ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 1, Spring 1999

ON NECESSARY CONDITIONS FOR THE EXISTENCE OF ODD PERFECT NUMBERS

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ABSTRACT. Based on a well-known theorem of Euler, which gives a necessary condition for the existence of odd perfect numbers, the author presents two apparently new necessary conditions. The method of proof also reveals a couple of interesting related results.

1. Introduction. As usual, $\mathbf{Z} := \{0, \pm 1, \pm 2, \dots\}$, $\mathbf{N} := \{0, 1, 2, \dots\}$ and $\mathbf{P} := \mathbf{N} \backslash \{0\}$. Further, recall that the sum-of-divisors function σ is defined by

$$\sigma(n) := \sum_{d \mid n} d, \quad n \in \mathbf{P}.$$

A positive integer n is called *perfect* if and only if $\sigma(n) = 2n$. The existence, or nonexistence, of odd perfect numbers is perhaps the oldest open question of number theory, e.g., see [**2**, p. 12]. There are, however, several theorems which give necessary conditions for the existence of odd perfect numbers. Doubtless, the best known of them is the following theorem due to Euler [**1**, p. 231].

Theorem 1.1. *If $n \in \mathbf{P} \backslash \{1\}$ and n is an odd perfect number, then canonically*

$$n = p^e p_1^{2e_1} p_2^{2e_2} \cdots p_r^{2e_r},$$

where $p, p_1, p_2, \dots, p_r$ are distinct odd primes and $p \equiv e \equiv 1 \pmod 4$.

Clearly, this theorem requires that the given odd integer $n (> 1)$ be factored or, at the very least, partially factored as

$$n = p^e m^2,$$

where $p \equiv e \equiv 1 \pmod 4$ and $m \in \mathbf{P}$. However, even this partial factoring can be decidedly difficult. Based on the obvious consequence

Received by the editors on April 4, 1997.
AMS *Mathematics Subject Classification.* Primary 11A05, Secondary 11A99.
Key words and phrases. Perfect numbers, the existence of odd perfect numbers.

of Euler's result that any odd perfect number must be the form $4m+1$, we here propose to give a couple of necessary conditions which are entirely additive in character. These are: Theorem 1.2, stated below, and its corollary, 2.3.

Theorem 1.2. *For each* $m \in \mathbf{P}$, *if* $4m+1$ *is an odd perfect number, then*

$$(4m+1)^2 + \sum_{k=1}^{2m} \sigma(8m+3-2k)\sigma(2k-1) \equiv 0 \pmod 2.$$

In Section 2 we prove this theorem and observe a couple of additional results that originate in our method. Several arithmetical functions arise naturally. These we collect in the following definition.

Definition 1.3. (i) For each $n \in \mathbf{N}$,

$$r_4(n) := |\{(x_1, x_2, x_3, x_4) \in \mathbf{Z}^4 | n = x_1^2 + x_2^2 + x_3^2 + x_4^2\}|.$$

(ii) For each $n \in \mathbf{P}$, $b(n) :=$ the exponent of the exact power of 2 dividing n, and then $0d(n) := n2^{-b(n)}$ is the odd part of n.

(iii) The arithmetical function Ψ_{12} is defined by

$$\sum_1^\infty \Psi_{12}(n)x^n := x\prod_1^\infty (1-x^{2n})^{12}, \qquad |x| < 1.$$

In part (iii) of the foregoing definition x is to be regarded as a complex variable. The function Ψ_{12} is due to Ramanujan [**5**, p. 155].

2. Proofs. Our proofs depend on the following two identities:

$$\prod_1^\infty (1-x^{2n})(1+tx^{2n-1})(1+t^{-1}x^{2n-1}) = \sum_{-\infty}^\infty x^{n^2}t^n, \tag{2.1}$$

which is valid for each pair of complex numbers t, x such that $t \neq 0$ and $|x| < 1$.

$$\begin{aligned}(2.2)\quad \prod_{1}^{\infty}&(1-x^{2n})^2(1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1})\\ &\qquad\cdot(1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1})\\ &= \sum_{-\infty}^{\infty} x^{2m^2}a^{2m}\sum_{-\infty}^{\infty} x^{2n^2}b^{2n}\\ &\quad + x\sum_{-\infty}^{\infty} x^{2m(m+1)}a^{2m+1}\sum_{-\infty}^{\infty} x^{2n(n+1)}b^{2n+1},\end{aligned}$$

which is valid for each triple of complex numbers a, b, x such that $a \neq 0$, $b \neq 0$ and $|x| < 1$. The first of these two identities, the triple-product identity, is a celebrated result; and, elementary proofs of it abound, e.g., see [**4**, pp. 282–283]. The second identity, due to the author, is not so widely known. But there is an accessible proof of it in [**3**].

Our first step in the proof of Theorem 1.2 is to establish the following lemma.

Lemma 2.1. *For each $m \in \mathbf{N}$,*

$$\begin{aligned}(2.3)\quad \Psi_{12}(2m+1) &= \sum_{k=1}^{2m+1}(-1)^{k-1}r_4(2m+1-k)2^{b(k)}k\sigma(0d(k))\\ &\quad + 4\sum_{k=1}^{2m} 2^{b(2m+1-k)+b(k)}\sigma(0d(2m+1-k))\sigma(0d(k))\\ &\quad + 4\sum_{k=1}^{2m-1}(-1)^k r_4(k)\sum_{j=1}^{2m-k} 2^{b(2m+1-k-j)+b(j)}\\ &\qquad\qquad \cdot\sigma(0d(2m+1-k-j))\sigma(0d(j)),\end{aligned}$$

$$\begin{aligned}(2.4)\quad &\sum_{k=1}^{2m+2}(-1)^k r_4(2m+2-k)2^{b(k)}k\sigma(0d(k))\\ &\quad + 4\sum_{k=1}^{2m+1}2^{b(2m+2-k)+b(k)}\sigma(0d(2m+2-k))\sigma(0d(k))\\ &\quad + 4\sum_{k=1}^{2m}(-1)^k r_4(k)\sum_{j=1}^{2m+1-k}2^{b(2m+2-k-j)+b(j)}\\ &\qquad\qquad \cdot\sigma(0d(2m+2-k-j))\sigma(0d(j))=0.\end{aligned}$$

Proof. To prove Lemma 2.1 we first appeal to identity (2.1) to express each series on the right side of identity (2.2) as an infinite product.

$$\begin{aligned}&\prod_1^\infty(1-x^{2n})^2(1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1})\\ &\qquad\cdot(1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1})\\ &\quad=\prod_1^\infty(1-x^{4n})^2(1+a^2x^{4n-2})(1+a^{-2}x^{4n-2})\\ &\qquad\cdot(1+b^2x^{4n-2})(1+b^{-2}x^{4n-2})\\ &\qquad+x(a+a^{-1})(b+b^{-1})\prod_1^\infty(1-x^{4n})^2(1+a^2x^{4n})\\ &\qquad\qquad\cdot(1+a^{-2}x^{4n})(1+b^2x^{4n})(1+b^{-2}x^{4n}).\end{aligned}$$

Next, in the foregoing identity, we let $a=b$ and then let $a\to ia$ to get

$$\begin{aligned}(2.5)\quad &\prod_1^\infty(1-x^{2n})^2(1+x^{2n-1})^2(1-a^2x^{2n-1})(1-a^{-2}x^{2n-1})\\ &\quad=\prod_1^\infty(1-x^{4n})^2(1-a^2x^{4n-2})^2\\ &\qquad\cdot(1-a^{-2}x^{4n-2})^2-(a-a^{-1})^2x\\ &\qquad\cdot\prod_1^\infty(1-x^{4n})^2(1-a^2x^{4n})^2(1-a^{-2}x^{4n})^2.\end{aligned}$$

Now in (2.5) we let (i) $x \to -x$, (ii) multiply the resulting identity and (2.5), and (iii) in the product of the two identities let $x \to x^{1/2}$ to get

$$\begin{aligned}
(2.6)\quad & \prod_1^\infty (1-x^n)^4(1-x^{2n-1})^2(1-a^4x^{2n-1})(1-a^{-4}x^{2n-1}) \\
&= \prod_1^\infty (1-x^{2n})^4(1-a^2x^{2n-1})^4(1-a^{-2}x^{2n-1})^4 \\
&\quad - (a-a^{-1})^4 x \prod_1^\infty (1-x^{2n})^4 \\
&\qquad \cdot (1-a^2x^{2n})^4(1-a^{-2}x^{2n})^4.
\end{aligned}$$

With z a complex variable and D_z denoting differentiation with respect to z, we define the operator Θ_z by $\Theta_z := zD_z$. We then operate on both sides of (2.6) with Θ_a^4; and, thereafter let $a \to 1$. Since there is a considerable amount of formal algebra in our proof, we adopt the following abbreviations.

$$G = G(a,x) := x\prod_1^\infty (1-x^{2n})^4(1-a^2x^{2n})^4(1-a^{-2}x^{2n})^4,$$

$$E = E(a,x) := \prod_1^\infty (1-x^{2n})(1-a^2x^{2n-1})(1-a^{-2}x^{2n-1}),$$

$$F = F(a,x) := \prod_1^\infty (1-x^{2n})(1-a^4x^{2n-1})(1-a^{-4}x^{2n-1})$$

and

$$\alpha(x) := \prod_1^\infty (1-x^{2n})^3(1-x^{2n-1})^6.$$

Of course, $F(a,x) = E(a^2,x)$ and $\alpha(x) = E(1,x)^3$. But we use different symbols to help us keep track of proceedings. Identity (2.6) now reads

$$(a-a^{-1})^4 G(a,x) = E(a,x)^4 - \alpha(x)F(a,x).$$

Easily,

$$\Theta_a^4\{(a-a^{-1})^4 G(a,x)\}|_{a=1} = 2^4 \cdot 4!\, G(1,x).$$

We now operate on the right side of the foregoing identity, and thereafter let $a \to 1$.

Recall that, for any complex number z and any $k \in \mathbf{N}$, $(z)_k := z(z-1)\cdots(z-k+1)$. Then, appealing (via identity (2.1)) to the series representations of $E(a,x)$ and $F(a,x)$, we get

$$\Theta_a^4 E(a,x)^4|_{a=1} = 3\cdot(4)_2\cdot 2^4\cdot E(1,x)^2\left\{\sum_{-\infty}^{\infty}(-1)^n n^2 x^{n^2}\right\}^2 + 4\cdot 2^4\cdot E(1,x)^3\cdot\sum_{-\infty}^{\infty}(-1)^n n^4 x^{n^2},$$

$$\Theta_a^4\{-\alpha(x)F(a,x)\}|_{a=1} = -2^8\cdot E(1,x)^3\cdot\sum_{-\infty}^{\infty}(-1)^n n^4 x^{n^2}.$$

Hence,

$$2G(1,x) = -E(1,x)^3\sum_{-\infty}^{\infty}(-1)^n n^4 x^{n^2} + 3E(1,x)^2\left\{\sum_{-\infty}^{\infty}(-1)^n n^2 x^{n^2}\right\}^2. \tag{2.7}$$

Put

$$f(x) := \sum_{-\infty}^{\infty}(-1)^n x^{n^2}.$$

Then we easily recognize

$$\Theta_x f(x) = \sum_{-\infty}^{\infty}(-1)^n n^2 x^{n^2}$$

and

$$\Theta_x^2 f(x) = \sum_{-\infty}^{\infty}(-1)^n n^4 x^{n^2}.$$

However, identity (2.1) yields

$$f(x) = \prod_1^{\infty}(1-x^n)(1-x^{2n-1}).$$

Hence,

$$\begin{aligned}\Theta_x(f(x) &= f(x)\left[-\sum_1^\infty \frac{nx^n}{1-x^n} - \sum_1^\infty \frac{(2n-1)x^{2n-1}}{1-x^{2n-1}}\right]\\ &= f(x)\left[-\sum_1^\infty 2^{b(n)+1}\sigma(0d(n))x^n\right]\end{aligned}$$

and

$$\begin{aligned}\Theta_x^2 f(x) = f(x)\bigg\{\bigg[-\sum_1^\infty 2^{b(n)+1}\sigma(0d(n))x^n\bigg]^2\\ - \sum_1^\infty 2^{b(n)+1} n\sigma(0d(n))x^n\bigg\}.\end{aligned}$$

Now, substituting these values for $\Theta_x f(x)$ and $\Theta_x^2 f(x)$ back into identity (2.7), we get

$$\begin{aligned}G(1,x) = E(1,x)^4\bigg\{\sum_1^\infty 2^{b(n)} n\sigma(0d(n))x^n\\ + 4\bigg[\sum_1^\infty 2^{b(n)}\sigma(0d(n))x^n\bigg]^2\bigg\}.\end{aligned}\tag{2.8}$$

We now recall that

$$\begin{aligned}E(1,x)^4 &= \prod_1^\infty (1-x^{2n})^4(1-x^{2n-1})^8\\ &= \left\{\sum_{-\infty}^\infty (-1)^n x^{n^2}\right\}^4\\ &= \sum_0^\infty (-1)^n r_4(n)x^n.\end{aligned}$$

Now, substituting this last series for $E(1,x)^4$ into the right side of (2.8), thereafter expanding and equating coefficients of like powers of x, we prove Lemma 2.1. □

On the strength of a well-known formula of Jacobi, we now eliminate r_4 from the identities (2.3) and (2.4). This formula is $r_4(n) = 8(2 + (-1)^n)\sigma(0d(n))$ for each $n \in \mathbf{P}$. For a proof, see [**4**, pp. 311–314].

Corollary 2.2. *For each* $m \in \mathbf{N}$,

$$\begin{aligned}\Psi_{12}(2m+1) = (2m+1)\sigma(2m+1) &\\ + 8\sum_{k=1}^{2m}(-1)^{k-1}\{2+(-1)^{2m+1-k}\} &\\ \cdot\,\sigma(0d(2m+1-k))2^{b(k)}k\sigma(0d(k)) &\\ + 4\sum_{k=1}^{2m}2^{b(2m+1-k)+b(k)} &\\ \cdot\,\sigma(0d(2m+1-k))\sigma(0d(k)) &\\ + 32\sum_{k=1}^{2m-1}(-1)^k\{2+(-1)^k\} &\\ \cdot\,\sigma(0d(k))\sum_{j=1}^{2m-k}2^{b(2m+1-k-j)+b(j)} &\\ \cdot\,\sigma(0d(2m+1-k-j))\sigma(0d(j)), &\end{aligned}\tag{2.9}$$

$$\begin{aligned}&2^{b(2m+2)}(2m+2)\sigma(0d(2m+2))\\ &+ 8\sum_{k=1}^{2m+1}(-1)^k\{2+(-1)^{2m+2-k}\}\\ &\qquad\cdot\,\sigma(0d(2m+2-k))2^{b(k)}k\sigma(0d(k))\\ &+ 4\sum_{k=1}^{2m+1}2^{b(2m+2-k)+b(k)}\sigma(0d(2m+2-k))\sigma(0d(k))\\ &+ 32\sum_{k=1}^{2m}(-1)^k\{2+(-1)^k\}\sigma(0d(k))\sum_{j=1}^{2m+1-k}2^{b(2m+2-k-j)+b(j)}\\ &\qquad\cdot\,\sigma(0d(2m+2-k-j))\sigma(0d(j)) = 0.\end{aligned}\tag{2.10}$$

Identity (2.10) is the key to completing the proof of Theorem 1.2. We recall that, for each $n \in \mathbf{P}$, $b(2n) = b(n) + 1$ and $0d(2n) = 0d(n)$.

Then, in (2.10) we let $m \to 4m$ and cancel a factor of 4 to get

$$\begin{aligned}
&(4m+1)\sigma(4m+1) - 2(4m+1)\sigma(4m+1)^2 \\
&\quad + 2\sum_{k=1}^{4m}(-1)^k\{2+(-1)^{8m+2-k}\} \\
&\qquad\qquad \cdot \sigma(0d(8m+2-k))2^{b(k)}k\sigma(0d(k)) \\
&\quad + 2\sum_{k=4m+2}^{8m+1}(-1)^k\{2+(-1)^{8m+2-k}\} \\
&\qquad\qquad\quad \cdot \sigma(0d(8m+2-k))2^{b(k)}k\sigma(0d(k)) \\
&\quad + \sum_{k=1}^{8m+1} 2^{b(8m+2-k)+b(k)}\sigma(0d(8m+2-k))\sigma(0d(k)) \\
&\quad + 8\sum_{k=1}^{8m}(-1)^k\{2+(-1)^k\}\sigma(0d(k)) \sum_{j=1}^{8m+1-k} 2^{b(8m+2-k-j)+b(j)} \\
&\qquad\qquad\qquad\qquad \cdot \sigma(0d(8m+2-k-j))\sigma(0d(j)) = 0.
\end{aligned}$$

Now we explicitly assume that $4m+1$ is perfect, so that $\sigma(4m+1) = 2(4m+1)$. For the first and second sigma-sums of the foregoing identity, the terms corresponding to even values of the index k are clearly divisible by 2. For the first sigma-sum, the sum (including the factor 2) corresponding to odd values of the index k is

$$S_1 := -2\sum_{j=0}^{2m-1} \sigma(8m+2-2j-1)\{2j+1\}\sigma(2j+1).$$

And, for the second sigma-sum, the sum corresponding to odd values of the index k is

$$\begin{aligned} S_2 := & -2 \sum_{j=2m+1}^{4m} \sigma(8m+2-2j-1)\{2j+1\}\sigma(2j+1) \\ = & -2 \sum_{j=0}^{2m-1} \sigma(8m+2-(8m+1-2j)) \\ & \cdot \{8m+1-2j\}\sigma(8m+1-2j) \\ = & -2 \sum_{j=0}^{2m-1} \sigma(2j+1)\{8m+2-2j-1\} \\ & \cdot \sigma(8m+2-2j-1). \end{aligned}$$

Hence,

$$S_1 + S_2 = -2(8m+2) \sum_{j=0}^{2m-1} \sigma(2j+1)\sigma(8m+2-2j-1).$$

Regarding the third sigma-sum, we observe that

$$\begin{aligned} & \sum_{k=1}^{8m+1} 2^{b(8m+2-k)+b(k)} \sigma(0d(8m+2-k))\sigma(0d(k)) \\ & = \sigma(4m+1)^2 + 2\sum_{k=1}^{2m} \sigma(8m+3-2k)\sigma(2k-1) \\ & \quad + \text{ a sum divisible by 4.} \end{aligned}$$

Now we cancel a factor of 2 in the identity and observe that

$$(4m+1)^2 + \sum_{k=1}^{2m} \sigma(8m+3-2k)\sigma(2k-1) \equiv 0 \pmod 2.$$

This completes the proof of Theorem 1.2.

Corollary 2.3. *For each $m \in \mathbf{P}$, if $4m+1$ is perfect and*

$$S := \{k \in \mathbf{P} \mid k \le 2m; 8m+3-2k \quad \textit{and} \quad 2k-1 \textit{ are squares}\},$$

then $|S| \equiv 1 \pmod 2$.

Proof. We recall that, for $n \in \mathbf{P}$, $\sigma(n)$ is odd if and only if n is a square or twice a square. Hence, for each $k \in \{1, 2, \dots, 2m\}$, the term $\sigma(8m+3-2k)\sigma(2k-1)$ is odd if and only if both $8m+3-2k$ and $2k-1$ are squares. Thus, because of Theorem 1.2 we must have $|S| \equiv 1 \pmod 2$. $\square$

Concluding remarks. The arithmetical function Ψ_{12} of identity 2.3 arises naturally in our proof of Theorem 1.2. It is one of a family of such functions first introduced by Ramanujan [**5**, p. 155]. As a matter of fact, for each positive divisor α of 24, the arithmetical function Ψ_α is defined by

$$\sum_{1}^{\infty} \Psi_\alpha(n) x^n := x \prod_{1}^{\infty} (1 - x^{24n/\alpha})^\alpha,$$

an identity which is valid for each complex number x such that $|x| < 1$. Of course, $\Psi_{24} = \tau$, the celebrated Ramanujan tau function. Formulas for both Ψ_6 and Ψ_{12} are presented and discussed elsewhere.

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