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LEGENDRE EXPANSIONS OF POWER SERIES

JAMES GUYKER

ABSTRACT. We estimate the Legendre coeffiients of power series representations and the rates of pointwise and mean square convergence of their Legendre series expansions. Our main result is based on showing that the nth coefficient of the Legendre expansion of x^m does not exceed $2/n$.

1. Introduction. In an application of the Gram-Schmidt procedure to curve fitting, polynomials are expressed in terms of a family of orthogonal polynomials. It is natural to attempt orthogonal expansions of functions defined by power series over a given interval by converting their succesive partial sums. (Many such functions, especially ones with no closed forms, arise as solutions of linear differential equations with power series as coefficients.) In particular, least squares approximations with respect to the simplest inner product are obtained by finding the first few terms of Legendre series expansions. For this case we show that the conversions are easily accomplished and derive straightforward error estimates for the rates of pointwise and mean square convergence. We then illustrate the errors with the standard Maclaurin series representations of calculus.

Given an integrable function $f(x)$ on $[-1,1]$, the unique polynomial which minimizes $\int_{-1}^{1}(f(x)-p(x))^2\,dx$ over all polynomials $p(x)$ of degree at most n is

$$P_n(x)=\sum_{j=0}^{n} b_j p_j(x)$$

where $p_j(x)=(2^j j!)^{-1}(d^j/dx^j)(x^2-1)^j$ is the classical Legendre polynomial (from Rodrigues's formula) and

$$b_j=\frac{2j+1}{2}\int_{-1}^{1} f(x)p_j(x)\,dx,$$

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see [**2, 10**]. In particular, if f is continuous on $[-1,1]$, then $\lim_n \int_{-1}^{1}(f(x)-P_n(x))^2\,dx = 0$ by the Weierstrass approximation theorem. Moreover, if both f and f' are sectionally continuous on $(-1,1)$, then $f(x)=\lim_n P_n(x)$ for each x in $(-1,1)$ [**8, 11–13, 15**]. In this article we approximate the Legendre coefficients b_j from elementary vector products for functions f which are defined by power series on $(-1,1)$ and obtain bounds on the rates of these limits. As usual, functions over intervals (a,b) are reduced to this case via the transformation $x \to (2x-b-a)/(b-a)$. A key identity in our development is an analogue of the Abel partial summation formula [**1**, Theorem 10.16]: for any numbers z_n and w_n

$$(1)\qquad \sum_{i=m}^{n} z_i w_i = \left(\sum_{i=m}^{n} z_i\right) w_m + \sum_{j=m}^{n-1}\left(\sum_{i=j+1}^{n} z_i\right)(w_{j+1}-w_j).$$

2. Coefficient estimates. Suppose that $f(x)=\sum a_i x^i$ where $\sum(a_{2i}/(2i+1))$ and $\sum(a_{2i+1}/(2i+2))$ both converge. Then $f(x)$ is defined for each x in $(-1,1)$. The proof is as follows:

$$\sum_i a_{2i+t}x^{2i+t} = \sum_i \frac{a_{2i+t}}{2i+t+1}(2i+t+1)x^{2i+t}, \quad t=0,1,$$

where $(2i+t+1)|x|^{2i+t}$ decreases for $i > N = -(1/2)(t+1+(1/\ln|x|))$ and converges to zero. By (1) we have that, for $n > N$,

$$\left|\sum_{2i+t>n} a_{2i+t}x^{2i+t}\right| \le 2(2n+t+1)|x|^{2n+t}\varepsilon(n,t),$$

where

$$(2)\qquad \varepsilon(n,t) = \max\left\{\left|\sum_{2i+t>k}\frac{a_{2i+t}}{2i+t+1}\right| : k \ge n\right\},$$

and hence $\lim_n \sum_{2i+t>n} a_{2i+t}x^{2i+t} = 0$. Therefore, $\sum a_i x^i$ converges.

The Legendre expansion of x^k is well known [**11, 13, 15**]:

$$x^{2i+t} = \sum_{j=0}^{i} m_{i,2j+t}p_{2j+t}(x), \quad t=0,1,$$

where

$$m_{ij} = \frac{2j+1}{2i+2} \prod_{k=1}^{[(j+2)/2]} \frac{2i+4-2k}{2i-(-1)^j+2k} \tag{3}$$

($[\cdot]$ is the greatest integer function).

The following result shows that the vector $\langle b_j \rangle$ of Legendre coefficients of $f(x)$ may be obtained from an easily generated matrix product.

Theorem 1. *Let* $f(x) = \sum a_i x^i$ *where* $\sum(a_{2i}/(2i+1))$ *and* $\sum(a_{2i+1}/(2i+2))$ *converge. Let* j *be a nonnegative integer and* $t = t(j) = j - 2[j/2]$. *Then* $b_j = \sum_i a_{2i+t} m_{ij}$ *and we have the estimate*

$$\left| b_j - \sum_{2i+t \le n} a_{2i+t} m_{ij} \right| \le (2j+1)\varepsilon(n,t).$$

Proof. By Abel's test [**1**, Theorem 10.18], for $k = 0, 1, \ldots,$ the series

$$\sum_i \frac{a_{2i+t+s}}{2i+2k+2t+s+1} = \sum_i \frac{a_{2i+t+s}}{2i+t+s+1} \, \frac{2i+t+s+1}{2i+2k+2t+s+1}, \quad s = 0, 1,$$

converge and therefore so do $\sum_i((-1)^{si} a_{i+t}/(i+2k+2t+1))$. Hence, by Abel's theorem [**3**, p. 325],

$$\begin{aligned} b_j &= \frac{2j+1}{2}\left(\int_{-1}^{0} f(x) p_j(x)\, dx + \int_0^1 f(x) p_j(x)\, dx \right) \\ &= (2j+1) \sum_{k=0}^{[j/2]} \frac{p_j^{(2k+t)}(0)}{(2k+t)!} \sum_i \frac{a_{2i+t}}{2i+2k+2t+1}. \end{aligned}$$

It follows that $b_j = \lim_n b_{nj}$ where $\sum_{k=0}^n b_{nk} p_k(x)$ is the Legendre expansion of the partial sum $\sum_{i=0}^n a_i x^i$. Therefore, b_j is of the given form since

$$\sum_{i=0}^n a_i x^i = \sum_{t=0}^{1} \sum_{2i+t \le n} \sum_{j=0}^{i} a_{2i+t} m_{i,2j+t} p_{2j+t}(x)$$

and $\{p_k(x) : 0 \leq k \leq n\}$ is linearly independent.

For the estimate,

$$\begin{aligned} |b_j - b_{nj}| &= \left| \sum_{2i+t>n} a_{2i+t} m_{ij} \right| \\ &= \left| \sum_{2i+t>n} \frac{a_{2i+t}}{2i+t+1}(2i+t+1)m_{ij} \right| \end{aligned}$$

where $(2i+t+1)m_{ij}$ is monotonically increasing with limit $2j+1$. Thus we have that $|b_j - b_{nj}|$ does not exceed $(2j+1)\varepsilon(n,t)$ by (1) and (2). $\square$

Remark. Let $\langle c_i \rangle$ be any bounded, monotone sequence. If $\sum a_i x^i$ satisfies Theorem 1 with coefficient error estimate $(2j+1)\varepsilon(n,t)$, then by (1) so does $\sum a_i c_i x^i$ with error estimate bounded by $(2j+1)(|c_{n+1}| + |c - c_{n+1}|)\varepsilon(n,t)$ where $c = \lim c_i$.

When the convergence is absolute, $\varepsilon(n,t)$ is at most $\sum_{2i+t>n} |a_{2i+t}|/(2i+t+1)$. In particular, if the terms of the sequence a_{2i+t} have a common sign and $f(x) = \sum a_i x^i$ has a closed form, then estimates to $\varepsilon(n,t)$ may be compared to its exact value

$$\left| \frac{1}{2}\left(\int_0^1 f(x)\,dx + (-1)^t \int_{-1}^0 f(x)\,dx \right) - \sum_{i=0}^{[(n-t-1)/2]} \frac{a_{2i+t}}{2i+t+1} \right|.$$

Examples 1. Consider the transformation to $[-1,1]$ of the exponential function e^x defined on $[a,b]$:

$$\exp\left(\frac{a+b+(b-a)x}{2}\right) = \exp\left(\frac{a+b}{2}\right) \sum \left(\frac{b-a}{2}\right)^i \frac{x^i}{i!}.$$

By the ratio test, if $n \geq r = (b-a)/2$, then

$$\begin{aligned} \varepsilon(n,t) &= \frac{e^a}{2r}(e^r - 1)(e^r + (-1)^t) - e^{a+r} \sum_{i=0}^{[(n-t-1)/2]} \frac{r^{2i+t}}{(2i+t+1)!} \\ &\leq \frac{e^{a+r} r^{n+1}}{n!((n+2)(n+1) - r^2)}. \end{aligned}$$

(Similarly for $\sinh x$ and $\cosh x$.)

2. For $|u| > 1$ and positive integer k, we have that $(u^k/(u^k - x^k)) = \sum_i (x^{ki}/u^{ki})$ on $(-1, 1)$ has estimate

$$\varepsilon(n,t) \leq \frac{1}{n+2} \frac{1}{|u|^{n+1}} \frac{|u|^k}{|u|^k - 1}$$

by the geometric sum formula.

3. $\sec x = \sum E_i^* x^{2i}$ on $[-1, 1]$ where $E_0^* = 1$ and

$$E_i^* = \sum_{j=1}^{i} (-1)^{j-1} \frac{E_{i-j}^*}{(2j)!}.$$

($(2i)! E_i^*$ is the Euler number E_i.) Then E_i^* is monotonically decreasing and $((3+\sqrt{3})/12)^i \leq E_i^* \leq (1/2)^i$ for all i. This will follow from the inequalities

$$\left(\frac{3+\sqrt{3}}{12}\right) E_{i-1}^* \leq E_i^* \leq \left(\frac{1}{2}\right) E_{i-1}^*$$

for $i \geq 1$. We first verify the lower inequality by induction. The case $i = 1$ is clear so let $i > 1$ and assume that it is true for all subscripts $i' < i$. Setting $w = (3+\sqrt{3})/12$, we have that

$$\begin{aligned} E_i^* - wE_{i-1}^* &= \left(\left(\frac{1}{2} - w\right) E_{i-1}^* - \left(\frac{1}{4!}\right) E_{i-2}^* \right) \\ &\quad + \sum_{j=2}^{[i/2]} \left(\frac{E_{i-2j+1}^*}{(4j-2)!} - \frac{E_{i-2j}^*}{(4j)!} \right) + (i - 2[i/2]) \frac{E_0^*}{(2i)!} \\ &\geq \left(\left(\frac{1}{2} - w\right) w - \frac{1}{24} \right) E_{i-2}^* \\ &\quad + \sum_{j=2}^{[i/2]} \left(\frac{w}{(4j-2)!} - \frac{1}{(4j)!} \right) E_{i-2j}^* + (i - 2[i/2]) \frac{E_0^*}{(2i)!} \\ &\geq 0 \end{aligned}$$

since each term is nonnegative.

Finally the upper inequality is a direct consequence of the lower since $(1/2)E^*_{i-1} - E^*_i$ is equal to

$$\sum_{j=1}^{[(i-1)/2]} \left(\frac{E^*_{i-2j}}{(4j)!} - \frac{E^*_{i-2j-1}}{(4j+2)!} \right) + (i-1-2[(i-1)/2]) \frac{E^*_0}{(2i)!}$$
$$\geq \sum_{j=1}^{[(i-1)/2]} \left(\frac{w}{(4j)!} - \frac{1}{(4j+2)!} \right) E^*_{i-2j-1} + (i-1-2[(i-1)/2]) \frac{E^*_0}{(2i)!}.$$

Therefore,

$$\varepsilon(n,0) = \ln\left(\frac{\cos 1}{1-\sin 1} \right) - \sum_{i=0}^{[(n-1)/2]} \frac{E^*_i}{2i+1}$$
$$\leq \frac{(1/2)^{(n-1)/2}}{n+2}$$

as in Example 2.

4. $\tan x = (\sec x)(\sin x) = \sum_{i\geq 1} B^*_{2i} x^{2i-1}$ on $[-1,1]$ where the definitions of B^*_{2i} and E^*_i are closely related:

$$B^*_{2i} = \sum_{j=1}^{i} (-1)^{j-1} \frac{E^*_{i-j}}{(2j-1)!}.$$

In fact, it will follow that $0 \leq 2E^*_i - B^*_{2i} \leq ((3-\sqrt{3})/3)(1/2)^{i-1}$. (The value of $(-1)^{i+1} B^*_{2i}(2i)!/(2^{2i}(2^{2i}-1))$ is the Bernoulli number B_{2i}.) Then B^*_{2i} is monotonically decreasing with $(\sqrt{3}/3)((3+\sqrt{3})/12)^{i-1} \leq B^*_{2i} \leq (1/2)^{i-1}$ since it follows as in Example 3 that

$$\frac{\sqrt{3}}{3} E^*_{i-1} \leq B^*_{2i} \leq E^*_{i-1},$$

and hence

$$B^*_{2i} \leq E^*_{i-1} \leq \frac{1}{2} E^*_{i-2} \leq \frac{\sqrt{3}}{3} E^*_{i-2} \leq B^*_{2(i-1)}.$$

Thus,

$$\varepsilon(n,1) = \ln(\sec x) - \sum_{i=0}^{[(n-2)/2]} \frac{B^*_{2i+2}}{2i+2}$$
$$\le \frac{1}{\sqrt{2}-1}\frac{(1/2)^{(n-1)/2}}{n+2}.$$

5. Let $f(x) = \sum a_i x^i$ on $(-1,1)$ where $|a_i| \le 1/i^p$ for some $p > 0$, e.g., $\ln(1+x)$ and $\operatorname{arctanh} x$. Note that the integrals in the least squares problem for $\ln(1+x)$ are improper. By the integral test, $\varepsilon(n,t) \le 1/(2p(n-1)^p)$.

6. For real $h > -1$, the binomial series $(1+x)^h = \sum \binom{h}{i} x^i$ defined on $(-1,1)$ satisfies Theorem 1 by Raabe's test [**1, 3, 14**] and, for $n > h+3$,

$$\varepsilon(n,t) = \left| \frac{2^h - t}{h+1} - \sum_{i=0}^{[(n-t-1)/2]} \frac{C_i}{2i+t+1} \right|$$

where $C_0 = h^t$ and, with $j = 2i+t$,

$$C_i = \frac{(h-j+2)(h-j+1)}{j(j-1)} C_{i-1}.$$

Convergence is slow for values of h near -1, e.g., $\varepsilon(40,0) \approx .0895$ for $h = -.5$, but is increasingly better for larger values of h, $\varepsilon(40,0) \approx .000000622$ for $h = 2.3$.

7. For $\arcsin x = x + \sum_{i\ge 1}(1\cdot 3\cdots(2i-1)/(2\cdot 4\cdots(2i)))(x^{2i+1}/(2i+1))$ on $[-1,1]$, we have that

$$\varepsilon(n,1) = \frac{\pi-2}{2} - \sum_{i=0}^{[(n-2)/2]} \frac{D_i}{(2i+1)(2i+2)}$$

where $D_0 = 1$ and $D_i = ((2i-1)/(2i))D_{i-1}$, e.g., $\varepsilon(41,1) \approx .000967$.

When Leibniz's alternating series test holds for $\sum a_{2i+t}/(2i+t+1)$, it follows that $\varepsilon(n,t) \le \max\{|a_{n+1}|/(n+2), |a_{n+2}|/(n+3)\}$:

8. For $\sin x = \sum (-1)^i (x^{2i+1}/(2i+1)!)$ on $[-1,1]$, $\varepsilon(n,1) \le (1/(n+2)!)$. (Similarly, for $\cos x$, $\arctan x$ and $\operatorname{arcsinh} x$.)

9. $\operatorname{sech} x = \sum(-1)^i E_i^* x^{2i}$ and $\tanh x = \sum_{i\geq 1}(-1)^{i+1} B_{2i}^* x^{2i-1}$ on $[-1,1]$ satisfy Leibniz's test by the arguments in Examples 3 and 4, respectively. Therefore, $\varepsilon(n,t) \leq (1/2)^{(n-t+1)/2}/(n+2)$ for both series.

10. $(1+x^2)^{-1} = \sum(-1)^i x^{2i}$ on $(-1,1)$ has estimate $\varepsilon(n,0) \leq 1/(n+2)$. (Similarly for $(\operatorname{arcsinh} x)' = (1+x^2)^{-1/2}$.)

When a bound M on the partial sums of $\sum a_{2i+t}$ is known, $\varepsilon(n,t) \leq 4M/(n+2)$ by (1).

11. Consider the series $f(x) = \sum \sin((i+1)\theta)x^i$ on $(-1,1)$ where θ is a fixed angle that is not a multiple of π. By [**1**, Theorem 10.19], $|\sum_{i\leq k}\sin(2i\theta)|$ and $|\sum_{1\leq i\leq k}\cos(2i\theta)|$ are bounded by $|\sin\theta|^{-1}$ for every k, and thus $\varepsilon(n,t) \leq (8-4t)/((n+2)|\sin\theta|)$.

3. Convergence. For pointwise convergence, we will need coefficient error estimates $e(n,j)$ such that $e(n,j)/\sqrt{j}$ is summable with respect to j. They will readily follow from the next result.

Proposition. *Let m_{ij} be defined by* (3) *for all nonnegative integers i and j. Then, for each j, we have that $m_{ij} \leq 2/(j+1)$ for all i, and $\lim_i m_{ij} = 0$. Moreover, $m_{ij} \geq m_{i+1,j}$ if and only if $i \geq I(j) = [1+(j-2)(j+3)/4]$.*

Proof. It is easily checked that $\lim_i m_{ij} = 0$ and that $m_{ij} \geq m_{i+1,j}$ if and only if $i \geq I(j)$.

Since $m_{i0} = (2i+1)^{-1}$ and $m_{I(1),1} = 1$, in order to verify $m_{ij} \leq 2/(j+1)$, it suffices to show that

$$(j+2)m_{I(j+1),j+1} \leq (j+1)m_{I(j),j} \tag{4}$$

for all $j \geq 1$. Consider first the case when j is a multiple of 4. For convenience, let $y_j = j^2+3j$. Then $I(j) = (y_j - 2j - 4)/4$ and $I(j+1) = y_j/4$. Since $m_{I(j+1),j+1} = m_{I(j+1)-1,j+1}$, in this case (4) is equivalent to

$$\prod_{i=1}^{j/2} \frac{(y_j-4i)(y_j-4i-2)}{(y_j-4i+2j+2)(y_j-4i-2j)} \leq \frac{(j+1)(2j+1)(y_j+2j+2)}{(j+2)(2j+3)(y_j-2)}$$

which may be rewritten in the form

$$\prod_{i=1}^{j/2} \frac{x_i}{x_i + 8(j-2i)} \le \frac{y}{y + (3j^2 - 7j - 14)} \tag{5}$$

where $y \ge 2x_i \ge 0$ for all i. To show (5), let us cross multiply and observe that

$$\begin{aligned} y\prod(x_i + 8(j-2i)) - y\prod x_i &\ge \sum_{k=1}^{j/2} 8y(j-2k)\prod_{i\ne k} x_i \\ &\ge 16\Big(\prod x_i\Big)\sum_{k=1}^{j/2}(j-2k) \\ &= (4j^2 - 8j)\prod x_i. \end{aligned}$$

(5) is now clear since it is reduced to checking that $3j^2 - 7j - 14 \le 4j^2 - 8j$.

Suppose next that $j = 4k - 1$ for some positive integer k. Using the same notation as in case 1, we have that (4) is equivalent to

$$\prod_{i=1}^{(j-1)/2} \frac{x_i}{x_i + 8(j-2i)} \le \frac{y}{y + (3j^2 - 7j - 14)}$$

which may be established as above with slight modification.

Now let $j = 4k - 2$ for some $k \ge 1$. In this case, (4) is equivalent to

$$\prod_{i=1}^{j/2} \frac{(y_j - 4i + 2)(y_j - 4i)}{(y_j - 4i + 2j + 4)(y_j - 4i - 2j + 2)} \le \frac{(j+1)(2j+1)(y_j + 2j + 4)}{(j+2)(2j+3)y_j}$$

which is of the form

$$\prod_{i=1}^{j/2} \frac{x_i^*}{x_i^* + 8(j - 2i + 1)} \le \frac{y^*}{y^* + (3j^2 + j - 4)} \tag{6}$$

where $y^* \geq 2x_i^* \geq 0$ for all i. Cross multiplying as in case 1, we have that

$$\begin{aligned} y^* \prod (x_i^* + 8(j - 2i + 1)) - y^* \prod x_i^* &\geq \sum_{k=1}^{j/2} 8y^*(j - 2k + 1) \prod_{i \neq k} x_i^* \\ &\geq 16\Big(\prod x_i^* \Big) \sum_{k=1}^{j/2} (j - 2k + 1) \\ &= 4j^2 \prod x_i^*. \end{aligned}$$

Hence (6) follows since $3j^2 + j - 4 \leq 4j^2$.

Finally, consider the case $j = 4k - 3$ for $k \geq 1$. Here (4) is equivalent to

$$\prod_{i=1}^{(j-1)/2} \frac{x_i^*}{x_i^* + 8(j - 2i + 1)} \leq \frac{y^*}{y^* + (3j^2 + j - 4)}$$

which is analogous to (6). $\square$

Suppose now that $f(x) = \sum a_i x^i$ where $\sum a_{2i}$ and $\sum a_{2i+1}$ both converge. (Equivalently, $f(x)$ is continuous on $[-1, 1]$ by Abel's theorem.) Then, by the above remark, $f(x)$ satisfies the hypotheses of Theorem 1, and the proof of the following result contains a refinement of the coefficient error estimate in this case.

Theorem 2. *Let $f(x) = \sum a_i x^i$ where $\sum a_{2i}$ and $\sum a_{2i+1}$ converge, and let $P_n(x) = \sum_{j=0}^{n} b_j p_j(x)$. Then, for each n,*

$$\left(\int_{-1}^{1} (f(x) - P_n(x))^2 \, dx \right)^{1/2} \leq \left(\frac{8}{2n + 3} \right)^{1/2} (\varepsilon'(n, 0) + \varepsilon'(n, 1))$$

and for x in $(-1, 1)$,

$$\begin{aligned} |f(x) - P_n(x)| \leq{}& 2(\varepsilon'(n, 0) + \varepsilon'(n, 1))|x|^{n+1} + \frac{2\varepsilon'(n, 0)}{n + 2} \\ &+ \left(3 - \frac{2}{\sqrt{n}} \right) \max\{\varepsilon'(n, 0), \varepsilon'(n, 1)\} \left(\frac{32\pi}{1 - x^2} \right)^{1/2}, \end{aligned}$$

where $\varepsilon'(n,t) = \max\{|\sum_{2i+t>k} a_{2i+t}| : k \geq n\}$.

Proof. The mean squares estimate is direct. By the least squares property of $P_n(x)$,

$$\begin{aligned}\int_{-1}^{1} (f(x) - P_n(x))^2\,dx &\leq \int_{-1}^{1} \bigg(\sum_{i>n} a_i x^i\bigg)^2 dx \\ &\leq \int_{-1}^{1} \bigg(\bigg|\sum_{2i>n} a_{2i} x^{2i}\bigg| + \bigg|\sum_{2i+1>n} a_{2i+1} x^{2i+1}\bigg|\bigg)^2 \\ &\leq \int_{-1}^{1} 4(\varepsilon'(n,0) + \varepsilon'(n,1))^2 x^{2(n+1)}\,dx \\ &= \bigg(\frac{8}{2n+3}\bigg)(\varepsilon'(n,0) + \varepsilon'(n,1))^2.\end{aligned}$$

Similarly, with $t = j - 2[j/2]$, we have

$$\begin{aligned}|f(x) - P_n(x)| &\leq \bigg|\sum_{i>n} a_i x^i\bigg| + \bigg|\bigg(\sum_{i=0}^{n} a_i x^i\bigg) - P_n(x)\bigg| \\ &\leq 2(\varepsilon'(n,0) + \varepsilon'(n,1))|x|^{n+1} \\ &\quad + \bigg|\bigg(\sum_{2i>n} a_{2i} m_{i0}\bigg) p_0(x)\bigg| \\ &\quad + \bigg|\sum_{j=1}^{n}\bigg(\sum_{2i+t>n} a_{2i+t} m_{ij}\bigg) p_j(x)\bigg|.\end{aligned} \tag{7}$$

A well-known bound on $p_j(x)$ for $j \geq 1$ and $-1 < x < 1$ is given by $p_j^2(x) \leq \pi/(2j(1-x^2))$ [**8**, p. 210], [**12**, p. 63]. Moreover, by (1) and the proposition,

$$\bigg|\sum_{2i+t>n} a_{2i+t} m_{ij}\bigg| \leq 4\varepsilon'(n,t) m_{I(j),j} \leq 8\varepsilon'(n,t)/(j+1).$$

Hence,

$$\begin{aligned}&\bigg|\sum_{j=1}^{n}\bigg(\sum_{2i+t>n} a_{2i+t} m_{ij}\bigg) p_j(x)\bigg| \\ &\qquad \leq \bigg(\sum_{j=1}^{n} \frac{1}{j^{3/2}}\bigg) \max\{\varepsilon'(n,0), \varepsilon'(n,1)\} \bigg(\frac{32\pi}{1-x^2}\bigg)^{1/2}\end{aligned}$$

so our pointwise estimate follows from (7) and the integral test. □

By (1), $\varepsilon(n,t) \le \varepsilon'(n,t)/(n+2)$. More importantly, for the examples given above, ε' may be approximated by the same methods used for ε: If $\sum |a_{2i+t}| < \infty$, then $\varepsilon'(n,t) \le \sum_{2i+t>n} |a_{2i+t}|$. In particular, if $\{a_{2i+t}\}$ has a common sign and $f(x) = \sum a_i x^i$ has a closed form, then a useful identity is

$$\varepsilon'(n,t) = \left| \frac{1}{2}(f(1) + (-1)^t f(-1)) - \sum_{i=0}^{[(n-t-1)/2]} a_{2i+t} \right|.$$

Furthermore, if $\sum a_{2i+t}$ satisfies Leibniz's test, then $\varepsilon'(n,t) \le \max\{|a_{n+1}|, |a_{n+2}|\}$.

Examples (revisited). 1. If $n \ge r = (b-a)/2$, then

$$\begin{aligned}\varepsilon'(n,t) &= (e^b + (-1)^t e^a)/2 - e^{a+r} \sum_{i=0}^{[(n-t-1)/2]} \frac{r^{2i+t}}{(2i+t)!} \\ &\le \frac{e^{a+r} r^{n+1}}{(n-1)!((n+1)n - r^2)}.\end{aligned}$$

2, 8, 9. $\varepsilon'(n,t)$ is bounded by $n+2$ times the bound given for $\varepsilon(n,t)$.

3.

$$\varepsilon'(n,0) = \sec 1 - \sum_{i=0}^{[(n-1)/2]} E_i^* \le (1/2)^{(n-1)/2}.$$

4.

$$\begin{aligned}\varepsilon'(n,1) &= \tan 1 - \sum_{i=0}^{[(n-2)/2]} B_{2i+2}^* \\ &\le \frac{(1/2)^{(n-1)/2}}{\sqrt{2}-1}.\end{aligned}$$

5.

$$\varepsilon'(n,t) \le \frac{1}{2(p-1)(n-1)^{p-1}}$$

where we now must assume $p > 1$.

6. For $n > h+3$,

$$\varepsilon'(n,t) = \left| 2^{h-1} - \sum_{i=0}^{[(n-t-1)/2]} C_i \right|$$

where now $h > 0$. Convergence is slow for h near zero, but steadily improves as h increases.

7. As in Example 6, the series for $\arcsin x$ satisfies the hypotheses of Theorem 2 by Raabe's test, and

$$\varepsilon'(n,1) = \frac{\pi}{2} - \sum_{i=0}^{[(n-2)/2]} \frac{D_i}{2i+1}.$$

Unfortunately, convergence is very slow: $\varepsilon'(100,1) \approx .08$.

Remark. The error analyses used for our examples were based on the usual convergence tests from elementary calculus. For finer remainder estimates, see the articles [**4, 5, 6**].

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Department of Mathematics, State University College of New York at Buffalo, Buffalo, New York 14222-1095
E-mail address: guykerj@buffalostate.edu