A}_{S^*}$, and note that, for each $u \in S^*$, $\langle f, P^*u \rangle \geq 0$ for all $f \in S$. Then, by Lemma 1.1, we have that $P^*u \in S^*$ for all $u \in S^*$. Now, since $P^*u = \mathbf{u}'\langle \mathbf{v}', u \rangle \in S^*$ for $u \in S^*$, it follows that, via a change of basis, we may rewrite P as

$P = \sum_{i=1}^n u_i \otimes v_i$ where $u_i \in S^*$, $i = 1, \dots, n$. Thus, for $u \in S^*$,

$$\mathbf{u}A\boldsymbol{\lambda}_u = \mathbf{u}\langle \mathbf{v}, \mathbf{u}\rangle \boldsymbol{\lambda}_u = \mathbf{u}\langle \mathbf{v}, u\rangle = P^* u \in S^*.$$

$\Leftarrow$. Let $\mathbf{v} \in V^n$ such that $\langle \mathbf{v}, \mathbf{u}\rangle = A$. Then $P = \sum_{i=1}^n u_i \otimes v_i$ is shape-preserving, since, for $f \in S$ and $u \in S^*$, we have

$$\langle Pf, u\rangle = \langle f, P^* u\rangle = \langle f, \mathbf{u}A\boldsymbol{\lambda}_u\rangle \geq 0. \qquad \square$$

Note 3. The preceding characterization theorem has an interesting geometric interpretation. For a fixed cone, S^*, the question of whether or not a particular action A preserves this shape is actually a question concerning the existence of subcones of S^* that have a particular set of n generators. Specifically, $\mathcal{A}_{S^*} \neq \varnothing$ if and only if there exists a subcone S_A^* of S^*, possessing n "A-cone" edges, i.e., n elements $(u_1, \dots, u_n) = \mathbf{u} \in (S^*)^n$ such that $S_A^* = \{\mathbf{u}A\boldsymbol{\lambda}_u \mid u \in S^*\}$. The following corollaries further the geometric insights into the shape-preserving problem. For example, in certain settings (as we shall see) the notion of "A-cone" edges simplifies to actual edges, in the sense that *nonnegative* linear combinations of $\{u_1, \dots, u_n\}$ will recover AS^*. This is a sufficient condition for existence.

Corollary 2.1. *If there exists $\mathbf{u} \in (S^*)^n$ such that $A\boldsymbol{\lambda}_u$ has nonnegative entries for all $u \in S^*$, then $\mathcal{A}_{S^*} \neq \varnothing$.*

Proof. Let $\mathbf{v} = (v_1, \dots, v_n)^T \in V^n$ such that $(\langle v_i, u_j\rangle) = A$, and let $P = \sum_{i=1}^n u_i \otimes v_i$. Then, for $x \in X$ such that $\langle x, u\rangle \geq 0$, for all $u \in S^*$, we have $\langle Px, u\rangle = \langle x, \mathbf{u}\rangle\langle \mathbf{v}, u\rangle = \langle x, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{u}\boldsymbol{\lambda}_u\rangle = \langle x, \mathbf{u}\rangle A\boldsymbol{\lambda}_u \geq 0$ since $u_i \in S^*$, $i = 1, \dots, n$. Thus P is shape-preserving. $\square$

Example 2.1. Consider the "quadratics" $V = [1, t, t^2]$ in $C[0,1]$, and let $S^* = \{\delta_t, t \in [0,1]\}$. Setting $u = \delta_t$, we have $u_{|V} = \mathbf{u}_{|V}\boldsymbol{\lambda}_u$, where

$$\mathbf{u} = (\delta_0, \delta_{1/2}, \delta_1)$$

and

$$\boldsymbol{\lambda}_u = (1 - 3t + 2t^2, 4(t - t^2), 2t^2 - t)^T.$$

Thus, if $A = I$, the (interpolating at t_i, $i = 1, 2, 3$) projection operator $P = \sum_{i=1}^{3} \delta_{t_i} \otimes v_i$ $(v_i(t) = \prod_{j \neq i}(t - t_j)/\prod_{j \neq i}(t_i - t_j)) : C[0, 1] \to V$ does not preserve positivity if $(t_1, t_2, t_3) = (0, 1/2, 1)$, since both the first and third elements of $\boldsymbol{\lambda}_u$ are sometimes negative on $[0, 1]$. In fact, the argument works for any choice of t_i to show that there is *no* interpolating projection onto the quadratics which preserves positivity. (It is well known that there is no projection onto the quadratics which preserves positivity, see, e.g., Example 3.6.)

On the other hand, if

$$A = \begin{pmatrix} 1 & 1/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/4 & 1 \end{pmatrix},$$

then

$$A\boldsymbol{\lambda}_u = (1 - 2t + t^2, 2(t - t^2), t^2)^T,$$

and all three $(A\boldsymbol{\lambda}_u)_i$ are always nonnegative on $[0, 1]$. Thus, by Corollary 2.1, the operator $P = \sum_{i=1}^{3} u_i \otimes v_i$ where $(\langle v_i, u_j \rangle) = A$ (where $\mathbf{v} = ((1 - t)^2, 2t(1 - t), t^2)^T$), preserves positivity. Note that P is the classical Bernstein operator onto the quadratics.

Note 4. The example above illustrates the observation that, in order to determine a set of action operators preserving a certain given shape, one may proceed as follows: for each $\mathbf{u} \in (S_0^*)^n$ consider $\Lambda_{\mathbf{u}} := \{\boldsymbol{\lambda}_u : u \in S_0^*\}$ and suppose $R_{\mathbf{u}} := \{\mathbf{a} : \mathbf{a} \cdot \boldsymbol{\lambda}_u \geq 0 \text{ for all } \boldsymbol{\lambda}_u \in \Lambda_{\mathbf{u}}\}$ is nonempty. Then $\mathcal{A}_{S^*}$ is not empty for any "action" matrix A whose rows are members of $R_{\mathbf{u}}$.

The following corollary, Corollary 2.2, is also quite useful in practice since it gives conditions in R^n relating to existence. In addition, we will see that, in the projection case (the identity action), the corollary extends to a characterizing theorem, Theorem 2.2, below.

Definition 2.1. We will say that the cone S^* is *simplicial* if S_0^* consists of independent elements. Thus, if S^* has finite dimension m, then S^* is simplicial is equivalent to $|E(S^*)| = m$.

Example 2.2. S^* of Example 1.2 is simplicial. For an example of a nonsimplicial shape, consider $X = L^1[0, 1]$ and $S_0^* = \{\phi_t\} \cup \{\psi_t\}$ with

$\langle f, \phi_t \rangle = (\int_0^t f(s)\,ds)$ with $t \in (0,1]$ and $\langle f, \psi_t \rangle = (\int_t^1 f(s)\,ds)$ with $t \in [0,1)$. Note that, for $t_1 < t_2$, $\phi_{t_1} = \phi_{t_2} + \psi_{t_2} - \psi_{t_1}$.

Definition 2.2. Let $(v_1, \dots, v_n)^T = \mathbf{v}$ be a basis for V, and let $S^*_{|\mathbf{v}}$ denote the subset of R^n given by $\{\langle \mathbf{v}, u \rangle \mid u \in S^*\}$. Since $AS^*_{|\mathbf{v}_1} \subset S^*_{|\mathbf{v}_1} \Leftrightarrow AS^*_{|\mathbf{v}_2} \subset S^*_{|\mathbf{v}_2}$ where $\mathbf{v}_1$ and $\mathbf{v}_2$ are two arbitrary bases, we will let $S^*_{|V}$ also stand for $S^*_{|\mathbf{v}}$ where $\mathbf{v}$ is an arbitrary basis.

Corollary 2.2. *In order for $\mathcal{A}_{S^*} \neq \varnothing$, it is necessary that $AS^*_{|V} \subset S^*_{|V}$. If, in addition, $AS^*_{|V}$ is contained in a simplicial subcone of $S^*_{|V}$, then this is sufficient for $\mathcal{A}_{S^*} \neq \varnothing$.*

Proof. If $P = \sum_{i=1}^n u_i \otimes v_i \in \mathcal{A}_{S^*}$, then by Lemma 1.1 we have $P^* u \in S^*$ for all $u \in S^*$. Thus $(P^*u)_{|\mathbf{v}} \in S^*_{|\mathbf{v}}$. But

$$(P^*u)_{|\mathbf{v}} = \langle \mathbf{v}, P^*u \rangle = \langle P\mathbf{v}, u \rangle = \langle \langle \mathbf{v}, \mathbf{u} \rangle \mathbf{v}, u \rangle = A \langle \mathbf{v}, u \rangle,$$

and hence $AS^*_{|\mathbf{v}} \subset S^*_{|\mathbf{v}}$. Now suppose that there exists $\{u_{1|_V}, \dots, u_{n|_V}\} \subset S^*_{|\mathbf{v}}$ such that $AS^*_{|\mathbf{v}} \subset \operatorname{cone}(u_{1|_V}, \dots, u_{n|_V})$. Set $\mathbf{u} = (u_1, \dots, u_n) \in (S^*)^n$. Note that

$$A\langle \mathbf{v}, u \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \mathbf{c}_u$$

where $\mathbf{c}_u$ is the vector of nonnegative coefficients guaranteed by the simplicial condition. Since $S^*_{|\mathbf{v}}$ has n independent elements, the matrix $M = \langle v_i, u_j \rangle$ is nonsingular. Thus, solving for $\mathbf{c}_u$, we have

$$\mathbf{c}_u = M^{-1} A \langle \mathbf{v}, u \rangle = M^{-1} A \langle \mathbf{v}, \mathbf{u} \rangle \boldsymbol{\lambda}_u = M^{-1} A M \boldsymbol{\lambda}_u.$$

The positive entries of $M^{-1}AM\boldsymbol{\lambda}_u$ imply that $\mathbf{u}M^{-1}AM\boldsymbol{\lambda}_u \in S^*$ for all $u \in S^*$. Thus $\mathcal{A}_{S^*} \neq \varnothing$ by Theorem 2.1. $\square$

Example 2.3. An illustrative example of Corollary 2.2 is obtained by considering again the Bernstein action onto the quadratics discussed in Example 2.1 above. This example is discussed in detail in the first part of Example 3.8 below.

2.1. *The projection action.* Let $\mathcal{P}_{S^*}$ denote the set of shape-preserving projections from X onto V. We will show that Corollary 2.2, in the case of projections, results in a simple geometric characterization of $\mathcal{P}_{S^*}$ (recall that the action matrix for a projection is the identity). This characterization will then lead us to a result concerning uniqueness.

Definition 2.3. The shape S^*, generated by the set S_0^*, is said to be *proper, with respect to* V, if $S^*_{|_V}$ is closed (in $X^*_{|_V}$). In addition, we say that a proper shape S^* is *strictly proper, with respect to* V, if distinct elements of S_0^* do not agree on V.

Note 5. To determine whether S^* is proper, with respect to V, it is of course sufficient to determine whether the set of nonzero elements of $(S_0^*)_{|V}$ is closed.

We give an example where S_0^* is not proper in the following.

Example 2.4. Let $X = L^1[0,1]$. Let $V = [v_1, v_2]$ where $v_1 = 1$ and $v_2 = t$. Define the following 'average-value' shape by $S^* = \overline{\text{cone}}\,(S_0^*)$ where $S_0^* = \{\phi_t\}$ and $\langle f, \phi_t\rangle = (\int_0^t f(s)\,ds)$ with $t \in (0,1]$. Then $S^*_{|_V}$ is not closed. Specifically, note that the sequence $\{(\phi_{1/n})_{|V}/n\}$ converges to $\delta_{0|_V}$. There does not exist, however, an element of S^* that restricts to δ_0 on V. Indeed, for such a functional to exist, it would have to be nonnegative against every function of the form $f(t) = mt - g(t)$ where $m > 0$ and $g(t)$ is a nonnegative function pointwise bounded by mt, since such a function $f(t)$ has shape. However, such a functional must vanish on $v_2(t) = t$, since it restricts to δ_0 in V, and thus it must vanish against all such $g(t)$. We know that such a functional is not identically zero, since it is one against the identically one function. But then it follows immediately that such a functional would not be bounded.

The following example is one where S_0^* is strictly proper, with respect to V.

Example 2.5. Let $X = C[-1,1] \supset V = [1,t]$, and consider the 'positive' shape given by $S_0^* = \{\delta_t\}$ where δ_t denotes point evaluation at $t \in [-1,1]$. Since

$$(\delta_t)_{|V} = \frac{1+t}{2}(\delta_1)_{|V} + \frac{1-t}{2}(\delta_{-1})_{|V},$$

we see that S_0^* is strictly proper, with respect to V.

Theorem 2.2. *Let S^* be simplicial and proper, with respect to V. Then $\mathcal{P}_{S^*} \neq \varnothing$ if and only if the cone $S^*_{|V}$ is simplicial.*

Proof. $\Leftarrow$. This direction follows immediately from Corollary 2.2.

$\Rightarrow$. We will show that $|E(S^*_{|V})| = n$. Let $P = \mathbf{u} \otimes \mathbf{v} \in \mathcal{P}_{S^*}$ and, from Lemma 2.1, we have $P^*S^* \subset S^*$. Note that $(P^*u)_{|V} = u_{|V}$ since P is a projection, and, since P^*X^* is n-dimensional, it follows that $(P^*u)_{|V} = (P^*w)_{|V}$ if and only if $P^*u = P^*w$ in X^*. Thus, there is a bijection between the n-dimensional cones P^*S^* and $S^*_{|V}$ given by $P^*u \Leftrightarrow u_{|V}$. This implies that $|E(S^*_{|V})| = |E(P^*S^*)|$ and we now show $|E(P^*S^*)| = n$. Since S^* proper, it follows again by the Krein-Milman theorem that the compact convex set $S^*_{|V} \cap B(X^*_{|V})$ is the closed convex hull of its extreme points, and hence, via the identification of P^*S^* and $S^*_{|V}$, there exists an independent subset $\{P^*w_1, \dots, P^*w_n\}$ such that each $P^*w_i \in E(P^*S^*)$. Note that we make the usual identification of a point on the edge with the edge itself. We will now show that it is impossible for there to be any other edges. Note that, for each i, $P^*w_i \in S^*$ and, as such, may be written as a (possibly infinite) nonnegative combination of elements of S_0^*, i.e., with $N^* = V^\perp \cap S_0^*$, we have

$$P^*w_i = \int_{S_i^*} u\,d\mu_i + \int_{N^*} u\,d\mu_i \tag{1}$$

where μ_i is a positive measure with $\operatorname{supp}(\mu_i) \cap \sim (N^*) = S_i^*$. Now, taking P^* of both sides of (1), we find that $P^*w_i = \int_{S_i^*} P^*u\,d\mu_i$, since P^* is a projection. However, since $P^*w_i \in E(P^*S^*)$, this is only possible if $P^*u = P^*w_i$ for all $u \in S_i^*$. Whence it follows that $u \in S_i^*$ only if $u_{|V} = (w_i)_{|V}$ and thus $S_i^* \cap S_j^* = \varnothing$, $i \neq j$. Now, suppose there exists $P^*w_{n+1} \in E(P^*S^*)$ such that $P^*w_{n+1} \neq P^*w_i$, $i = 1, \dots, n$. Then P^*w_{n+1} has a representation as in (1), while the n-dimensionality of P^*X^* implies the existence of constants c_i, $i = 1, \dots, n$, such that

$$
\begin{aligned}
\int_{S^*_{n+1}} u\, d\mu_{n+1} + \int_{N^*} u\, d\mu_{n+1} &= P^* w_{n+1} \\
&= c_1 P^* u_1 + \cdots + c_n P^* u_n \\
&= \int_{S^*_1 \cup \cdots \cup S^*_n} u\, d\mu + \int_{N^*} u\, d\mu,
\end{aligned} \tag{2}
$$

where the (signed) measure $\mu = \sum_{i=1}^n c_i \mu_i$. However, S^*_{n+1} and $S^*_1 \cup \cdots \cup S^*_n$ are disjoint and so (2) contradicts the independence of the set S^*_0. Thus $|E(P^*S^*)| = n$. $\square$

Note 6. If $P = \mathbf{u} \otimes \mathbf{v} = \sum_{i=1}^n u_i \otimes v_i$ preserves shape and has range V, then from Lemma 2.1 we see that, without loss (after a possible change of basis), we may assume that $u_i \in S^*$, $i = 1, \dots, n$. As noted before, this fact gives us much insight into the make-up of shape-preserving operators, i.e., the functionals of an operator preserving shape S^* must be, without loss, in S^* themselves. But now we see, in addition, that if S^* is simplicial and proper and a shape-preserving projection exists, then, in fact, the u_i can be chosen from $E(S^*)$, i.e., are just (positive) scalar multiples of elements in S^*_0.

For a strictly proper shape $S^* = \overline{\text{cone}}\,(S^*_0)$, distinct elements of S^*_0 do not agree on V. This gives an immediate uniqueness result.

Theorem 2.3. *Let S^* be simplicial and strictly proper. If $\mathcal{P}_{S^*} \neq \varnothing$, then $\mathcal{P}_{S^*} = \{P\}$.*

Proof. Let $E = E(S^*_{|V})$. From Theorem 2.2, we have $|E| = n$ and $E = \{u_{1|V}, \dots, u_{n|V}\}$, where each $u_{i|V}$ is an edge of $S^*_{|V}$. Since $S^*_{|V} = \text{cone}\,((S^*_0)_{|V})$, $E \subset (S^*_0)_{|V}$, and thus each $u_{i|V} \in (S^*_{|V})_0$ extends uniquely to a $u_i \in S^*_0$. Then for $P \in \mathcal{P}_{S^*}$, we see from the above proof that $P^* u_i = u_i$ for $i = 1, \dots, n$. From here it follows that P is unique. $\square$

Remark. Although the focus of this paper is the case where V is finite-dimensional, the preceding theorem, Theorem 2.2, extends to the case where V is infinite-dimensional. In this case the definition of 'proper' is amended to '$S^*_{|V}$ is weak*-closed, in $X^*_{|V}$.'

We will see many applications and examples of the preceding theory. It will also be demonstrated that, even in the setting of proper shapes, there exist actions that give rise to nonunique operators and 'nonsimplicial' cones.

3. Applications and examples. As a first application, we give a condition for which the necessary inclusion condition given in Corollary 2.2 extends to a characterizing condition.

Theorem 3.1. *Let S^* be proper and $\mathcal{P}_{S^*} \neq \varnothing$. Then $\mathcal{A}_{S^*} \neq \varnothing$ if and only if $AS^*_{|V} \subset S^*_{|V}$.*

Proof. This follows immediately from Theorem 2.2 and Lemma 2.2. $\square$

Next we extend the previous uniqueness result concerning projections. Here the nullspace of S^*, $(S^*)^{\perp}$, will play a role; indeed, the linear subspace $(S^*)^{\perp}$ often results in $\mathcal{P}_{S^*}$ being a linear manifold, as in [**3, 6**] and [**7**]. In this setting, the question of minimal (norm) shape-preserving projections can be addressed using classical techniques from approximation theory, also see [**5**]. This theorem is also appropriate for settings where a given shape is not *total* over V.

Theorem 3.2. *Let V be an n-dimensional subspace of X, with basis $v_1, \dots, v_n$. Let S^* be a simplicial shape, strictly proper with respect to $[v_1, \dots, v_k]$, $k < n$, and such that $\{v_{k+1}, \dots, v_n\} \subset (S^*)^{\perp}$. If $\mathcal{P}_{S^*} \neq \varnothing$, then $\mathcal{P}_{S^*}$ is a linear manifold, i.e., $\mathcal{P}_{S^*} = P_1 + \mathcal{D}$, where P_1 is a shape-preserving projection onto V and $\mathcal{D}$ is a subspace of $\mathcal{B}(X, V)$.*

Proof. By Theorem 2.3, let $P_0 = \sum_{i=1}^{k} u_i \otimes v_i$ be the unique shape-preserving projection onto the subspace $[v_1, \dots, v_k]$. Now, with $\{v_{k+1}, \dots, v_n\} \subset (S^*)^{\perp}$, let $P_1 = P_0 + \sum_{i=k+1}^{n} u_i \otimes v_i$ for any $\{u_{k+1}, \dots, u_n\} \subset [v_1, \dots, v_k]^{\perp} \cap X^*$ such that $(\langle v_i, u_j \rangle) = I_{n-k}$ for

$i, j = k+1, \dots, n$. By Lemma 2.1, and Note 2, P_1 is a projection. Then define the manifold $\mathcal{P} = \{P \mid P = P_1 + \Delta\}$, where $\Delta \in \mathcal{D} = \mathrm{sp}\,\{\delta \otimes v \mid \delta \in V^\perp \text{ and } v \in [v_{k+1}, \dots, v_n]\}$. Thus $\mathcal{P}$ is a manifold of shape-preserving projections with respect to V. The fact that P_0 is unique implies $\mathcal{P}_{S^*} = \mathcal{P}$. $\square$

Example 3.1. We begin with projections in a very simple setting and then apply Theorem 3.2 to obtain an interesting result. For n a positive integer, let $X = C^n[0,1]$ and $V = [v_1] = [t^n]$. Our shape will (initially) be 'one-dimensional' as well. Let $\delta_{t_0}^{(n)}$ denote the evaluation of the nth-derivative at t_0, and define $S_0^* = \{\delta_{t_0}^{(n)}\}$ for fixed $t_0 \in [0,1]$; generating S^* via S_0^*, we (trivially) define a proper shape. By Theorem 2.2, the shape-preserving projection is given by $P_0 = u_1 \otimes v_1$ where $u_1 = \delta_{t_0}^{(n)}/(n!)$. Of course, $\prod_{n-1} = [1, t, \dots, t^{n-1}] \subset (S^*)^\perp$, and so $\mathcal{P}_{S^*}$, the set of all shape-preserving projections onto $\prod_n = \prod_{n-1} \oplus [t^n]$, is a manifold with the following representation:

$$\mathcal{P}_{S^*} = (\delta_{t_0}^{(n)}/(n!) \otimes t^n + \delta_0^{(n-1)}/(n-1)! \otimes t^{n-1} + \cdots + \delta_0 \otimes 1) + \mathcal{D}$$

where

$$\mathcal{D} = \{\phi \otimes v \mid \phi \in (\textstyle\prod_n)^\perp \text{ and } v \in \textstyle\prod_{n-1}\}.$$

This argument holds for any $t_0 \in [0,1]$ and thus the shape defined by $S_0^* = \{\delta_t^{(n)}, t \in [0,1]\}$ (while not proper with respect to $\prod_n$) can be preserved by a projection in any of the above described manifolds. In summary, then, we say there exist n-convex-preserving projections onto $\prod_n$.

The next application involves the consideration of shape-preserving operators onto two-dimensional subspaces.

Corollary 3.1. *Let V be a two-dimensional subspace, and let S^* be a proper shape. Then $\mathcal{A}_{S^*} \neq \varnothing$ if and only if $AS^*_{|V} \subset S^*_{|V}$. In particular, $\mathcal{P}_{S^*} \neq \varnothing$.*

Proof. Note in this case that we do not require S^* to be simplicial. Indeed, since there is a basis with shape by Lemma 1.2, $S^*_{|V}$ is a closed cone with exactly two edges, which is unique to subspaces of dimension two, and $\mathcal{P}_{S^*} \neq \varnothing$ by Theorem 2.2. The remaining follows from Theorem 3.1. $\square$

Example 3.2. Let $X = C[-1,1]$. Let $V = [v_1, v_2]$ where $v_1 = 1$ and $v_2 = t+1$. Define the 'positive' shape given by $S_0^* = \{\delta_t\}$ where δ_t denotes point evaluation at $t \in [-1,1]$. One will note that this shape is proper, with respect to V. Indeed, $S^*_{|V}$ is simplicial since

$$(\delta_t)_{|V} = \frac{1+t}{2}(\delta_1)_{|V} + \frac{1-t}{2}(\delta_{-1})_{|V}.$$

By Corollary 3.1, $\mathcal{P}_{S^*} \neq \varnothing$. One can actually construct the (unique) shape-preserving projection by following the 'recipe' from Corollary 2.2. Setting

$$\mathbf{u} = \boldsymbol{\psi} M^{-1},$$

where $\boldsymbol{\psi} = (\delta_1, \delta_{-1})$ and $M = (\langle v_i, \delta_{-2j+3}\rangle)$, we find $u_1 = \delta_{-1}$ and $u_2 = (\delta_1 - \delta_{-1})/2$. Note that this projection is, of course, the well-known (minimal) norm-1 interpolating projection onto the lines.

It is possible, of course, to define a reasonable shape that cannot be preserved by a projection onto a two-dimensional subspace. The following example demonstrates one such situation and shows that the assumption of "proper" in Theorem 2.2 cannot be dropped.

Example 3.3. Let $X = L^1[0,1]$. Let $V = [v_1, v_2]$, where $v_1 = 1$ and $v_2 = t$. Define the following 'average-value' shape by $S^* = \overline{\text{cone}}\,(S_0^*)$ where $S_0^* = \{\phi_t\}$ and $\langle f, \phi_t\rangle = (\int_0^t f(s)\,ds)/t$ with $t \in (0,1]$. In Example 2.4 we showed that S is not proper, with respect to V. Now suppose that there did exist $P = u_1 \otimes 1 + u_2 \otimes t \in \mathcal{P}_{S^*}$. If $u_1 \notin S^*$, then, by Lemma 1.1, there exists $f_0 \in S$ such that $\langle f_0, u_1\rangle < 0$ and this implies that $Pf_0 \notin S^*$, $Pf_0 = \langle f_0, u_1\rangle + \langle f_0, u_2\rangle t$, whence $\langle Pf_0, \phi_t\rangle < 0$ for t sufficiently close to 0. Thus we must have $u_1 \in S^*$. However, the orthogonality condition, $\langle t^{i-1}, u_j\rangle = \delta_{ij}$, $i,j = 1,2$, implies that $(u_1)_{|V} = (\delta_0)_{|V}$, and it was shown in Example 2.4 that there does not exist an element of S^* that restricts as such. Hence, $\mathcal{P}_{S^*} = \varnothing$.

Example 3.4. Let $V = [v_1, \ldots, v_n]$ be an n-dimensional subspace and $S^* = \text{cone}\,(\{\phi_1, \ldots, \phi_n\})$ so that S^* is proper over V. Then Corollary 2.2 tells us that any action A such that $AS^*_{|V} \subset S^*_{|V}$ guarantees $\mathcal{A}_{S^*} \neq \varnothing$. In particular, of course, there will always exist a shape-preserving projection.

In the remaining examples, we work with preserving some standard shapes.

Example 3.5. Let $A = I_n$. For integer $n \geq 2$, let $X = C^{n-1}[0,1]$ and $V = \prod_n = [1, t, \ldots, t^n]$. Define the shape $S_0^* = \{\delta_t^{(n-1)}\}$ where $\delta_t^{(n-1)}$ denotes evaluation of the $(n-1)$st derivative at t with $t \in [0,1]$. Note, for example, that for $n = 2$ we are preserving monotonicity onto the quadratics; $n = 3$ corresponds to preserving convexity onto the cubics, etc. We will show quite easily that $\mathcal{P}_{S^*} \neq \varnothing$. To begin, we observe that our shape is not total, with respect to V. Indeed, note $\langle t^r, \delta_\tau^{(n-1)} \rangle = 0$ for $r \leq n-2$. However, with respect to $V' = [t^{n-1}, t^n]$, the shape is total, as well as strictly proper. Furthermore, $S^*_{|V'}$ is simplicial since

$$(\delta_t^{(n-1)})_{|V} = t(\delta_1^{(n-1)})_{|V} + (1-t)(\delta_0^{(n-1)})_{|V}$$

and thus, using Theorem 2.2 and Theorem 2.3, we have the existence of a unique shape-preserving projection onto V'. We can construct the projection, onto V', by setting

$$(u_1, u_2) = (\phi_1, \phi_2)\begin{pmatrix} 1 & -1/n \\ 0 & 1/n \end{pmatrix}$$

where

$$\phi_i = \frac{\delta_{i-1}^{(n-1)}}{(n-1)!}$$

and writing $P_0 = u_1 \otimes t^{n-1} + u_2 \otimes t^n$. Because $\prod_{n-2} \subset (S^*)^\perp$, shape-preserving projections onto $\prod_n$ are plentiful; for example,

$$P_1 = P_0 + \frac{1}{(n-2)!}\delta_0^{(n-2)} \otimes t^{n-2} + \cdots + \delta_0 \otimes 1$$

is one such operator.

Example 3.6. (See also [**6**].) Let $A = I_3$, $X = C[0,1]$, $V = \Pi_2 = [1, t, t^2]$ and $S_0^* = \{\delta_t\}$ where $t \in [0,1]$. We will show, using Theorem 2.2, that there does not exist any 'positive-preserving' projections onto the quadratics by demonstrating that the cone $S^*_{|V}$ is not simplicial. It is easy to see that the shape generated by S^*_0 is simplicial and proper. As we have seen from the proof of Theorem 2.2, the edges of $S^*_{|V}$ are contained in the rays generated by $(S^*_0)_{|V}$. Fix $(\delta_{t_1}, \delta_{t_2}, \delta_{t_3}) \subset S^*_0$ with distinct t_1, t_2, t_3. Fix $\delta_t \in S^*_0$ with t different than any t_i, $i = 1, 2, 3$. Set $\mathbf{v} = (v_1, v_2, v_3)^T = (1, t, t^2)^T \subset V^3$, and let

$$M = (\langle \delta_{t_i}, v_j \rangle) = \begin{pmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{pmatrix}.$$

Then $(\delta_t)_{|V} = (\delta_{t_1}, \delta_{t_2}, \delta_{t_3})\boldsymbol{\lambda}_{\delta_t}$ where

$$\begin{aligned} \boldsymbol{\lambda}_{\delta_t} &= M^{-1}\langle \delta_t, \mathbf{v} \rangle \\ &= \begin{pmatrix} (t^2 - t(t_2 + t_3) + t_2 t_3)/(t_2 t_3 - t_2 t_1 + t_1^2 - t_3 t_1) \\ (t^2 - t(t_1 + t_3) + t_1 t_3)/(t_1 t_3 - t_2 t_1 + t_2^2 - t_3 t_2) \\ (t^2 - t(t_2 + t_1) + t_2 t_1)/(-t_1 t_3 + t_2 t_1 + t_3^2 - t_3 t_2) \end{pmatrix}. \end{aligned}$$

Now we can simply observe that $\boldsymbol{\lambda}_{\delta_t}$ does not have positive entries for all t. Indeed, note that the numerator of the first entry is positive at $t = 0$ and negative at $t = (t_2 + t_3)/2$. Therefore,the cone $(S^*)_{|V}$ is not simplicial and this implies by Theorem 2.2 that $\mathcal{P}_{S^*} = \varnothing$ (it can be shown that, in fact, the cone $S^*_{|V}$ has infinitely many edges). Note that, in the above, we of course have $S^*_0 \subset X^*$; indeed, the above argument holds for any X such that this is true. For example, there cannot exist a positivity-preserving projection onto Π_2 from $X = C^m[0, 1]$, $m \geq 0$, where $\|f\|_X = \max_{i=0,\dots,m} \sup_{t\in[0,1]} \|f^{(i)}(t)\|$, see Example 3.3.

Example 3.7. In the previous example we demonstrated that positivity could not be preserved from $X = C^m[0, 1]$ onto Π_2, for any $m = 0, 1, 2, \dots$. However, by making the shape slightly more restrictive, the theory reveals a projection preserving the new shape. Fix $\mathbf{v} = (1, t, t^2)^T$ and again consider the cone $S^*_{|\mathbf{v}} \subset \mathbf{R}^3$ from Example 3.6. Clearly, every $\langle \mathbf{v}, \delta_t \rangle$ belongs to an edge of this cone. However, let $\phi \in X^*$ be any functional such that $\langle \mathbf{v}, \phi \rangle^T = (1, 1, 0)^T$. Then every ray generated by $\langle \mathbf{v}, \delta_t \rangle$ is in the convex hull of $\{\langle \mathbf{v}, \delta_0 \rangle, \langle \mathbf{v}, \delta_1 \rangle, \langle \mathbf{v}, \phi \rangle\}$. Defining $T^* = \operatorname{cone}(S^* \cup \{\phi\})$, we have that $(T^*)_{|V}$ is simplicial. For another example, let $m \geq 1$ and define $\phi = \delta_0 + \delta_0'$. The shape given by $\{\delta_t\}_{t\in[0,1]} \cup \{\phi\}$ can be preserved onto Π_2 by a projection.

In the next few examples, we will be using actions other than the identity. One will note, however, that, though the actions used are still in some sense 'close' to the identity, the existence results are quite different from the similar projection cases.

One such action in which we will be interested is given by the second-degree Bernstein operator considered above in Example 2.1. With respect to the basis $1, t, t^2$, we find the action matrix to be

$$B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

Thus, this is the action given by the restriction of the second-degree Bernstein operator to the quadratics.

Example 3.8. Let $A = B_2$, $X = C[0,1]$, $V = \Pi_2 = [1, t, t^2]$ and $S_0^* = \{\delta_t\}$ where $t \in [0,1]$. We will show that $\mathcal{A}_{S^*} \neq \varnothing$ by appealing to Corollary 2.2. Let $\mathbf{v} = (1, t, t^2)^T$ and note

$$A(S_0^*)_{|V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} = \begin{pmatrix} 1 \\ t \\ (1/2)(t+t^2) \end{pmatrix}.$$

We attempt to find $\{u_{1|_V}, u_{2|_V}, u_{3|_V}\} \subset S^*_{|V}$ so that $A(S_0^*)_{|V} \subset$ cone $(u_{1|_V}, u_{2|_V}, u_{3|_V})$ (by examining the parametric curve $A(S_0^*)_{|V}$, this choice of $u_{i|_V}$'s is clear). Let $\mathbf{u} = (\delta_{0|_V}, \delta_{1/2_{|V}}, \delta_{1|_V})$ and write $A(S_0^*)_{|V} = A\langle \mathbf{v}, \delta_t\rangle = \mathbf{u}\mathbf{c}_{\delta_t} = \langle \mathbf{v}, \mathbf{u}\rangle \mathbf{c}_{\delta_t}$ and, solving for $\mathbf{c}_{\delta_t}$, we have

$$\begin{aligned} \mathbf{c}_{\delta_t} &= (\langle \mathbf{v}, \mathbf{u}\rangle)^{-1} A \langle \mathbf{v}, \delta_t\rangle \\ &= \begin{pmatrix} 1 & -3 & 2 \\ 0 & 4 & -4 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} \\ &= \begin{pmatrix} t^2 - 2t + 1 \\ 2(t - t^2) \\ t^2 \end{pmatrix}. \end{aligned}$$

This vector has positive entries for all t and thus Corollary 2.2 implies $\mathcal{A}_{S^*} \neq \varnothing$. To actually construct a shape-preserving A-action operator, simply set $\mathbf{u} = (\delta_0, \delta_{1/2}, \delta_1)$, that is, extend the $\mathbf{u}$ above to all of X, and set $P = \mathbf{u} \otimes (\langle \mathbf{v}, \mathbf{u} \rangle)^{-1} A \mathbf{v}$, where $A = B_2$ above. It is easy to check that this is shape-preserving as well as A-action. This operator comes about naturally from the above proof and, as one can easily verify, this operator is precisely the second-degree Bernstein operator. Thus, it should preserve monotonicity as well. To check this via our theory, and to make a point about uniqueness, we first recast the monotonicity problem slightly. Let $X = C^1$, $V = [t, t^2]$, $S_0^* = \{\delta_t'\}$ and action A' be given by

$$A' = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}.$$

Note that $u_1 = 2(\delta_{1/2} - \delta_0)$ and $u_2 = 2(\delta_1 - \delta_{1/2})$ are both elements of $S^* = \overline{\text{cone}}\,(S_0^*)$. Thus, with $\mathbf{u} = (u_{1|_V}, u_{2|_V})$ and $\mathbf{v} = (t, t^2)^T$, we again write $A' \langle \mathbf{v}, \delta_t' \rangle = \mathbf{u} \mathbf{c}_{\delta_t'}$ and find

$$\mathbf{c}_{\delta_t'} = \begin{pmatrix} 3/2 & -1 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} t \\ t^2 \end{pmatrix} = \begin{pmatrix} t - t^2/2 \\ t^2/2 \end{pmatrix}.$$

By Corollary 2.2, $\mathcal{A}_{S^*} \neq \varnothing$. Extending $\mathbf{u}$ to all of X, we find that the operator $P' = \mathbf{u} \otimes (\langle \mathbf{v}, \mathbf{u} \rangle)^{-1} A' \mathbf{v}$ is a shape-preserving A'-action operator. Now simply choosing $u \in X^*$ so that $P = P' + u \otimes 1$ has the B_2 action, we find that P is a monotonicity-preserving A-action, Bernstein action, operator from $C^1[0,1]$ to Π_2. In fact, $P = P' + \delta_0 \otimes 1$ is the Bernstein operator.

The next example demonstrates that Theorem 2.3 is specific to projections.

Example 3.9. Recall from the last example that $S_0^* = \{\delta_t'\}$ on $V = [t, t^2]$ in $X = C^1$ defines a strictly proper shape. We then defined a shape-preserving A'-action operator, P'. Now, in a similar manner, we will construct a second A'-action operator that is also shape-preserving. Let $\mathbf{u} = (\delta_0', \delta_1')$ and $\mathbf{v} = (t, t^2)^T$. Set $N = \langle \mathbf{v}, \mathbf{u} \rangle$

and define $Q = \mathbf{u} \otimes N^{-1}A'\mathbf{v}$. To show Q is shape-preserving, note that

$$\begin{aligned} Q^*\delta'_t &= \mathbf{u}\langle N^{-1}A'\mathbf{v}, \delta'_t\rangle \\ &= \mathbf{u}\begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 2t \end{pmatrix} \\ &= \mathbf{u}\frac{1}{4}\begin{pmatrix} 3-2t \\ 1+2t \end{pmatrix}, \end{aligned}$$

which, for every $t \in [0,1]$, is a nonnegative combination of δ'_0 and δ'_1 and, as such, is an element of S^*. This implies Q is shape-preserving. Clearly $Q \neq P'$ and we have demonstrated that, even with a strictly proper shape, nonuniqueness exists for actions other than the identity.

The next example further illustrates the geometric nature of shape preservation. We consider a family of actions that are 'close' to the identity action and look for those actions which can preserve positivity onto the first degree trigonometric polynomials.

Example 3.10. Let $X = C[0,\pi]$, $V = [1, \sin t, \cos t]$ and $S_0^* = \{\delta_t\}$, $t \in [0,\pi]$. Thus, the shape which we wish to preserve is positivity. The cone $S^*_{|_V}$ is not a simplicial cone of V^*; in fact, each $(\delta_t)_{|_V}$ belongs to an edge. Thus, no projection can preserve positivity onto V. However, we might ask if there exist actions 'close' to the identity that can preserve the shape S^*. For $c > 0$, let

$$A_c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}.$$

We want to apply Corollary 2.2 and thus we let $\mathbf{v} = (1, \sin t, \cos t)^T$ and define $S^*_{|_V} = \{\langle \mathbf{v}, u\rangle | u \in S^*\} \subset R^3$. Each edge of this cone is a ray through a point $(1, \sin t, \cos t)$, $t \in [0,\pi]$, i.e., each edge is formed by taking all nonnegative scalar multiples of each (column) vector $\langle \mathbf{v}, \delta_t\rangle$. Indeed, this 'half-circle' set of vectors $E = (S_0^*)_{|_V} = \{(1, \sin t, \cos t)^T\}$, $t \in [0,\pi]$ generates the cone $S^*_{|_V}$. Note that E and A_cE, matrix multiplication of each $(1, \sin t, \cos t)^T \in E$ by A_c, form 'concentric half-circles.' Hence, by Corollary 2.2, we must have $c < 1$ if we want to preserve shape. With the sufficiency of Corollary 2.2 in mind, we seek the largest concentric half-circle that can be inscribed in the unit half-circle so that the inscribed half-circle is contained in the convex hull of three elements of the unit half-circle. The answer is, obviously, a half-circle of radius $r = 1/\sqrt{2}$. Thus, if $c \leq 1/\sqrt{2}$, Corollary 2.2 implies $\mathcal{A}_{cS^*} \neq \varnothing$.

We would like to conclude with an example that gives a shape-preserving A-action such that $AS^*_{|V}$ cannot be contained in a simplicial subcone of $S^*_{|V}$.

Example 3.11. Let X be a Banach space with three-dimensional subspace $V = [v_1, v_2, v_3]$ and dual space X^*. We define the shape using four dual elements. Choose $\phi_1, \phi_2, \phi_3 \in X^*$ so that $\langle v_i, \phi_j \rangle = \delta_{ij}$. Choose a fourth element ϕ_4 so that

$$\langle v_1, \phi_4 \rangle = -1 \quad \text{and} \quad \langle v_2, \phi_4 \rangle = \langle v_3, \phi_4 \rangle = 1,$$

thus $S_0^* = \{\phi_i\}_{i=1}^4$. Let the action be given by

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

To show $\mathcal{A}_{S^*} \neq \varnothing$, we appeal to Theorem 2.1; thus we must find $\mathbf{u} = (u_1, u_2, u_3) \in (X^*)^3$ such that $\mathbf{u}A\boldsymbol{\lambda}_{\phi_i} \in S^*$ for $i = 1, \dots, 4$. Let

$$u_1 = \phi_1 + \phi_2, \qquad u_2 = \phi_1 + \phi_3 \quad \text{and} \quad u_3 = \phi_2 + \phi_4.$$

Note that $A = \langle \mathbf{v}, \mathbf{u} \rangle$. From here it follows that $A\boldsymbol{\lambda}_{\phi_i} = \langle \mathbf{v}, \phi_i \rangle$. Thus

$$\begin{aligned} \mathbf{u}A\boldsymbol{\lambda}_{\phi_1} &= \phi_1 + \phi_2, \\ \mathbf{u}A\boldsymbol{\lambda}_{\phi_2} &= \phi_1 + \phi_3, \\ \mathbf{u}A\boldsymbol{\lambda}_{\phi_3} &= \phi_2 + \phi_4, \end{aligned}$$

and

$$\mathbf{u}A\boldsymbol{\lambda}_{\phi_4} = \phi_3 + \phi_4.$$

Thus $\mathcal{A}_{S^*} \neq \varnothing$. The reason this example is of interest is that $A\boldsymbol{\lambda}_{\phi_4}$ has negative entries, i.e., geometrically, the subcone $AS^*_{|V}$, of $S^*_{|V}$, has four edges and it cannot be contained in a simplicial (three-edged) subcone of $S^*_{|V}$.

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Department of Mathematics, University of California, Riverside, CA 92521-0135
E-mail address: blc@ucrmath.ucr.edu

Department of Mathematics, Murray State University, Murray, KY 42071
E-mail address: mike@banach.mursuky.edu