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ON EQUAL SUMS OF CUBES

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ABSTRACT. The complete solution in positive or negative rationals of the Diophantine equation $x^3 + y^3 = u^3 + v^3$ was found by Euler. However, the complete solution in integers of this equation and of the related equation $x^3+y^3+z^3 = t^3$ has not been found earlier. This paper gives a complete solution of these equations in positive or negative integers as well as a complete solution in positive integers only.

Introduction. The complete solution of the Diophantine equation

$$x^3 + y^3 = u^3 + v^3 \tag{1}$$

in rational numbers, whether positive or negative, was first given by Euler. Writing z, t for $-u, v$ respectively in (1), we get the related equation

$$x^3 + y^3 + z^3 = t^3. \tag{2}$$

Euler's solution has been presented by Hardy and Wright [**1**, pp. 199–200] who have stated that these equations give rise to a number of different problems, since we may look for solutions in (a) integers or (b) rationals and we may or may not be interested in the signs of solutions. They have further indicated that the complete solution of these equations in integers is not known. Hua Loo Keng [**2**, p. 290], while giving the complete rational solution of the equations, has also stated that, "The solutions to the equation $x^3+y^3+z^3+w^3 = 0$ present a very interesting problem. Unfortunately we still have not obtained a formula for all the solutions."

When we are not interested in the signs of the solutions, both equations (1) and (2) are equivalent to the equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0. \tag{3}$$

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We give below a new solution of (3) which will be shown to be a complete solution in integers whether positive or negative. This solution will then be used to obtain:

(i) the complete solution of (1) in positive integers, and

(ii) the complete solution of (2) in positive integers.

As the equation (3) is homogeneous, any integer solution (x_1, x_2, x_3, x_4) of (3) leads to another integer solution (kx_1, kx_2, kx_3, kx_4) where $k \neq 0$ is an integer. There is, therefore, no loss of generality in considering the equation (3) with $(x_1, x_2, x_3, x_4) = 1$. Similar remarks apply to equations (1) and (2).

Theorem 1. *The complete integer solution of the equation*

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0, \quad (x_1, x_2, x_3, x_4) = 1$$

is given by

$$(4) \qquad \begin{cases} dx_1 = (a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4) + (2a + b)c^3 \\ dx_2 = -\{a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4 - (a - b)c^3\} \\ dx_3 = c(-a^3 + b^3 + c^3) \\ dx_4 = -\{(2a^3 + 3a^2b + 3ab^2 + b^3)c + c^4\}, \end{cases}$$

where a, b, c are arbitrary integers and $d \neq 0$ is an integer so chosen that $(x_1, x_2, x_3, x_4) = 1$.

Proof. To solve the equation (3), we write

$$(5) \qquad x_1 = (a + b)\theta + a, \qquad x_2 = a\theta - b, \qquad x_3 = c\theta, \qquad x_4 = -c.$$

Substituting in (3), we get the equation

$$(6) \qquad \{(a + b)\theta + a\}^3 + (a\theta - b)^3 + (c\theta)^3 - c^3 = 0$$

or

$$(7) \qquad (\theta^2 + \theta + 1)[\{a^3 + (a + b)^3 + c^3\}\theta + a^3 - b^3 - c^3] = 0.$$

Thus, the only rational root of equation (6) is given by

$$(8) \qquad \theta = (b^3 + c^3 - a^3)\{a^3 + (a + b)^3 + c^3\}^{-1}$$

where we assume that $\{a^3+(a+b)^3+c^3\} \neq 0$. On substituting this value of θ in (4) we get a rational solution of (3) which on multiplying throughout by $\{a^3+(a+b)^3+c^3\}$ leads to the solution given by (4).

It is easily verified that (4) gives a solution of (3) even when $\{a^3+(a+b)^3+c^3\} = 0$. Thus, when a, b, c take integer values, (4) gives a solution in integers of the equation (3). We shall now show that this is a complete solution of (3) in integers. Thus, if (X_1, X_2, X_3, X_4) is any given solution of (1) in integers so that

$$X_1^3 + X_2^3 + X_3^3 + X_4^3 = 0, \tag{9}$$

we will show that there exist integers a, b, c such that, for these values of a, b, c, (4) generates the given solution (X_1, X_2, X_3, X_4). The solution $X_i = 0$, $i = 1, 2, 3, 4$, is generated by $a = b = c = 0$. If X_i are not all zero, then it is easily seen that at least two of the X_i will be nonzero. In view of the symmetry of equation (3), there is no loss of generality in taking $X_3 \neq 0$ and $X_4 \neq 0$. Further, we may, without loss of generality, also take $X_1 \neq -X_3$ (if necessary, by rearranging the X_i).

We choose

$$\begin{cases} a = X_2X_3 - X_1X_4 \\ b = X_1X_3 - X_2X_3 + X_2X_4 \\ c = X_3^2 - X_3X_4 + X_4^2 \end{cases} \tag{10}$$

and we write $\theta_1 = -X_3X_4^{-1}$. Then

$$\begin{aligned} (a+b)\theta_1 + a &= -X_1(X_3^2 - X_3X_4 + X_4^2)X_4^{-1}, \\ a\theta_1 - b &= -X_2(X_3^2 - X_3X_4 + X_4^2)X_4^{-1}, \\ c\theta_1 &= -X_3(X_3^2 - X_3X_4 + X_4^2)X_4^{-1}, \end{aligned}$$

and therefore we have

$$\begin{aligned} &\{(a+b)\theta_1 + a\}^3 + (a\theta_1 - b)^3 + (c\theta_1)^3 - c^3 \\ &\qquad = -(X_1^3 + X_2^3 + X_3^3 + X_4^3)(X_3^2 - X_3X_4 + X_4^2)X_4^{-3} \\ &\qquad = 0. \end{aligned}$$

This shows that equation (6) where a, b, c are integers given by (10) has the rational root θ_1. Since (6) can have only one rational root, the root given by (8) must be the same as θ_1. Thus we must have

$$\frac{b^3+c^3-a^3}{a^3+(a+b)^3+c^3} = -\frac{X_3}{X_4}$$

or

$$\frac{b^3+c^3-a^3}{X_3} = -\frac{a^3+(a+b)^3+c^3}{X_4} = k. \tag{11}$$

The conditions $X_3 \neq 0$, $X_4 \neq 0$, $X_1 \neq -X_3$ ensure that k is nonzero. Moreover, using the relations (10) and subsequently (9),it is easily established that k is a nonzero integer. Finally, using the relations (11), we find that when a, b, c are given by (10), the solution given by (4) is as follows:

$$\begin{aligned} dx_1 &= (a+b)(b^3+c^3-a^3) + a\{a^3+(a+b)^3+c^3\} \\ &= (X_1X_3 - X_1X_4 + X_2X_4)(kX_3) \\ &\quad - (X_2X_3 - X_1X_4)(kX_4) \\ &= kX_1(X_3^2 - X_3X_4 + X_4^2). \end{aligned}$$

Similarly,

$$\begin{aligned} dx_2 &= kX_2(X_3^2 - X_3X_4 + X_4^2) \\ dx_3 &= kX_3(X_3^2 - X_3X_4 + X_4^2) \\ dx_4 &= kX_4(X_3^2 - X_3X_4 + X_4^2). \end{aligned}$$

We take $d = k(X_3^2 - X_3X_4 + X_4^2)$. As $k \neq 0$, $X_3 \neq 0$, $X_4 \neq 0$, d is a nonzero integer. It follows that the solution of (3) where a, b, c are integers given by (10) is the given solution (X_1, X_2, X_3, X_4). This completes the proof. □

Theorem 2. *The complete solution in positive integers of the equation $x^3 + y^3 = u^3 + v^3$ where $(x, y, u, v) = 1$ is given by*

$$\begin{cases} dx = (a^4+2a^3b+3a^2b^2+2ab^3+b^4) + (2a+b)c^3 \\ dy = c(-a^3+b^3+c^3) \\ du = a^4+2a^3b+3a^2b^2+2ab^3+b^4 - (a-b)c^3 \\ dv = (2a^3+3a^2b+3ab^2+b^3)c + c^4, \end{cases} \tag{12}$$

where a, b, c are positive integers such that either (i) $a < b$ *or* (ii) $a > b$ *and c is chosen so that*

$$(a^3-b^3)^{1/3} < c < \left\{\frac{a^4+2a^3b+3a^2b^2+2ab^3+b^4}{a-b}\right\}^{1/3} \tag{13}$$

and $d > 0$ *is taken so that* $(x, y, u, v) = 1$.

We note that, given any two positive integers a, b *such that* $a > b$, *we can always find integers* c *satisfying the inequalities* (13). *As an example, we may take* $c = a$ *when* (13) *holds.*

Proof. The solution (12) of equation (1) is obtained by replacing x_1, x_2, x_3, x_4 by $x, -u, y, -v$, respectively, in equation (3) and its solution (4). It is accordingly the complete solution of (1). It is easy to verify that, when a, b, c are positive integers satisfying the above conditions, then (12) gives a solution of (1) in positive integers. On the other hand, let (X, Y, U, V) be any given solution of (1) in positive integers. On account of the symmetry of equation (1), there is no loss of generality in assuming that $X > Y$ and $U < V$. Using the results of the previous section, we find that the solution (X, Y, U, V) is generated by taking

$$\begin{aligned} a &= XV - UY \\ b &= UV + UY + XY \\ c &= V^2 + VY + Y^2. \end{aligned}$$

It is clear that a, b, c are positive integers. Further, the solution (x, y, u, v) generated by these values of a, b, c is given by

$$\begin{aligned} du &= a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4 - (a-b)c^3 \\ &= kU(Y^2 + YV + V^2) > 0, \end{aligned}$$

since $k = \{a^3 + (a+b)^3 + c^3\}/V > 0$. It follows that either (i) $a < b$ or (ii) $a > b$ and

$$c < \left\{\frac{a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4}{a-b}\right\}^{1/3}.$$

Similarly,

$$dy = c(-a^3 + b^3 + c^3) = kY(Y^2 + YV + V^2) > 0$$

shows that either (i) $a < b$ or (ii) $a > b$ and $c > (a^3 - b^3)^{1/3}$.

This finally proves that, given any solution (X, Y, U, V) in positive integers, there exist positive integers a, b, c satisfying the conditions of the theorem such that (12) generates the given solution (X, Y, U, V) in positive integers. This completes the proof. $\square$

Theorem 3. *The complete solution in positive integers of the equation* $x^3 + y^3 + z^3 = t^3$ *where* $(x, y, z, u) = 1$ *is given by*

$$\tag{14} \begin{cases} dx = c(-a^3 - b^3 + c^3) \\ dy = -a^4 + 2a^3b - 3a^2b^2 + 2ab^3 - b^4 + (a+b)c^3 \\ dz = a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4 + (2a-b)c^3 \\ dt = c\{a^3 + (a-b)^3 + c^3\} \end{cases}$$

where a, b, c *are positive integers such that* $a > b$ *and*

$$c > (a^3 + b^3)^{1/3},$$

and, as before, $d > 0$ *is taken so that* $(x, y, z, t) = 1$.

Proof. The solution (14) of equation (2) is derived by replacing x_1, x_2, x_3, x_4 by $z, y, x, -t$, respectively, in (3) and in its complete solution given by (4) and by replacing b by $-b$ in (4). It is therefore the complete solution of (2) in integers.

It is easily seen that, when a, b, c are positive integers such that $a > b$ and $c > (a^3 + b^3)^{1/3}$, the solution given by (14) is in positive integers. Conversely, let (X, Y, Z, T) be any given solution of (2) in positive integers. There is no loss of generality in assuming that $X < Y < Z$. As in the case of equation (3), we find that this solution is generated by the following values of the parameters,

$$\begin{aligned} a &= TZ + XY, \\ b &= TY + XY - XZ, \\ c &= T^2 + TX + X^2. \end{aligned}$$

It is easily seen that a, b, c are positive integers with $a > b$. Further, the solution generated by (14) gives

$$dx = c(c^3 - a^3 - b^3) = k(T^2 + TX + X^2)X > 0,$$

since $k = \{a^3 + (a-b)^3 + c^3\}/T > 0$. It follows that

$$c > (a^3 + b^3)^{1/3}.$$

Thus, there exist positive integers which satisfy the conditions of the theorem and which generate the given solution (X, Y, Z, T) in positive integers. This completes the proof. $\square$

REFERENCES

1. G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, Fifth edition, Oxford University Press, London, 1979.

2. Hua Loo Keng, *Introduction to number theory*, Springer-Verlag, Berlin, 1982.

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