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DIFFERENTIAL OPERATORS OVER C^*-ALGEBRAS

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1. Introduction. The study of differential operators over C^*-algebras was initiated by Miščenko and Kasparov in their work on the Novikov conjecture. The setup is as follows. Let M be a closed smooth manifold. Let A be a unital C^*-algebra. We consider A-linear partial differential operators on sections of bundles of finitely generated Hilbert A-modules over M. The goal is to develop an index theory where the index takes values in the K-groups of A. A prerequisite is the availability of a package of basic analytic facts concerning domains, ranges and compactness of operators. These are most conveniently expressed in terms of Sobolev spaces. This matter has been handled by several authors in a fairly ad hoc way. (This will be discussed further below.) The purpose of this paper is to give a rather general treatment, including the case of manifolds with boundary. This extension is somewhat speculative, since to date no work has been done on C^*-algebraic index theory on manifolds with boundary.

The main difficulty involved is one which is ubiquitous in working with operators over C^*-algebras: bounded operators between Hilbert A-modules often don't have everywhere-defined adjoints. When this happens, analysis on the usual pattern can't be carried out. We will establish the existence of adjoints for A-linear differential (and pseudodifferential) operators between Sobolev spaces by using classical results on the Dirichlet problem over $\mathbf{C}$.

Let ε_P be a trivial bundle over $\mathbf{R}^n$ with fiber a finitely generated Hilbert A-module P. The Sobolev space $W_k(\varepsilon_P)$ for an integer $k > 0$ consists of $\mathcal{L}^2$ sections of ε_P with k distributional derivatives in $\mathcal{L}^2$, with an inner product derived from these derivatives. For an A-module bundle V over a manifold with boundary M the definition is globalized using local trivializations and a partition of unity to obtain $W_k(V)$. For manifolds with boundary, one also has $W_k^0(V)$, the subspace satisfying zero Dirichlet boundary conditions in a weak sense, see Section 3.1. Most of this paper is concerned with these spaces. I don't know whether

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similar results hold for $W_k(V)$. In the case $A = \mathbf{C}$, this works well. For general A, it isn't clear that the inner products resulting from this pasting process have any useful properties with respect to adjoints. There seem to be no direct proofs of the adjointability of differential operators, or of the theorem of Rellich on compactness of inclusions, for example.

An alternative approach is via self-adjoint extensions of differential operators. Let D_0 be a formally self-adjoint A-linear elliptic differential operator of order $2k$ on V. Assume that D_0 is elliptic with respect to Dirichlet boundary conditions. (These are given by the normal derivatives of orders $0, \dots, k-1$.) Let D be its extension with domain $\mathcal{D}(D) = F_{2k}(V) := W_{2k}(V) \cap W_k^0(V)$. The basic facts we will generalize are as follows: if $A = \mathbf{C}$, then D is self-adjoint. If $(D_0 u, u) \geq (u, u)$, the spectrum of D is contained in $[1, \infty)$. In particular, D is bijective and we can define a Hilbert space $\mathcal{E}_+$ as $\mathcal{D}(D^{1/2})$ with the inner product $(u, v)_+ = (D^{1/2}u, D^{1/2}v)$. This turns out to be $W_k^0(V)$. For arbitrary A there is a problem. All general proofs of these facts require some sort of application of the Riesz representation theorem, which is equivalent to the existence of adjoints for bounded operators.

Particular cases of this problem, for closed manifolds, were dealt with by Kasparov [**11, 12**], and by Rosenberg, Skandalis and Weinberger [**19**, Section 1]. In this case the situation simplifies considerably. For closed manifolds $W_k(V) = W_k^0(V)$ so that $F_{2k}(V) = W_{2k}^0(V)$. If D_0 is of second order such that $\mathcal{D}(D^{1/2}) = W_1^0(V)$, then $\mathcal{D}(D^{k/2}) = W_k^0(V)$. Therefore it suffices to consider second order operators. The relevant results of these papers are mainly applicable to a covering space situation. Take $A = C^*(\pi_1(M))$, the group C^*-algebra, and ψ the flat bundle associated to the universal covering space of M by the representation $\pi_1(M) \to C^*(\pi_1(M))$. Let E be a complex vector bundle. The operators D_0 are obtained from differential operators on E extended to $\psi \otimes E$ using the flat connection of ψ. D_0 is actually of the form $T_0^{\#} T_0 + I$, with T_0 first order. ($T_0^{\#}$ denotes the formal adjoint.) In this situation, D has the desired properties.

The approach adopted in [**19**] is perturbative and has strongly influenced this paper. (This method has also been used by U. Bunke.) We give a very rough sketch of the ideas involved, which doesn't accurately describe the work in [**19**]. The essential point is that the principal symbol of D_0 is complex, in the sense that it is of the form $I_\psi \otimes \tau$, where

τ is a symbol on E. Such an operator is a direct summand of an operator $\tilde{D}_0$ on some $\varepsilon \otimes E$ with symbol $I_\varepsilon \otimes \tau$. $\tilde{D}_0$ is then a lower order perturbation of the extension to $\varepsilon \otimes E$ of a differential operator on E. Because of the special form of D_0, only bounded perturbations have to be considered.

For manifolds with boundary it is necessary to consider operators of arbitrary even order. Therefore, the more elaborate perturbation theory of Rellich and Kato is required. The conclusion is that, for operators D_0 with complex principal symbol, D has the same properties as in the case $A = \mathbf{C}$ above, Theorem 4.1. This suffices to describe all customary constructions of $W_k^0(V)$ in the form $\mathcal{E}_+$ for some D, Proposition 5.2. It is an interesting question whether the conclusions concerning D are true without the assumption of complex symbol. Theorem 4.3 gives an affirmative answer when A is commutative, i.e., for the case of families of operators.

The properties of the spaces $W_k^0(V)$ are developed within the framework of "scales of spaces." This originally was developed as a setting for the Friedrichs extension and generalizations, such as the Lax-Milgram theorem, which exploit the Riesz representation theorem to obtain weak solutions for differential operators. Here the situation is reversed. It follows from the above results that an extension of D realizes the isomorphism between $W_k^0(V)$ and its dual, which allows one to construct adjoints for many operators on $W_k^0(V)$. We show in Theorems 5.3 and 5.4 that a class of inner products on $W_k^0(V)$ which includes all of the usual ones are compatible, in the sense that the identity map between any two of them has an adjoint. Therefore, there is a preferred compatibility class of inner products which depends only on the smooth structure of the manifold. Further, a properly supported pseudodifferential operator between any two has an adjoint, and Rellich's theorem holds. I believe that many of these results are new even for closed manifolds.

Knowledge of Hilbert modules and related matters is assumed. The necessary material is in [**22**]. Proofs cited from the literature, unless otherwise noted, are either valid for operators on Hilbert modules or require only minor modifications.

The next section sets up a general framework of Hilbert module structures on domains of operators and establishes the basic duality and its consequences concerning adjoints. It also contains a discussion of connections with regular operators in the sense of Baaj and Julg. Sections 3 and 4 investigate the spectral properties of certain extensions of A-linear differential operators. The main results on Sobolev spaces are in Section 5.

2. Abstract Sobolev spaces. This section contains the construction of spaces defined by certain self-adjoint operators over C^*-algebras, which have many of the properties of Sobolev spaces. The framework was developed during the 1950's by a rather large number of mathematicians. The approach here is very similar to that of Berezanskii [**4**, Chapter 1], although the intent is quite different. Also see this reference for attributions. There is a close connection with the work of Baaj and Julg on unbounded Kasparov modules [**3**], which will be explained in Section 2.2.

2.1. Let A be a unital C^*-algebra and $\mathcal{E}$ a Hilbert A-module. We will work with unbounded A-linear operators D on $\mathcal{E}$ which are symmetric (so in particular have dense domain $\mathcal{D}(D)$), with real spectrum $\sigma(D)$ bounded below by some real α. (Shortly we will take $\alpha > 0$.) This implies that D is self-adjoint [**23**, Theorem 5.19]. However, there are self-adjoint operators on Hilbert modules whose spectrum isn't real [**9**, Section 2.9].

Lemma 2.1. *The conditions above are equivalent to the following: D is densely defined with a nonempty resolvent set $\rho(D)$ and $(Du, u) \geq \alpha(u, u)$ for all $u \in \mathcal{D}(D)$.*

Proof. Assuming the given conditions, let $\beta < \alpha$. Then $\sigma(D - \beta) \subset [\alpha - \beta, \infty)$, so $D - \beta$ is bijective and $\sigma((D - \beta)^{-1})$ is nonnegative. $(D - \beta)^{-1}$ is self-adjoint and bounded, so $(D - \beta)^{-1/2}$ exists and is self-adjoint. For $v \in \mathcal{E}$,

$$(v, (D - \beta)^{-1}v) = ((D - \beta)^{-1/2}v, (D - \beta)^{-1/2}v) \geq 0.$$

Then if $v = (D - \beta)u$,

$$(Du, u) - \beta(u, u) = ((D - \beta)u, u) \geq 0.$$

Therefore, $(Du,u) \geq \alpha(u,u)$.

If (Du,u) is self-adjoint, then D is symmetric [**23**, Theorem 4.18]. Since $\rho(D) \neq \varnothing$, by a standard continuation argument [**23**, Theorem 5.21], $\rho(D)$ contains the upper or lower half-plane. By [**1**, Theorem 12.8], the condition $(Du,u) \geq \alpha(u,u)$ implies that $\rho(D)$ contains $\{z \mid \mathcal{R}z < \alpha\}$. Continuation then shows that $\sigma(D) \subset [\alpha, \infty)$. $\square$

If either of the equivalent sets of conditions is satisfied, we will say that D is bounded below by α. Henceforth we assume that this holds for some $\alpha > 0$.

The proof in [**1**] also shows that $\|(D+\lambda)^{-1}\| \leq (\mathcal{R}\lambda + \alpha)^{-1}$, $\mathcal{R}\lambda \geq 0$. Making use of this, Kato [**13**, Theorem 3.35] shows that D has a unique square root with the same properties, except that it is bounded below by $\alpha^{1/2}$.

Let $\mathcal{E}_+$ be $\mathcal{D}(D^{1/2})$ equipped with the inner product $(u,v)_+ = (D^{1/2}u, D^{1/2}v)$. $\mathcal{E}_+$ is complete since $D^{1/2}$ is closed. Since $\mathcal{D}(D)$ is a core for $D^{1/2}$ [**13**, Lemma 3.38], $\mathcal{D}(D)$ is dense in $\mathcal{D}(D^{1/2})$ with respect to the same inner product. Then $(u,v)_+ = (Du,v)$ for $u,v \in \mathcal{D}(D)$. Note that $D^{1/2} : \mathcal{E}_+ \to \mathcal{E}$ is unitary. The inclusion $\mathcal{E}_+ \to \mathcal{E}$ is bounded, for if $u \in \mathcal{D}(D)$, $\|u\|_+^2 = \|(Du,u)\| \geq \alpha\|u\|^2$.

We might, as is customary, define the negative space $\mathcal{E}_-$ as the representable (anti-)dual $\mathcal{E}'_+$ of $\mathcal{E}_+$, but this can be given a more concrete description. $D^{1/2} : \mathcal{D}(D^{1/2}) \to \mathcal{E}$ is a bijection with bounded inverse. Equip $\mathcal{E}$ with the inner product $(u,v)_- = (D^{-1/2}u, D^{-1/2}v)$. Let $\mathcal{E}_-$ be the completion of $\mathcal{E}$ with respect to this. The inclusion $\mathcal{E} \to \mathcal{E}_-$ is bounded, for if $u \in \mathcal{E}$, $\|u\|_- = \|D^{-1/2}u\| \leq K\|u\|$ since $D^{-1/2}$ is bounded. $D : \mathcal{D}(D) \to \mathcal{E}$ is isometric for the inner products $(\cdot,\cdot)_+$ and $(\cdot,\cdot)_-$ since

$$(Du, Dv)_- = (D^{-1/2}Du, D^{-1/2}Dv) = (D^{1/2}u, D^{1/2}v),$$

so it extends to a unitary $\overline{D} : \mathcal{E}_+ \to \mathcal{E}_-$. There is a natural unitary $\varphi : \mathcal{E}_+ \to \mathcal{E}'_+$, $(\varphi u)(v) = (u,v)_+$. Then $\varphi\overline{D}^{-1} : \mathcal{E}_- \to \mathcal{E}'_+$ is unitary. If $w \in \mathcal{E}$, $w = Du$ with $u \in \mathcal{D}(D)$, then

$$(\varphi\overline{D}^{-1}w, v) = (u,v)_+ = (Du,v) = (w,v),$$

so $\varphi\overline{D}^{-1}$ is the bounded extension of $w \to (w,\cdot)$.

We will construct adjoints for a class of operators between these spaces, which includes differential (and pseudodifferential) operators in concrete cases. Let D_1 and D_2 be operators as above on Hilbert A-modules $\mathcal{E}_1$ and $\mathcal{E}_2$, and let $\mathcal{E}_{1,+}$ and $\mathcal{E}_{2,+}$ be the associated spaces. Suppose dense submodules S_1 of $\mathcal{E}_{1,+}$ and S_2 of $\mathcal{E}_{2,+}$ are given. Then, since $\mathcal{D}(D_i^{1/2})$ is dense in $\mathcal{E}_i$ and $\mathcal{E}_{i,+} \to \mathcal{E}_i$ is bounded, S_i is dense in $\mathcal{E}_i$. Let $T_0 : S_1 \to S_2$ be an A-linear homomorphism admitting a "formal adjoint" in the sense that there exists an A-linear $T_0^{\#} : S_2 \to S_1$ so that $(T_0 u, v)_2 = (u, T_0^{\#} v)_1$ if $u \in S_1$, $v \in S_2$.

Proposition 2.2. *If T_0 extends boundedly to $T : \mathcal{E}_{1,+} \to \mathcal{E}_{2,+}$, then $T_0^{\#}$ extends boundedly to $T^{\#} : \mathcal{E}_{2,-} \to \mathcal{E}_{1,-}$. In this situation T has the bounded adjoint for the $+$-inner products $T^* = \overline{D}_1^{-1} T^{\#} \overline{D}_2 : \mathcal{E}_{2,+} \to \mathcal{E}_{1,+}$.*

Proof. Since S_2 is dense in $\mathcal{E}_2$, $\mathcal{E}_2$ is dense in $\mathcal{E}_{2,-}$, and $\mathcal{E}_2 \to \mathcal{E}_{2,-}$ is bounded, it follows that S_2 is dense in $\mathcal{E}_{2,-}$. It is therefore sufficient to show that $T_0^{\#}$ is bounded from $\|\cdot\|_{2,-}$ to $\|\cdot\|_{1,-}$. (In what follows, omitted subscripts denote the generic case.) Since $D^{1/2}$ is self-adjoint, for $u, v \in S$,

$$\|(u,v)\| = \|(D^{1/2}u, D^{-1/2}v)\| \le \|D^{1/2}u\| \|D^{-1/2}v\| = \|u\|_+ \|v\|_-.$$

Therefore, (u, v) extends continuously $\mathcal{E}_+ \times \mathcal{E}_- \to A$, and

$$\sup_{\|u\|_+ = 1} \|(u,v)\| \le \|v\|_-.$$

To show equality, let $u = \overline{D}^{-1} v = D^{-1} v$ for $v \in S$. Since D is isometric $\|u\|_+ = \|v\|_-$. Then $\|(u,v)\| = \|(D^{-1}v, v)\| = \|v\|_-^2$, so $\|(u/\|u\|_+, v)\| = \|v\|_-$. Then S is dense in $\mathcal{E}$, and we conclude that

$$\|u\|_+ = \sup_{\|u\|_+ = 1} \|(u,v)\| = \|v\|_-, \quad u, v \in S. \tag{2.1}$$

As above, for $u \in S_1$, $v \in S_2$,

$$\|(u, T_0^{\#} v)\| \le \|T_0 u\|_{2,+} \|v\|_{2,-} \le C \|u\|_{1,+} \|v\|_{2,-}$$

by the boundedness of T_0. Then, by (2.1),

$$\|T_0^{\#}v\|_{1,-} = \sup_{\|u\|_{1,+}=1} \|(u, T_0^{\#}v)_1\| \leq C\|v\|_{2,-}$$

so $T_0^{\#}$ is bounded.

Since $\overline{D}_1^{-1}T^{\#}\overline{D}_2$ is bounded, it is enough to check the adjoint condition on a dense set. Since S_2 is dense in $\mathcal{E}_{2,-}$ and $\overline{D}_2 : \mathcal{E}_{2,+} \to \mathcal{E}_{2,-}$ is unitary, $\overline{D}_2^{-1}(S_2) = D_2^{-1}(S_2)$ is dense in $\mathcal{E}_{2,+}$. Then, for $u \in S_1$, $v \in S_2$,

$$\begin{aligned}(u, \overline{D}_1^{-1}T^{\#}\overline{D}_2 v)_{1,+} &= (u, D_1^{-1}T_0^{\#}D_2 v)_{1,+} \\ &= (D_1^{1/2}u, D_1^{1/2}D_1^{-1}T_0^{\#}D_2 v)_1 \\ &= (u, T_0^{\#}D_2 v) = (T_0 u, D_2 v) = (Tu, v)_{2,+}. \qquad \square\end{aligned}$$

We call two complete inner products on the same A-module *compatible* if the identity map has an adjoint. This is an equivalence relation; for, if k is an adjointable bijection of Hilbert modules, then k^* is bijective and $(k^{-1})^* = (k^*)^{-1}$. (This is a special case of [**22**, Theorem 15.3.8].) It follows easily from this that compatible inner products have the same adjointable and compact operators.

Now let D_1 and D_2 be two operators as above acting on the same Hilbert module $\mathcal{E}$.

Proposition 2.3. *If $\mathcal{D}(D_1) = \mathcal{D}(D_2)$, then $\mathcal{D}(D_1^{1/2}) = \mathcal{D}(D_2^{1/2})$, and the inner products $(\cdot,\cdot)_{1,+}$ and $(\cdot,\cdot)_{2,+}$ are compatible.*

Proof. Now $\mathcal{D}(D_1^{1/2}) = \mathcal{D}(D_2^{1/2})$ follows from [**23**, Theorem 9.4(b)], with a few modifications. The main inequality should be replaced by $\|Sf\| \leq 2c\|(T+I)f\|$, which is easily derived. A suitable proof of the subsidiary Theorem 9.1 can be found in [**18**, Theorem 10.12]. The fact that $\mathcal{D}((D+I)^{1/2}) = \mathcal{D}(D^{1/2})$ is contained in the proof of [**13**, Lemma 3.4]. Call the common domain $\mathcal{D}$. We verify the hypotheses of Proposition 2.2. Let $S_1 = S_2 = \mathcal{D}$ and $T_0 = I_{\mathcal{D}}$, so that $T_0^{\#} = I_{\mathcal{D}}$. It must be shown that $I_{\mathcal{D}}$ is bounded from $\|\cdot\|_{1,+}$ to $\|\cdot\|_{2,+}$. Since

$\mathcal{D}(D_1^{1/2}) = \mathcal{D}(D_2^{1/2})$, $D_2^{1/2}$ is $D_1^{1/2}$-bounded [**23**, Theorem 5.9]; that is, there exists a $C > 0$ so that $\|D_2^{1/2}u\| \leq C(\|u\| + \|D_1^{1/2}u\|)$. By assumption, for $u \in \mathcal{D}(D_1)$,

$$\|D_1^{1/2}u\| = \|(D_1^{1/2}u, D_1^{1/2}u)\|^{1/2} = \|(D_1u, u)\|^{1/2} \geq \alpha\|u\|$$

for some $\alpha > 0$. Since $\mathcal{D}(D_1)$ is a core for $D_1^{1/2}$, this holds for $u \in \mathcal{D}$. It follows that $\|u\|_{2,+} = \|D_2^{1/2}u\| \leq K\|D_1^{1/2}u\| = K\|u\|_{1,+}$. □

2.2. We have been considering operators D which are symmetric with real spectrum bounded below by some positive number. There is a construction due to Baaj and Julg [**3**] which essentially exhausts this class of operators. Although the following material will not be used in this paper, it seems helpful to clarify the connections. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert A-modules.

Definition 2.4. A regular operator from $\mathcal{E}_1$ to $\mathcal{E}_2$ is a closed A-linear mapping T such that $\mathcal{D}(T)$ and $\mathcal{D}(T^*)$ are dense, and T^*T+I has dense range.

The operator we are interested in is $D = T^*T + I$. A full account of the properties of regular operators was first published by Woronowicz. An exposition may be found in [**14**, Chapter 9]. The basic facts are as follows: If T is regular, then $T^{**} = T$ and $T^*T + I$ is surjective. T^* is also regular.

We show that D meets our requirements.

Proposition 2.5. *If T is regular, D is symmetric with real spectrum contained in $[1, \infty)$ and therefore is self-adjoint.*

Proof. That D is symmetric follows from [**14**, Proposition 9.9]. Since $(Du, u) \geq (u, u)$, D is bijective and D^{-1} is bounded, so $\rho(D) \neq \varnothing$. Lemma 2.1 shows that $\sigma(D) \subset [1, \infty)$. □

In the reverse direction, suppose that D is symmetric and $\sigma(D) \subset [\alpha, \infty)$ with $\alpha > 0$. By [**13**, Theorem 3.35], $D - \alpha$ has a self-adjoint square root with spectrum in $[0, \infty)$. Then $\alpha^{-1}D = (\alpha^{-1/2}(D - \alpha)^{1/2})^2 + I$, so up to constant multiples the operators we are concerned with arise from regular operators.

Define the T-inner product by $(u, v)_T = (Tu, Tv) + (u, v)$. The next proposition connects regular operators and the modules $\mathcal{E}_+$ defined in Section 2.1.

Proposition 2.6. *$\mathcal{D}(D^{1/2}) = \mathcal{D}(T)$ and $(\cdot, \cdot)_+ = (\cdot, \cdot)_T$ on this domain.*

Proof. By [**14**, Lemma 9.2], $\mathcal{D}(T^*T)$ is a core for T. By [**13**, Theorem 3.35], it is also a core for $(T^*T)^{1/2}$. Thus $\mathcal{D}(T)$ is the set of $u \in \mathcal{E}_1$ such that there exists $(u_i) \subset \mathcal{D}(T^*T)$ with $u_i \to u$ and (Tu_i) Cauchy. By [**13**, Lemma 3.41], $\mathcal{D}(D^{1/2}) = \mathcal{D}((T^*T + I)^{1/2}) = \mathcal{D}((T^*T)^{1/2})$. Therefore $\mathcal{D}(D^{1/2})$ is similarly described, with $((T^*T)^{1/2}u_i)$ Cauchy. But for $v \in \mathcal{D}(T^*T)$, $\|Tv\| = \|(T^*T)^{1/2}v\|$, so the domains are the same. On $\mathcal{D}(T^*T)$, $(\cdot, \cdot)_+ = (\cdot, \cdot)_T$, so this holds on the common domain. □

3. Differential operators over A.

3.1. Our goal is to extend to elliptic operators with coefficients in a C^*-algebra some of the classical results on the Dirichlet problem. We start by reviewing differential operators and Sobolev spaces on a manifold with boundary in a slightly more general context than is usual. Details and proofs may be found in [**8**, Chapter 1] and [**6**, Chapter 2]. Let A be a C^*-algebra with unit (perhaps $\mathbf{C}$). Let M^n be a compact connected Riemannian manifold with boundary ∂M and V a smooth Hermitian A-module bundle over M with fiber a finitely generated Hilbert A-module P. Let $C_0^\infty(V)$ be the smooth sections of V with support in $\overset{\circ}{M}$ and $C^\infty(V)$ the restrictions of elements of $C_0^\infty(\tilde{V})$ to M, where $\tilde{V}$ is any extension of V over $M \cup_{\partial M} (\partial M \times I)$.

Throughout this paper a *differential operator* on V will be an A-linear partial differential operator D_0 on $C^\infty(V)$ with C^∞ coefficients. This is the same as saying that locally D_0 may be written $\sum_{|\alpha| \leq k} a_\alpha(x)(\partial^{|\alpha|}/\partial x^\alpha)$, where the a_α are smooth A-linear bundle endomorphisms and the partials are defined with respect to A-linear local trivializations of V. An *extension* of D_0 will be an operator extending the action of D_0 on $C_0^\infty(V)$. We will also consider differential operators between smooth sections of two A-bundles.

The Hilbert module $\mathcal{L}^2(V)$ is the completion of $C_0^\infty(V)$ with respect to the norm $\|\cdot\|$ derived from the inner product $(u,v)_V = \int_M \langle u, v\rangle \, d\mu_M$, with $\langle \cdot, \cdot \rangle$ the fiberwise inner product on V and μ_M the measure coming from the inner product on M. Different inner products give equivalent norms, so that the $\mathcal{L}^2(V)$ for different choices are canonically identified as topological vector spaces. The Sobolev space $W_k(V)$ for an integer $\dot{k} > 0$ is the set of all $u \in \mathcal{L}^2(V)$ such that, for all A-linear differential operators T_0 of order $\leq k$, $T_0 u \in \mathcal{L}^2(V)$ in the sense of distributions. This means the following: let $T_0^\#$ be the formal adjoint of T_0. Then, for each $u \in W_k(V)$, there must exist $z \in \mathcal{L}^2(V)$ so that $(z, v) = (u, T_0^\# v)$ for all $v \in C_0^\infty(V)$.

Lemma 3.1. *$W_k(V)$ is independent of the smooth inner products on M and V.*

Proof. Let μ_1 and μ_2 be smooth measures on M, with $d\mu_2 = \rho d\mu_1$. For a fixed inner product on V and $u, v \in C_0^\infty(V)$,

$$(u,v)_2 = \int \langle u, v \rangle \rho \, d\mu_1 = \int \langle u, \rho v\rangle \, d\mu_1 = (u, \rho v)_1.$$

(Note that this can be read as saying that multiplication by ρ is formally adjoint to the identity on $C_0^\infty(V)$ from $(\cdot,\cdot)_1$ to $(\cdot,\cdot)_2$.) The same holds by continuity for $u, v \in \mathcal{L}^2(V)$. A brief computation using this shows that, if $T_0^\#$ is the formal adjoint of T with respect to $(\cdot,\cdot)_2$, then its formal adjoint with respect to $(\cdot,\cdot)_1$ is $\rho T_0^\# \rho^{-1}$. Let $u \in W_k(V)$ for $(\cdot,\cdot)_1$. For all $v \in C_0^\infty(V)$, $(u, T_0^\# v)_2 = (u, \rho T_0^\# \rho^{-1}(\rho v))_1$, and there exists $z \in \mathcal{L}^2(V)$ such that $(u, T_0^\# v)_2 = (z, \rho v)_1 = (z, v)_2$. Therefore, $u \in W_k(V)$ for $(\cdot,\cdot)_2$.

Suppose that V has two inner products, with that on M fixed. The formal adjoint of the map on $C_0^\infty(V)$ induced by the identity of V is the induced map of the adjoint bundle map. The rest of the argument is the same as before. $\square$

Let $\eta = H^n \times P$, a metrically trivialized bundle, where $H^n = \{x \in$

$R^n \mid x_n \geq 0\}$. For an integer $k > 0$, the Sobolev k-inner product on compactly supported elements of $W_k(\eta)$ is

$$(f,g)_k = \int_{H^n} \sum_{|\alpha|\leq k} \left\langle \frac{\partial^{|\alpha|} f}{\partial x^\alpha}, \frac{\partial^{|\alpha|} g}{\partial x^\alpha} \right\rangle d\mu_{H^n}. \tag{3.1}$$

Cover M with finitely many coordinate charts $\chi_i : U_i \to \tilde{U}_i$, where the U_i are open in M and the $\tilde{U}_i$ have compact closure in H^n and V is trivial over each U_i. Choose isometric bundle isomorphisms $\hat{\chi}_i : V|_{U_i} \to \eta|_{\tilde{U}_1}$. Let $\{\phi_i\}$ be a partition of unity subordinate to $\{U_i\}$. The Sobolev k-inner product on $W_k(V)$ is

$$(u,v)_{V,k} = \sum_i (\hat{\chi}_i^{-1} \circ (\phi u) \circ \chi_i, \hat{\chi}_i^{-1} \circ (\phi v) \circ \chi_i)_k.$$

$W_k(V)$ is a Hilbert module in this inner product. Varying all the choices made gives equivalent norms. $C^\infty(V)$ is dense in $W_k(V)$. Let $W_k^0(V)$ be the closure of $C_0^\infty(V)$ in $W_k(V)$. We set $W_0(V) = W_0^0(V) = \mathcal{L}^2(V)$. There are bounded inclusions $W_{k+1}(V) \to W_k(V)$ and similarly for the spaces $W_k^0(V)$. A differential operator of order k extends by distributional derivatives to bounded operators $W_{k+r}(V) \to W_r(V)$ and $W_{k+r}^0(V) \to W_r^0(V)$ for $r \geq 0$. The various standard inequalities will be used in what follows without further comment.

Let $F_{2k}(V) = W_{2k}(V) \cap W_k^0(V)$, $k \geq 0$. More precisely, $F_{2k}(V)$ is the set of elements of $W_{2k}(V)$ whose image under $W_{2k}(V) \to W_0(V)$ is in the image of $W_k^0(V) \to W_0(V)$. $F_{2k}(V)$ will always be considered with the $2k$-inner product. (The pullback of the two preceding maps gives an equivalent norm, but the inner product isn't convenient.) For the present, we assume that a fixed choice of the data $\{U_i\}$, $\{\tilde{U}_i\}$, $\{\chi_i\}$ and $\{\hat{\chi}_i\}$ is used to construct all Sobolev spaces.

3.2. We consider the Dirichlet problem with $A = \mathbf{C}$. Let E be a Hermitian $\mathbf{C}$-vector bundle over M and D_0 a formally self-adjoint differential operator of order $2k$ on E. Assume that D_0 is elliptic, in the sense of Lopatinski, with respect to Dirichlet boundary conditions [**10**, Chapter 20, **6**, Chapter 5]. This is a condition which only involves the principal symbol of D_0 and requires in particular that D_0 be elliptic in the usual sense. Let D be the extension of D_0 with the domain

$F_{2k}(E)$. The basic fact we wish to generalize is: D is symmetric with real spectrum [**6**, Chapter 5].

The largest known general class of symbols which are elliptic with respect to Dirichlet boundary conditions appears to be those which are strongly elliptic [**2**, Section I.2]. Let $\sigma_{2k}(x,\xi)$ be the principal symbol of D_0, for $x \in M$, $\xi \neq 0 \in T_x^*(M)$. Then we require that, for all $v \neq 0 \in E_x$, $(-)^k\langle\sigma_{2k}(x,\xi)v,v\rangle > 0$. (The usual condition that this expression be $> K|\xi|^{2k}$ for some $K > 0$ follows since σ_{2k} is homogeneous and M is compact.) In addition, self-adjoint scalar elliptic operators are of even order and are elliptic with respect to Dirichlet boundary conditions [**6**, Chapter 5, Section 4.5].

If D_0 is to be of use in constructing an inner product, we should require at least that $(D_0u,u) \geq \alpha(u,u)$ for some $\alpha \in \mathbf{R}$ and all $u \in C_0^\infty(E)$. In fact, this implies that an elliptic D_0 is strongly elliptic. (I don't know of a reference for this. Since it won't be used, the proof is omitted.) Conversely, if D_0 is strongly elliptic, then Gårding's inequality holds: There are $C_1 > 0$ and C_2 such that $(D_0u,u) \geq C_1\|u\|_k^2 - C_2\|u\|^2$ [**6**, Chapter 4, Section 8.7]. Therefore, D_0 is bounded below. Despite these remarks, we will not generally assume that D_0 is strongly elliptic.

In several places we will need a Dirichlet form. More generally, let B_0 be any differential operator of order $2k$ acting on sections of an A-bundle, V. Let $\{\phi_i\}$ be the partition of unity of Section 3.1. Then $B_0 = \sum B_0\phi_i$, and each $B_0\phi_i$ is the sum of terms $a_\alpha(x)(\partial^{|\alpha|}/\partial x^\alpha)\phi_i$. For $u,v \in C_0^\infty(V)$, we may replace $(a_\alpha(x)(\partial^{|\alpha|}/\partial x^\alpha)\phi_i u,v)$ by a form with no more than k differentiations on the left and the formal adjoints of no more than k differentiations and $a_\alpha(x)$ on the right. This is justified since $\phi_i u \in C_0^\infty(V|_{U_i})$. Summing all these forms gives a form which is equal to (B_0u,v) on $C_0^\infty(V)$. It extends to a form $\Phi(u,v)$ which is continuous for $u,v \in W_k^0(V)$.

Let B be the extension of B_0 to $F_{2k}(V)$.

Lemma 3.2. $\Phi(u,v) = (Bu,v)$ *for* $u,v \in F_{2k}(V)$.

Proof. Let $(u_i) \subset C_0^\infty$ such that $u_i \xrightarrow{k} u$ and $v \in C_0^\infty$. By definition,

$(Bu, v) = (u, B_0 v)$. Then

$$(u, B_0 v) = \lim(u_i, B_0 v) = \lim(B_0 u_i, v) = \lim \Phi(u_i, v) = \Phi(u, v).$$

Therefore, $(Bu, v) = \Phi(u, v)$. By continuity this holds for $v \in F_{2k}$. □

The following applies in particular to D.

Proposition 3.3. *If $\rho(B) \neq \varnothing$ and $(B_0 v, v) \geq \alpha(v, v)$ for some $\alpha \in \mathbf{R}$ and all $v \in C_0^\infty(V)$, then $\sigma(B) \subset [\alpha, \infty)$.*

Proof. We may take $\alpha = 0$. Let $u \in F_{2k}$ and $(u_i) \subset C_0^\infty$ with $u_i \xrightarrow{k} u$. Then

$$(Bu, u) = \Phi(u, u) = \lim \Phi(u_i, u_i) = \lim(B_0 u_i, u_i) \geq 0.$$

The conclusion now follows from Lemma 2.1. □

We will consider operators on $V \otimes E$, where E is a $\mathbf{C}$-vector bundle and V is an A-vector bundle. We first take $V = \varepsilon_P = \varepsilon = M \times P$.

We will use the notation $\odot$ for algebraic tensor product and $\otimes$ for completed tensor product of Hilbert modules. If P is an A-module and Q is a B-module, $P \otimes Q$ is the $A \otimes B$-module which is the completion of $P \odot Q$ for the inner product $(p_1 \otimes q_1, p_2 \otimes q_2) = (p_1, p_2) \otimes (q_1, q_2)$ [**5**, Section 13.5].

For the rest of this section, we will require that the bundle isomorphisms $\hat{\chi}_i$ used in the construction of the Sobolev inner products be of the form $\hat{\chi}_{i,\varepsilon} \otimes \hat{\chi}_{i,E} : (\varepsilon \otimes E)|_{U_i} \to (\eta_P \otimes \eta_{\mathbf{C}^r})|_{\tilde{U}_i}$, where $\hat{\chi}_{i,\varepsilon}$ is the identity on fibers. These will be called special inner products. The product structure of ε determines a homomorphism $s : P \odot C^\infty(E) \to C^\infty(\varepsilon \otimes E)$ by $s(p \otimes u)(x) = p \otimes u(x)$.

Lemma 3.4. *s extends to unitaries $s_k : P \otimes W_k(E) \to W_k(\varepsilon \otimes E)$. These restrict to unitaries $\mathring{s}_k : P \otimes W_k^0(E) \to W_k^0(\varepsilon \otimes E)$.*

Proof. By [**21**, Section 44.1], the image of s is dense in the C^∞ topology. Since $C^\infty(\varepsilon \otimes E) \to W_k(\varepsilon \otimes E)$ is continuous and $C^\infty(\varepsilon \otimes E)$

is dense in $W_k(\varepsilon \otimes E)$, the image of s is dense in $W_k(\varepsilon \otimes E)$. Under the stated conditions on the $\hat{\chi}_i$, a calculation shows that the inner products of $p_1 \otimes u_1$, $p_2 \otimes u_2$ and of their images are both given by $(p_1, p_2)_P(u_1, u_2)_{E,k}$. Therefore, completion gives a unitary. The proof of the other statement is similar. □

These maps are compatible with inclusions such as $W_k \to W_l$ in an evident sense. The diagrams involved commute on algebraic tensor products with smooth sections, and therefore commute, since the extensions between completions are unique. This type of argument proves the commutativity of a number of diagrams in this paper.

It follows from this lemma that $P \otimes W_k(E)$ and $P \otimes W_k^0(E)$ may be considered as subspaces of $P \otimes W_0(E)$. It is generally true that, if H_1 and H_2 are Hilbert spaces, $T : H_1 \to H_2$ is bounded and injective, and P is a finitely generated A-module, then $I \otimes T : P \otimes H_1 \to P \otimes H_2$ is injective. Since no further use of this fact will be made, the proof is omitted.

We will show eventually that there are unitaries $P \otimes F_{2k}(E) \to F_{2k}(\varepsilon \otimes E)$. The following lemma is the first step.

Lemma 3.5. *$F_{2k}(V)$ is a closed subspace of $W_{2k}(V)$.*

Proof. Let $(u_i) \subset F_{2k}$ be a Cauchy sequence for the $2k$-norm. Then $u_i \overset{2k}{\to} u \in W_{2k}$. Since $F_{2k} \to W_k^0$ is continuous, $u_i \overset{k}{\to} v \in W_k^0$. By continuity of the inclusions of W_{2k} and W_k^0 into W_0, $u_i \to u$ and v in W_0, so $u = v$ and $u \in F_{2k}$. □

Consider the diagram below

$$\begin{tikzcd}
 & P \otimes W_{2k}(E) \arrow[r, "s_{2k}"] \arrow[d] & W_{2k}(\varepsilon \otimes E) \arrow[d] \\
P \otimes F_{2k}(E) \arrow[ur, "I\otimes j"] \arrow[dr] & P \otimes W_0(E) \arrow[r, "s_0"] & W_0(\varepsilon \otimes E) \\
 & P \otimes W_k^0(E) \arrow[r, "s_k"'] \arrow[u] & W_k^0(\varepsilon \otimes E) \arrow[u]
\end{tikzcd}$$

The unlabeled maps are of the form *identity* $\otimes$ *inclusion*. Commutativity of the triangle is clear. The right side commutes by the remark following Lemma 3.4. Using that lemma, this shows that $s_{2k}(I \otimes j)$ maps $P \otimes F_{2k}(E)$ isometrically into $F_{2k}(\varepsilon \otimes E)$. We will show in Proposition 3.8 that it is onto. This is most economically done after it has been shown that the spaces are the domains of suitable operators.

3.3. Given a differential operator D_0 on E, we will construct an A-linear differential operator $\hat{D}_0$ on $\varepsilon \otimes E$ with similar properties. In fact, we will construct a homomorphism of the algebras of differential operators $\mathrm{Diff}\,(E) \to \mathrm{Diff}_A(\varepsilon \otimes E)$. This is a special case of a construction using connections in Section 4.

To a smooth endomorphism c of E, assign $\hat{c} = I_\varepsilon \otimes c$. Choose coordinate charts $\{U_i\}$ for M and isometric trivializations of the $E|_{U_i}$. A metric product structure on ε was assumed given, so that each $(\varepsilon \otimes E)|_{U_i}$ is trivialized. On U_i assign to $\partial/\partial x_i$ on $E|_{U_i}$,

$$\widehat{\frac{\partial}{\partial x_i}} = \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} \otimes I_E + I_V \otimes \frac{\partial}{\partial x_i} \quad \text{on} \quad (\varepsilon \otimes E)|_{U_i}.$$

These two assignments determine a homomorphism of the algebras of differential operators over U_i, which is independent of the choice of trivializations for E. Let D_0 be a differential operator of order k on E. If we use its local representations in the U_i to construct A-differential operators over the U_i, it follows from the transformation rule for $\partial/\partial x_i$ that the latter agree on each $U_i \cap U_j$, which gives the desired $\hat{D}_0$. Since

$$\widehat{\frac{\partial^\#}{\partial x_i}} = -\widehat{\frac{\partial}{\partial x_i}} = -\frac{\partial}{\partial x_i} = \widehat{\frac{\partial}{\partial x_i}}^{\#},$$

the composition rule for formal adjoints shows that the correspondence preserves adjoints. The principal symbol of $\partial/\partial x_i$ on $E|_{U_i}$ or $\varepsilon \otimes E|_{U_i}$ at $x \in M$ is right multiplication by ξ^i for $\xi \in T^*M_x$. Since $(r \otimes e)\xi^i = r \otimes e\xi^i$ for $r \in \varepsilon_x$, $e \in E_x$, there is the equality of principal symbols $\sigma_k(\hat{D}_0) = I_\varepsilon \otimes \sigma_k(D_0)$. Therefore $\hat{D}_0$ is elliptic if and only if D_0 is. (The definition of ellipticity for A-linear pseudodifferential operators is the direct analog of the usual one, see [**16**, Section 3].

Let D_0 have order $2k$ and $\hat{D}$ be the extension of $\hat{D}_0$ to $F_{2k}(\varepsilon \otimes E)$. The following shows that $\hat{D}$ is symmetric if D_0 is formally self-adjoint. Let V be any Hermitian A-bundle.

Lemma 3.6. *Let B_0 be a formally self-adjoint differential operator of order $2k$ on V and B its extension to $F_{2k}(V)$. Then B is symmetric.*

Proof. Let Φ be the form of Lemma 3.2, so that $\Phi(u,v) = (Bu, v)$ for $u, v \in F_{2k}$. Since B_0 is formally self-adjoint, it can be shown in exactly the same way that $\Phi(u,v) = (u, Bv)$, so $(Bu, v) = (u, Bv)$. □

3.4. From now on, we assume that D_0 is formally self-adjoint and elliptic with respect to Dirichlet boundary conditions; therefore, D is symmetric with real spectrum. We aim to prove that the spectra of D and $\hat{D}$ are identical. We will discuss the following diagram, with $\lambda \in \mathbf{C}$.

$$\begin{array}{ccc} P \otimes F_{2k}(E) & \xrightarrow{s_{2k}} & F_{2k}(\varepsilon \otimes E) \\ {\scriptstyle I\otimes D - \lambda}\downarrow & & \downarrow{\scriptstyle \hat{D}-\lambda} \\ P \otimes W_0(E) & \xrightarrow[s_0]{} & W_0(\varepsilon \otimes E) \end{array} \tag{3.3}$$

$I \otimes D$ is defined to be the closure of $I \odot D$ considered as an unbounded operator on $P \otimes W_0(E)$ with domain $P \odot F_{2k}(E)$. D is symmetric, so $I \odot D$ is symmetric and therefore closable. We will show first that $\mathcal{D}(I \otimes D) = P \otimes F_{2k}(E)$.

We observe that the extension of the inclusion $P \odot F_{2k}(E) \to P \odot W_0(E)$ to $P \otimes F_{2k}(E) \to P \otimes W_0(E)$ has as image the elements u such that there exists a sequence $(u_i) \subset P \odot F_{2k}(E)$ which is Cauchy for the norm of $P \odot F_{2k}(E)$ and $u_i \to u$. If $\lambda \in \mathbf{C}$ then $\mathcal{D}(\overline{I \odot D - \lambda}) = \mathcal{D}(\overline{I \odot D})$. By definition, $u \in \mathcal{D}(\overline{I \odot D - \lambda})$ if and only if there exists $(u_i) \subset P \odot F_{2k}(E)$ such that $u_i \to u$ and $((I \odot D - \lambda)u_i)$ is Cauchy. Choose $\lambda \notin \sigma(D)$. Then $D - \lambda : F_{2k}(E) \to W_0(E)$ is bounded and bijective. By the open mapping theorem $D - \lambda$ has a bounded inverse. Then $I \odot D - \lambda = I \odot (D - \lambda)$ is also bounded and bijective with bounded inverse, so that (u_i) is Cauchy in the norm of $P \odot F_{2k}(E)$ if and only if $((I \odot D - \lambda)u_i)$ is Cauchy.

To show that the diagram commutes it is sufficient to consider D and $\hat{D}$ acting on the larger spaces W_{2k}. We can also replace the completed tensor products with the dense subspaces given by algebraic tensor products. Since $C^\infty(E)$ is dense in $W_{2k}(E)$, it is enough to consider elements $p \otimes u$ with $p \in P$ and $u \in C^\infty(E)$. For these, commutativity is direct from the definition of $\hat{D}_0$. The following argument is given since the proofs of more general statements in the literature are much more involved.

Lemma 3.7. *The spectra of D and $I \otimes D$ are the same.*

Proof. Let $\lambda_0 \notin \sigma(D)$. Since $A \mapsto I \otimes A$ is a faithful representation of $\mathcal{L}(W_0(E))$ into $\mathcal{L}(P \otimes W_0(E))$, $\sigma(I \otimes D - \lambda_0)^{-1}) = \sigma((D - \lambda_0)^{-1})$. If T is an operator which is closable and injective, and T^{-1} is closable, then $\overline{T^{-1}} = \overline{T}^{-1}$. Take $T = I \odot (D - \lambda_0)$, so that $\overline{T} = I \otimes (D - \lambda_0)$. Then $T^{-1} = I \odot (D - \lambda_0)^{-1}$ and $\overline{T^{-1}} = I \otimes (D - \lambda_0)^{-1}$. Therefore,

$$
\begin{aligned}
I \otimes (D - \lambda_0)^{-1} &= \overline{I \odot (D - \lambda_0)^{-1}} = \overline{I \odot (D - \lambda_0)}^{-1} \\
&= \overline{I \odot D - \lambda_0}^{-1} = (I \otimes D - \lambda_0)^{-1},
\end{aligned}
$$

so $\lambda_0 \notin \sigma(I \otimes D)$. Then, by [**23**, Example 5.27], $\lambda \in \sigma(I \otimes D)$ if and only if $(\lambda - \lambda_0)^{-1} \in \sigma((I \otimes D - \lambda_0)^{-1})$ if and only if $(\lambda - \lambda_0)^{-1} \in \sigma((D - \lambda_0)^{-1})$ if and only if $\lambda \in \sigma(D)$. □

Proposition 3.8. 1. *The map s_{2k} of Lemma* 3.4 *is an isomorphism from $P \otimes F_{2k}(E)$ to $F_{2k}(\varepsilon \otimes E)$.*

2. *The spectra of D and $\hat{D}$ are the same, in particular, $\sigma(\hat{D})$ is real.*

Proof. It has been shown that, in diagram (3.2), s_0 is an isomorphism and s_{2k} a monomorphism, and that $s_{2k}(I \otimes j)$ is a monomorphism of $P \otimes F_{2k}(E)$ into $F_{2k}(\varepsilon \otimes E)$. Choose $\lambda \in \mathbf{C} - \mathbf{R}$, so $\lambda \notin \sigma(I \otimes D)$. It follows that $\hat{D} - \lambda$ is surjective. $\hat{D}$ is symmetric, Lemma 3.6, so $\hat{D} - \lambda$ is injective with bounded inverse [**23**, Theorem 5.18]. Therefore $\hat{D} - \lambda$ is bijective and s_{2k} is an isomorphism. For (2) it remains to show that $\sigma(\hat{D}) = \sigma(I \otimes D)$. From the diagram, for any $\lambda \in \mathbf{C}$, $\hat{D} - \lambda$ is bijective if and only if $I \otimes D - \lambda$ is. If $\hat{D} - \lambda : F_{2k}(\varepsilon \otimes E) \to W_0(\varepsilon \otimes E)$ is bijective, then since it is bounded, by the open mapping theorem its inverse is bounded into $F_{2k}(\varepsilon \otimes E)$ and so into $W_0(\varepsilon \otimes E)$; therefore, $\lambda \notin \sigma(\hat{D})$. In the same way, $I \otimes D - \lambda$ is bijective if and only if $\lambda \notin \sigma(I \otimes D)$. Therefore $\sigma(\hat{D}) = \sigma(I \otimes D)$. □

3.5. Operators with "coefficients in A" will be introduced in earnest by considering lower order perturbations of the operator $\hat{D}$. The theory of Rellich and Kato will show that such perturbations preserve certain properties of the spectrum.

Let S and T be A-linear operators between Hilbert modules. Recall that S is *T-bounded* if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and there exists $C \geq 0$ such that $\|Su\| \leq C(\|Tu\| + \|u\|)$ for $u \in \mathcal{D}(T)$. The infimum of the numbers $a \geq 0$ such that there exists a $b \geq 0$ with $\|Su\| \leq a\|Tu\| + b\|u\|$ is the *T-bound* of S. If this is zero, we say that S is *infinitesimally small* with respect to T.

The perturbation result we need is as follows [**18**, Theorem 10.13]. Let S and T be symmetric with $\mathcal{D}(S) = \mathcal{D}(T)$ with $S - T$ T-bounded with bound less than one. Then S has real spectrum if and only if T does. We take $T = \hat{D}$. Let $\tilde{D}_0$ be a formally self-adjoint elliptic differential operator of order $2k$ on $\varepsilon \otimes E$ with the same principal symbol as $\hat{D}_0$. Let $S = \tilde{D}$ be the extension of $\tilde{D}_0$ to $F_{2k}(\varepsilon \otimes E)$. $\tilde{D}$ is symmetric by Lemma 3.6.

Lemma 3.9. *$\tilde{D} - \hat{D}$ is infinitesimally small with respect to $\hat{D}$.*

Proof. $\hat{D}$ is closed since $\rho(\hat{D}) \neq \varnothing$. Since $\mathcal{D}(\hat{D}) \subset W_{2k}(\varepsilon \otimes E)$, $\hat{D}$ satisfies an inequality of the form $\|u\|_{2k} \leq C(\|\hat{D}u\| + \|u\|)$, $K \geq 0$, [**1**, Theorem 15.2]. It is therefore sufficient to show that, for every $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that $\|(\tilde{D} - \hat{D})u\| \leq \varepsilon\|u\|_{2k} + K_\varepsilon\|u\|$ for all $u \in \mathcal{D}(\hat{D})$. $\tilde{D}_0 - \hat{D}_0$ is a differential operator of order $\leq 2k - 1$ and $\tilde{D} - \hat{D}$ is its extension to $F_{2k}(\varepsilon \otimes E)$. Therefore, there exists $L > 0$ so that $\|(\tilde{D} - \hat{D})u\| \leq L\|u\|_{2k-1}$, $u \in F_{2k}(\varepsilon \otimes E)$. The estimate now follows from the interpolation property: given $\eta > 0$ there exists $C_\eta > 0$ such that $\|u\|_{2k-1} \leq \eta\|u\|_{2k} + C_\eta\|u\|$. □

It follows that $\tilde{D}$ has real spectrum.

If the principal symbol of a differential operator on $\varepsilon \otimes E$ is of the form $\sigma = I \otimes \tau$, where τ is a symbol on E, we say that the operator has *complex symbol*. If, further, τ is elliptic for Dirichlet boundary conditions, we say that the operator has *complex Dirichlet symbol*.

Now let $\tilde{D}_0$ be any $2k$th order formally self-adjoint differential operator on $\varepsilon \otimes E$ with complex Dirichlet symbol $I \otimes \tau$. Then there exists a $2k$th order formally self-adjoint elliptic differential operator D_0 on E with $\sigma_{2k}(D_0) = \tau$ [**17**, Theorem 4.3.2], so D_0 is elliptic with respect to Dirichlet boundary conditions. Then $\sigma_{2k}(\hat{D}_0) = I \otimes \tau$. The preceding discussion yields the following.

Theorem 3.10. *Let E be a Hermitian* $\mathbf{C}$*-vector bundle on M and ε a trivial Hermitian A-vector bundle. Let $\tilde{D}_0$ be a formally self-adjoint differential operator of order $2k$ on $\varepsilon \otimes E$ with complex Dirichlet symbol. Then the extension $\tilde{D}$ of $\tilde{D}_0$ to $F_{2k}(\varepsilon \otimes E)$ is symmetric with real spectrum.*

4. Operators on nontrivial A-bundles. Results similar to those of the preceding section can be proved for operators with complex principal symbols on nontrivial A-bundles. The most important examples are operators with coefficients in an A-bundle. We will continue the notation of the preceding section.

4.1. Let V be a Hermitian A-vector bundle on M and D_0^1 a formally self-adjoint differential operator of order $2k$ on $V \otimes E$. Let D^1 be the extension of D_0^1 to $F_{2k}(V \otimes E)$.

Theorem 4.1. *If D_0^1 has complex Dirichlet symbol, then D^1 is symmetric with real spectrum. If $(D_0^1 u, u) \geq \alpha(u, u)$ for some $\alpha \in \mathbf{R}$ and all $u \in C_0^\infty(V \otimes E)$, then D^1 has spectrum in $[\alpha, \infty)$.*

Proof. Using polar decomposition, one can choose an A-bundle W and an isometric isomorphism $r : V \oplus W \to \varepsilon$ for some trivial A-bundle ε. Regard r as an identification. Let $\sigma_{2k}(D_0^1) = I_V \otimes \tau$. There exists a formally self-adjoint differential operator D_0^2 on $W \otimes E$ with $\sigma_{2k}(D_0^2) = I_W \otimes \tau$. Let $\tilde{D}_0 = D_0^1 \oplus D_0^2$. Then $\sigma_{2k}(\tilde{D}_0) = I_\varepsilon \otimes \tau$ is a complex Dirichlet symbol. Let $\tilde{D}$ and D^2 be the extensions of $\tilde{D}_0$ and D_0^2 to the appropriate F_{2k}-spaces. By Theorem 3.10, $\tilde{D}$ is symmetric with real spectrum.

We will show that $\tilde{D}$ and $D^1 \oplus D^2$ are unitarily equivalent. Then, since the spectrum of an orthogonal direct sum of operators is the union of the spectra, the spectrum of D^1 is real. It is also easy to see that an orthogonal direct summand of a symmetric operator is symmetric, so D^1 is symmetric. If the boundedness condition holds, then D^1 has spectrum in $[\alpha, \infty)$ by Proposition 3.3. The remainder of the proof doesn't depend on a tensor product decomposition, so we will replace $V \otimes E$ by V and $W \otimes E$ by W for notational simplicity.

We first show that the domains of $\tilde{D}$ and $D^1 \oplus D^2$ are the same. r induces an isomorphism $C^\infty(V) \oplus C^\infty(W) \to C^\infty(\varepsilon)$. The construction of the Sobolev inner products made use of trivializations of the bundles over coordinate patches. We will assume that these have been chosen so that those for ε are the sums of those for V and W. Note that this choice does not affect the generality of the conclusion. The inner products are then direct sums, so that there are unitaries $t_k : W_k(V) \oplus W_k(W) \to W_k(\varepsilon)$. The same holds for the W_k^0-spaces. These are compatible with inclusions for different values of k, since extensions from smooth sections to Sobolev spaces are unique.

Lemma 4.2. *t_0 restricts to an isomorphism $F_{2k}(V) \oplus F_{2k}(W) \to F_{2k}(\varepsilon)$.*

Proof. Every $u \in F_{2k}(\varepsilon)$ corresponds uniquely to the sum of some $v \in W_0(V)$ and $w \in W_0(W)$. It is therefore sufficient to show that such a representation is possible with v and w in the W_{2k}-spaces and in the W_k^0-spaces. These follow using the above isomorphisms for the subscripts $2k$ and k. $\square$

To continue with the proof of Theorem 4.1, consider m:

$$
\begin{aligned}
&(W_0(V) \oplus W_0(V)) \oplus (W_0(W) \oplus W_0(W)) \\
&\qquad \longrightarrow (W_0(V) \oplus W_0(W)) \oplus (W_0(V) \oplus W_0(W)) \\
&\qquad \xrightarrow{t_0 \oplus t_0} W_0(\varepsilon) \oplus W_0(\varepsilon),
\end{aligned}
$$

where the first map interchanges terms. This takes the direct sum of the graphs of two operators to the graph of their direct sum. It follows from Lemma 4.2 that m takes $\mathcal{D}(D^1) \oplus \mathcal{D}(D^2)$ to $\mathcal{D}(\tilde{D})$. The diagram

$$\begin{tikzcd} F_{2k}(V) \oplus F_{2k}(W) \arrow[r, "t_{2k}"] \arrow[d, "D^1 \oplus D^2"'] & F_{2k}(\varepsilon) \arrow[d, "\tilde{D}"] \\ W_0(V) \oplus W_0(W) \arrow[r, "t_0"'] & W_0(\varepsilon) \end{tikzcd}$$

is commutative. It is enough to check this with F_{2k} replaced by W_{2k} and then on the dense subspaces of C^∞ sections. For these it is clear since $\tilde{D}_0 = D_0^1 \oplus D_0^2$. It is then evident that m takes the sum of the graphs of D^1 and D^2 to the graph of $\tilde{D}$. □

4.2. We construct operators satisfying the hypotheses of Theorem 4.1 by taking complex operators with coefficients in an A-bundle. The construction is related to one in [**17**, IV.9]. Consider a first order differential operator T_0 on E. We initially work on a coordinate chart U for M over which E is trivialized. Choose a connection ∇^E for E. Since $\nabla^E_{\partial/\partial x_i}$ differs from $\partial/\partial x_i$ by an operator of order zero, we may write

$$T_0 = A_0 + \sum A_i \nabla^E_{\partial/\partial x_i}$$

on U. Using a partition of unit for M puts T_0 in the form $B_0 + \sum B_j \nabla^E_{X_j}$ where the X_j are vector fields. Let ∇^V be an A-linear connection for V and ∇ the connection on $V \otimes E$ determined by $\nabla(u \otimes v) = \nabla^V u \otimes v + u \otimes \nabla^E v$. Let

$$\hat{T}_0 = I_V \otimes B_0 + \sum (I \otimes B_j) \nabla_{X_j}, \tag{4.1}$$

which is clearly a differential operator. Since we can write $\hat{T}_0 = I_V \otimes T_0 + \sum \nabla^V_{X_j} \otimes B_j$, it doesn't depend on the choice of ∇^E. Then, since $\sigma_1(\nabla_{\partial/\partial x_i}) = \sigma_1(\partial/\partial x_i)$, it follows as in Section 3.3 that $\sigma_1(\hat{T}_0) = I_V \otimes \sigma_1(T_0)$. If ∇^V is unitary, the correspondence $T_0 \Rightarrow \hat{T}_0$ preserves adjoints. This results from a computation using the skew-adjointness of the connections, taking ∇^E unitary.

For an arbitrary unitary connection we don't obtain a correspondence of higher order operators, since the different $\nabla_{\partial/\partial x_i}$ don't commute. However we may consider polynomials in a fixed first order operator T_0. If $p(t)$ is a monic polynomial of degree k with real coefficients, then define $D_0 = p(T_0)$ and $\hat{D}_0 = p(\hat{T}_0)$. Then, by the composition rule for symbols, $\sigma_k(\hat{D}_0) = \sigma_k(p(\hat{T}_0)) = I_V \otimes \sigma_1(T_0)^k = I_V \otimes \sigma_k(D_0)$. A similar computation shows that adjoints are preserved.

If ∇^V is flat, i.e., curvature zero, one obtains a homomorphism $\mathrm{Diff}\,(E) \to \mathrm{Diff}_A(V \otimes E)$. Indeed, this situation is locally the one we considered in Section 3.3. ∇^V determines (by parallel transport) a class of local trivializations of V with respect to which $\nabla^V_{\partial/\partial x_i} = \partial/\partial x_i$. Using (4.1) with the connection on $E|_U$ determined by partial differentiation gives $\widehat{\partial/\partial x_i} = (\partial/\partial x_i) \otimes I_E + I_V \otimes (\partial/\partial x_i) = \partial/\partial x_i$ on $(V \otimes E)|_U$. Then, as in Section 3.3, if D_0 is of order k, $\sigma_k(\hat{D}_0) = I \otimes \sigma_k(D_0)$ and adjoints are preserved if ∇^V is unitary.

Therefore, in both cases $\hat{D}_0$ has complex principal symbol, and Theorem 4.1 may be applied with $D_0^1 = \hat{D}_0$ provided that T_0 or D_0 is formally self-adjoint and elliptic and $p(t)$ is of even degree, then $p(T_0)$ is formally self-adjoint and strongly elliptic. $p(t) = (t^2 + I)^k$ gives a lower bound of one and is often used for $T_0 = d + \delta$ on differential forms.

4.3. The conclusions of Theorem 4.1 hold for commutative C^*-algebras without assuming that the symbol of the operator, which will be denoted D_0, is complex. The idea of the proof is to show that D_0 is equivalent to a continuous family of operators over $\mathbf{C}$.

So far we have avoided giving a definition of elliptic boundary conditions over a C^*-algebra A. The most frequently used description over $\mathbf{C}$ is that of Hörmander [**10**, Definition 20.1.1]. It makes sense in general and clearly agrees with our notion of complex Dirichlet symbol in the case of Dirichlet boundary conditions. However, for a commutative A it is rather easy to see that this definition is equivalent to assuming that the restriction of the operator, in the sense described below, to every point of the spectrum of A is elliptic for the restricted boundary conditions. In what follows we will assume the latter.

We begin with some preliminaries on continuous fields of Hilbert spaces [**7**, 10.1, 10.2] and families of differential operators. A fair number of routine verifications are omitted.

Let Y be a compact connected Hausdorff space and $\mathcal{E}$ a Hilbert $C(Y)$-module. For $y \in Y$, let $e_y : C(Y) \to \mathbf{C}$ be the evaluation. $\mathcal{E}_y = \mathcal{E} \otimes_{e_y} \mathbf{C}$

is a quotient $\mathcal{E}/\mathcal{M}_{\mathcal{E},y}$. The submodule $\mathcal{M}_{\mathcal{E},y}$ can be described in two ways: as $\{u \in \mathcal{E} \mid e_y(u,u) = 0\}$ or as $\overline{\mathcal{E}\mathcal{M}_y}$, where $\mathcal{M}_y$ is the kernel of e_y. The inner product on $\mathcal{E}_y$ is induced from that on $\mathcal{E}$ by e_y. Let $q_y : \mathcal{E} \to \mathcal{E}_y$ be the quotient map. If $\Gamma = \{(q_y(u)) \mid y \in Y, u \in \mathcal{E}\}$, it is immediate that $\{\mathcal{E}_y\}$ and Γ satisfy the axioms for a continuous field of Hilbert spaces. A homomorphism $T : \mathcal{E} \to \mathcal{F}$ of Hilbert $C(Y)$-modules, not necessarily with adjoint, induces homomorphisms $T_y = T \otimes_{e_y} \mathbf{C} : \mathcal{E}_y \to \mathcal{F}_y$ by $T_y(q_y u) = q_y(Tu)$.

Let V be a smooth bundle over M with fiber a finitely generated $C(Y)$-module P. There is a natural $\mathbf{C}$-bundle V_y over M such that $C(V_y) = C(V)_y$. By Swan's theorem, P may be realized as the continuous sections of a Hermitian $\mathbf{C}$-vector bundle over Y, so $C(P)_y \cong \mathbf{C}^r$ for some r. Let $p_y : C(P) \to \mathbf{C}^r$. If $\hat{\chi}_i$ are local trivializations of V as in Section 3.1, the bundles of kernels of the maps $V|_{U_i} \xrightarrow{\hat{\chi}_i} \tilde{U}_i \times P \xrightarrow{I\times p_y} \tilde{U}_i \times \mathbf{C}^r$ form a smooth subbundle of V. Let V_y be the quotient bundle and $r_y : V \to V_y$ the quotient map. Then

$$
\begin{array}{ccc}
V\,|_{U_i} & \xrightarrow{\tilde{\chi}_i} & \tilde{U}_i \times P \\
{\scriptstyle r_y|_{U_i}}\downarrow & & \downarrow{\scriptstyle I\times P_y} \\
V_y\,|_{U_i} & \xrightarrow[\tilde{\chi}_{y,i}]{} & \tilde{U}_i \times \mathbf{C}^r
\end{array}
\tag{4.2}
$$

defines local trivializations $\hat{\chi}_{y,i}$ for V_y. A $C(Y)$-valued fiber inner product for V defines $\mathbf{C}$-valued fiber inner product for V_y by evaluation at y. Let $r_{y^*} : C(V) \to C(V_y)$ be the induced homomorphism. The $\mathcal{L}^2$-inner products $(\cdot,\cdot)$ on $C(V)$ and $(\cdot,\cdot)_y$ on $C(V_y)$ are related by $(r_{y^*}u, r_{y^*}v)_y = e_y(u,v)$. u is in the kernel of r_{y^*} if and only if $u \in \mathcal{M}_{C(V),y}$. A linear splitting $\mathbf{C}^r \to P$ of p_y and a partition of unity determine a smooth right inverse s_y of r_y. Therefore, r_{y^*} is surjective, and $C(V_y) \cong C(V)_y$ as claimed. Since r_y and s_y are smooth, r_{y^*} is split surjective on $C^\infty(V)$ or $C_0^\infty(V)$.

We now turn to differential operators and Sobolev spaces. A smooth $C(Y)$-linear bundle endomorphism a of V descends to a smooth endomorphism a_y of V_y, which is bijective if a is. Given a differential operator D_0 of order k on V, define $\overline{D}_{0,y}$ on V_y by $\overline{D}_{0,y}(r_{y^*}u) = r_{y^*}(D_0 u)$ for $u \in C^\infty(V)$. $\overline{D}_{0,y}$ is differential, since in local coordinates (4.1) $(\partial/\partial x_j)(r_{y^*}u) = r_{y^*}(\partial u/\partial x_j)$. From what has been said it is evident that $\sigma_k(\overline{D}_{0,y}) = \sigma_k(D_0)_y$, so $\overline{D}_{0,y}$ is elliptic if D_0 is. If D_0 is formally self-adjoint, then $\overline{D}_{0,y}$ is also with respect to $(\cdot,\cdot)_y$.

If $(\cdot,\cdot)_k$ is a Sobolev k-inner product for V constructed using the local trivializations (4.1), then its evaluation $(\cdot,\cdot)_{k,y}$ is a k-inner product for V_y. The $C(Y)$-modules $W_k(V)$, $W_k^0(V)$ and $F_{2k}(V)$ define continuous fields with spaces $\{W_k(V)_y\}$, etc. We will show that there are isomorphisms of the type $W_k(V_y) \cong W_k(V)_y$. r_{y^*} and s_{y^*} are bounded on smooth sections between the k-norms. For r_{y^*} this is clear; for s_{y^*} the usual proofs that a smooth bundle map defines a bounded homomorphism apply. Therefore, r_{y^*} and s_{y^*} extend to maps between $W_k(V)$ and $W_k(V_y)$. Since $r_y s_y = I$, r_{y^*} is surjective. If $u \in W_k(V)$, $r_{y^*}u = 0$ is the same as $(r_{y^*}u, r_{y^*}u)_y = 0$ or $e_y(u,u) = 0$. Therefore the kernel of r_{y^*} is $\mathcal{M}_{W_k(V),y}$, and $W_k(V)_y$ is canonically isomorphic to $W_k(V_y)$. Since the inner products on both quotients are induced by evaluation at y, the isomorphism is isometric. The proof that $W_k^0(V)_y \cong W_k^0(V_y)$ is the same. For $F_{2k}(V)$, a diagram similar to (3.2) shows that $s_{y^*} : W_{2k}(V_y) \to W_{2k}(V)$ maps $F_{2k}(V_y)$ into $F_{2k}(V)$. Then $r_{y^*} : F_{2k}(V) \to F_{2k}(V_y)$ is surjective and the same argument as before applies.

Theorem 4.3. *Let D_0 be a formally self-adjoint differential operator of order $2k$ on a bundle V of finitely generated modules over a commutative C^*-algebra with unit, which is elliptic with respect to Dirichlet boundary conditions. Then the extension D of D_0 to $F_{2k}(V)$ is symmetric with real spectrum. If $(D_0u,u) \geq \alpha(u,u)$ for some $\alpha \in \mathbf{R}$ and $u \in C_0^\infty(V)$, then $\sigma(D) \subset [\alpha,\infty)$.*

Proof. D is symmetric by Lemma 3.6. Therefore, for $\lambda \in \mathbf{C} - \mathbf{R}$, $D - \lambda$ is injective with continuous inverse [**23**, Theorem 5.18]. We will show that $D - \lambda$ has dense range, so that $\lambda \notin \sigma(D)$. We consider two operators from $F_{2k}(V_y)$ to $W_0(V)$. One is D_y and the other is $\overline{D}_y$, the extension of $\overline{D}_{0,y}$. These are identical: D extends to an operator D' on $W_{2k}(V)$ acting by distributional derivatives, so D_y extends to D'_y on $W_{2k}(V_y)$. $\overline{D}_y$ extends to $\overline{D}'_y$ on $W_{2k}(V_y)$. By construction, $D'_y = \overline{D}'_y$ on $C^\infty(V_y)$. Since $C^\infty(V_y)$ is dense in $W_{2k}(V_y)$, $D'_y = \overline{D}'_y$. Therefore $D_y = \overline{D}_y$.

$\overline{D}_y$ is self-adjoint by the ellipticity assumption, so $D_y - \lambda$ is surjective.

Proposition 10.2.2 of [**7**], the implication $(i) \Rightarrow (ii')$, asserts that $(D-\lambda)F_{2k}(V)$ is dense in $W_0(Y)$.

The second conclusion is a consequence of Proposition 3.3. □

5. Concrete Sobolev spaces. This section makes the connection between the abstractly defined spaces $\mathcal{E}_+$ of Section 2 and the concrete Sobolev spaces of Section 3. We show that the Sobolev inner products $(\cdot,\cdot)_k$ give rise to symmetric operators D on $F_{2k}(V)$ with strictly positive spectrum. The domain of $D^{1/2}$ is $W_k^0(V)$ and on this $(\cdot,\cdot)_k = (\cdot,\cdot)_+$ which was defined to be $(D^{1/2}\cdot, D^{1/2}\cdot)$. It follows that the Sobolev inner products are all compatible. This gives a preferred compatibility class of inner products on $W_k^0(V)$, depending only on the smooth structure of M. As consequences, differential or properly supported pseudodifferential operators between Sobolev spaces have adjoints, and Rellich's theorem holds.

5.1. For the purposes of this section, a tensor product decomposition of the bundle isn't relevant, since the operator in question is essentially scalar. Therefore, we will consider $W_k^0(V)$ for any A-bundle V, which is to be thought of as $V \otimes \varepsilon_{\mathbf{C}}$. In this context a symbol is complex when it acts by complex scalars. We will show that there exists a differential operator D_0 of order $2k$ which satisfies the hypotheses of Theorem 4.1 such that $(\cdot,\cdot)_k = (D_0\cdot,\cdot)$ on $C_0^\infty(V)$. For this discussion, we assume that a fixed choice of the inner products $(\cdot,\cdot)_k$ has been made.

Lemma 5.1. *There exists a formally self-adjoint differential operator D_0 with strongly elliptic complex symbol, which is bounded below by some positive constant, such that if $u, v \in C_0^\infty(V)$, then $(u,v)_k = (D_0u, v)$.*

Proof. For $k=0$, where $(\cdot,\cdot)_0 = (\cdot,\cdot)$ we have $D_0 = I$, so assume $k>0$. Let f and g be smooth sections of the trivial bundle η over H^n with compact supports in the interior of H^n. We can rewrite the expression (3.1) as

$$(f,g)_k = \int_{H^n} \langle Ff, g\rangle \, d\mu_{H^n}$$

by integration by parts, where $F = \sum_{|\alpha|\le k}(-)^\alpha(\partial^{|2\alpha|}/\partial x^{2\alpha})$. F is formally self-adjoint and has the complex principal symbol $(-)^k\sum_{|\alpha|=k}\xi^{2\alpha}$, which is strongly elliptic. We defined

$$(u,v)_{V,k} = \sum_i (\hat{\chi}_i^{-1}\circ(\phi u)\circ\chi_i, \hat{\chi}_i^{-1}\circ(\phi v)\circ\chi_i)_k.$$

If $u, v \in C_0^\infty(V)$, a single term of the integrand for this pulls back under χ_i to

$$\langle \rho_i\phi_i(\hat{\chi}_i^{-1}\circ F(\hat{\chi}_i\circ(\phi u)\circ\chi_i^{-1})\circ\chi_i, v\rangle\, d\mu_M$$

(after some rearrangement), where $(d\chi_i)'\, d\mu_{H^n} = \rho_i\, d\mu_M$. Let D_0u be the sum of the expressions on the left of the bracket so that $(D_0u, v) = (u,v)_k$. Formal self-adjointness follows immediately from the symmetry of $(\cdot,\cdot)_{H^n,k}$. We check that D_0 is bounded below by some positive number. By definition, $(f,f)_{H^n,k} \ge (f,f)_{H^n}$, so as above

$$\begin{aligned}(u,u)_k &\ge \sum_i (\hat{\chi}_i^{-1}\circ(\phi_i u)\circ\chi_i, \hat{\chi}_i^{-1}\circ(\phi_i u)\circ\chi_i)_{H^n}\\ &= \sum_i\int_M (\phi_i u, \phi_i u\rangle\rho_i\, d\mu_M\\ &= \int_M\left(\sum\rho_i\phi_i^2\right)\langle u,u\rangle\, d\mu_M.\end{aligned}$$

The sum is bounded below on M so the estimate follows.

We show that D_0 is a differential operator with strongly elliptic complex symbol. G_i defined by $G_i s = \hat{\chi}_i^{-1}\circ F(\hat{\chi}_i\circ s)$ operates on $C^\infty((\chi_i^{-1})^*V|_{\overline{U}_i})$, which is the set of smooth functions s from $\tilde{U}_i$ to the total space of V such that $\pi_V s(x) = \chi_i^{-1}x$. If we define $\hat{\chi}_{i*} : C^\infty((\chi_i^{-1})^*V|_{\overline{U}_i}) \to C^\infty(\eta|_{\overline{U}_i})$ by $\hat{\chi}_{i*}s = \hat{\chi}\circ s$, then $G_i s = \hat{\chi}_{i*}^{-1}F\hat{\chi}_{i*}s$. Therefore, G_i is the composition of differential operators and is differential of order $2k$. Since $\sigma_{2k}(F)$ is complex and strongly elliptic, the same is true of $\sigma_{2k}(G_i)$.

The operator $Q_i u = G_i(u\circ\chi_i^{-1})$ on $V|_{U_i}$ is differential with $\sigma_{2k}(Q_i)(x,\xi) = \sigma_{2k}(G_i)(\chi_i(x), (d\chi_i)'\xi)$ [**15**, p. 186]. From this it is clear that $\sigma_{2k}(Q_i)$ is also complex and strongly elliptic. Including the scalar factors gives $\rho_i\phi_i Q_i\phi_i$ which is a differential operator on $V|_{U_i}$ with complex symbol which is strongly elliptic on the interior of the support of ϕ_i. Let R_i be its extension by 0 to M. Then $D_0 = \sum R_i$ has strongly elliptic complex symbol on M. □

Recall that a strongly elliptic symbol is elliptic for Dirichlet boundary conditions. It follows from Theorem 4.1 that the extension D of D_0 to $F_{2k}(V)$ is symmetric with positive spectrum.

Proposition 5.2. *$\mathcal{D}(D^{1/2}) = W_k^0(V)$ and $(\cdot,\cdot)_+ = (\cdot,\cdot)_k$ on this domain.*

Proof. We first show that $(u,v)_k = (u,v)_+$ for all $u,v \in F_{2k}$. From Section 2.1 $(u,v)_+ = (Du,v)$ for $u,v \in F_{2k}$. By Lemma 3.2 there is a Hermitian form Φ which is continuous on W_k^0 such that $(Du,v) = \Phi(u,v)$, $u,v \in F_{2k}$. Therefore, $(u,v)_+ = \Phi(u,v)$, $u,v \in F_{2k}$. By Lemma 5.1, $(u,v)_k = (Du,v)$, $u,v \in C_0^\infty$, so $(u,v)_k = \Phi(u,v)$, $u,v \in C_0^\infty$. Since both sides are continuous on W_k^0, this equality holds for $u,v \in W_k^0$. Therefore, $(u,v)_k = (u,v)_+$ for $u,v \in F_{2k}$.

F_{2k} is a core for $D^{1/2}$ so $\mathcal{D}(D^{1/2})$ is the set of $u \in W_0$ for which there exists $(u_i) \subset F_{2k}$ with $u_i \to u$ and u_i is Cauchy for $\|\cdot\|_+$. I claim that W_k^0 is the set of $u \in W_0$ for which there exists $(u_i) \subset C_0^\infty$ with $u_i \to u$ and (u_i) is Cauchy for $\|\cdot\|_k$. If this holds and T_0 is a differential operator of order $\le k$, (T_0u_i) is Cauchy in W_0, so for some $z \in W_0$, $(T_0u_i) \to z$. Then, for $v \in C_0^\infty$,

$$(z,v) = \lim(T_0u_i, v) = \lim(u_i, T_0^{\#}v) = (u, T_0^{\#}v)$$

and $u \in W_k$, so $u \in W_k^0$. Conversely, if $u \in W_k^0$, there exists $(u_i) \subset C_0^\infty$ such that $u_i \xrightarrow{k} u$, which clearly implies the stated condition.

From these descriptions and the facts that $C_0^\infty \subset F_{2k}$ and $\|\cdot\|_k = \|\cdot\|_+$ on F_{2k}, it follows that $W_k^0 \subset \mathcal{D}(D^{1/2})$. On the other hand, suppose $(u_i) \subset F_{2k}$, $u_i \to u$ and (u_i) is Cauchy for $\|\cdot\|_+$. Since C_0^∞ is dense in W_k^0 and $F_{2k} \subset W_k^0$, we may choose $(v_i) \subset C_0^\infty$ such that $\|u_i - v_i\|_+ < 1/i$. Then $v_i \to u$ and (v_i) is Cauchy for $\|\cdot\|_+$, so $\mathcal{D}(D^{1/2}) \subset W_k^0$.

The equality $(\cdot,\cdot)_+ = (\cdot,\cdot)$ follows since it holds on the core F_{2k} of $D^{1/2}$. □

5.2. We now derive consequences concerning Sobolev spaces. For the properties of properly supported pseudodifferential operators, which include differential operators, see [**20**, Chapter II]. Let V and W be any Hermitian A-bundles on M.

Theorem 5.3. 1. *All Sobolev k-inner products on $W_k^0(V)$ are compatible.*

2. *Let T_0 be a properly supported pseudodifferential operator of order $k-l$ from V to W. For any choice of such inner products, the extension $T : W_k^0(V) \to W_l^0(W)$ has an adjoint.*

Proof. 1. We check the hypotheses of Proposition 2.2. Given two such inner products, $\mathcal{E}_1$ and $\mathcal{E}_2$ are W_0 with the $\mathcal{L}^2$-inner products. D_1 and D_2 are the operators with domain $F_{2k}(V)$ derived from the k-inner products. By Proposition 5.2, $\mathcal{D}(D_1^{1/2}) = \mathcal{D}(D_2^{1/2}) = W_k^0(V)$ and $(\cdot,\cdot)_{1,+}$ and $(\cdot,\cdot)_{2,+}$ are the k-inner products. We take $S_1 = S_2 = C_0^\infty(V)$. It was noted in Section 3.1 that the identity $j : S_1 \to S_2$ has a formal adjoint. Since all k-norms are equivalent, j extends to the identity $W_k^0(V) \to W_k^0(V)$. By Proposition 2.2, this has an adjoint for the two k-inner products.

2. Properly supported pseudodifferential operators have properly supported formal adjoints, so Proposition 2.2 again applies. □

The next theorem is more general and applies in particular to operators constructed using Theorem 4.1. For this we assume that inner products have been chosen for M and V, so an inner product $(\cdot,\cdot)$ on $W_0(V)$ is defined. Let D be a symmetric operator with domain $F_{2k}(V)$ and positive spectrum.

Theorem 5.4. 1. *For all such choices of inner products and D, $\mathcal{D}(D^{1/2}) = W_k^0(V)$, and the inner products $(\cdot,\cdot)_+$ on $W_k^0(V)$ are compatible.*

2. *If two such operators D_1 and D_2 have domains $F_{2k}(V)$ and $F_{2l}(W)$, respectively, then the analog of* (2) *in Theorem* 5.3 *holds for*

the inner products $(\cdot,\cdot)_{1,+}$ *and* $(\cdot,\cdot)_{2,+}$.

Proof. If D is the operator associated to some k-inner product for a given inner product $(\cdot,\cdot)$ on $W_0(V)$, $\mathcal{D}(D^{1/2}) = W_k^0(V)$ by Proposition 5.2. For any other D satisfying the conditions for $(\cdot,\cdot)$, conclusion (1) follows from Proposition 2.3. Theorem 5.3 provides the connection between different choices of $(\cdot,\cdot)$.

Assertion (2) is immediate from (1) and Theorem 5.3. □

Our remaining results can be similarly generalized. We now prove an extension of Rellich's theorem.

Proposition 5.5. *The inclusions* $W_k^0(V) \to W_l^0(V)$ *are compact for* $k > l$, *for any Sobolev inner products on these spaces.*

Proof. Let $W_k^0(\varepsilon)$ have a special inner product as in Section 3.2. Then, by Lemma 3.4,

$$\begin{array}{ccc} P \otimes W_k^0 & \longrightarrow & W_k^0(\varepsilon) \\ {\scriptstyle I\otimes j}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ P \otimes W_l^0 & \longrightarrow & W_l^0(\varepsilon) \end{array}$$

with the horizontal arrows unitaries. The Sobolev spaces on the left are those for complex functions. By Rellich's theorem, j is compact. For any Hilbert module Q there are isomorphisms

$$\mathcal{L}(P) \otimes \mathcal{K}(Q) \cong \mathcal{K}(P) \otimes \mathcal{K}(Q) \cong \mathcal{K}(P \otimes Q),$$

the latter coming from tensor product of operators [**5**, Chapter 13, Section 5.1]. Therefore, $I \otimes j$, and thus m is compact. Since all Sobolev inner products are compatible, the same will hold if they are constructed as in Section 4.1, so that there are unitaries t_k : $W_k^0(V) \oplus W_k^0(W) \to W_k^0(\varepsilon)$. The conclusion then follows by composing $t_l^{-1} m t_k$ on the right with the inclusion of $W_k^0(V)$ and on the left by the projection onto $W_l^0(V)$. □

We finally deal with inclusions of codimension zero submanifolds.

Proposition 5.6. *Let N be a codimension zero submanifold with boundary of M. Then the inclusion of N induces adjointable maps $W_k^0(V \mid N) \to W_k^0(V)$.*

Proof. Consider the diagram

$$\begin{tikzcd}
W_k^0(V \mid N) \arrow[r, "r"] \arrow[d, "j"'] & W_k^0(V) \arrow[d, "m"] \\
W_k^0(\varepsilon \mid N) \arrow[r, "s"'] & W_k^0(\varepsilon)
\end{tikzcd}$$

The k-inner product on $C_0^\infty(V)$ restricts to one on $C_0^\infty(V \mid N)$ so that the inclusion of these spaces induces the inclusion r, and similarly for s. If a special inner product is used for ε, then s is identified with $I \otimes q : P \otimes W_{N,k}^0 \to P \otimes W_{M,k}^0$ where q is constructed like r and s. Since q is adjointable, so is s. On the other hand, using the inner products and maps t_k of Section 4.1, we may assume that j and m are inclusions of orthogonal summands. Then m^* is the projection onto $W_k^0(V)$ so $r = m^* s j$ and $r^* = j^* s^* m$. $\square$

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