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ON ENDOMORPHISM RINGS OF FREE MODULES

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ABSTRACT. Right (left) ideals in the ring of $n \times n$ matrices over a principal ideal domain are themselves principal. We separate this result into its arithmetic part and the part that simply reflects the fact that we are dealing with endomorphisms of a free module. We obtain, under some mild hypotheses, an extension of this classical result to right ideals in the endomorphism ring of free modules over arbitrary rings.

1. Introduction. It is a classical result that the ring of $n \times n$ matrices over a principal ideal domain R is itself a principal (left and right) ideal ring. An analogous result holds for the matrices over a division ring. Of course these matrix rings are just the endomorphism rings of the free modules of finite rank over the base rings. Ideals in endomorphism rings of infinite dimensional vectors spaces over division rings have been studied as well (e.g., [**2**], [**6**]). Also, for extensions of the result on principal ideal domains, see [**1**] for example.

In looking at the classical theorem explicitly from the point of view of the endomorphism rings, rather than arithmetically, we were led to a quite general result which has various classical theorems as special cases. This is given in Section 4, though we need to invoke a certain hypothesis which, while slightly peculiar, seems to be at the heart of things. It is the arithmetic structure of principal ideal domains that then certifies the condition in the classical case. We think that these developments help delineate the classical results to some extent, and extend those results as well, so we hope that this note might be of interest to others.

Throughout the paper, R will denote a ring, F a free left R-module and $\Lambda = \mathrm{End}_R(F)$ is the ring of endomorphisms of F. We operate on the left of F with elements of Λ as well as with elements of R.

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2. A correspondence. In this section we give a general correspondence between submodules of a free module and right ideals of its endomorphism ring. This is a very simple result and is probably well known, certainly so in special cases ([**2**], [**6**]). Variants of this correspondence may be found in [**5**] and in [**3**, Chapter 5, Sect. 7]. See also [**4**]. However the discussion below is short, self-contained, somewhat different, and allows us to put a slight spin on the result to fit our application. Therefore we include it here.

Let M be an R submodule of F and write

$$\Phi(M) = \{\lambda \in \Lambda \mid \lambda F \subseteq M\} = \mathrm{Hom}_R(F, M).$$

Clearly $\Phi(M)$ is a right ideal of Λ but it has a somewhat special property. Thus write J for $\Phi(M)$ for a moment and suppose that $\sigma \in \mathrm{Hom}\,(F, JF)$. Here JF denotes the submodule $\sum_{\lambda \in J} \lambda F$ of F. Then

$$\sigma F \subseteq JF = \mathrm{Hom}\,(F, M)F \subseteq M$$

whence $\sigma \in J$. Thus $\mathrm{Hom}\,(F, JF) \subseteq J$ and, since J is obviously contained in $\mathrm{Hom}\,(F, JF)$, such ideals $\Phi(M)$ are *complete* in the sense of the following.

Definition 1. Let I be a right ideal of the ring Λ. We say that I is complete if $I = \mathrm{Hom}\,(F, IF)$.

The right ideal I of Λ is contained in the right ideal $\mathrm{Hom}\,(F, IF)$ in any case so that the latter is sort of a closure of I and it might be more natural to call I closed if they are equal. However we will reserve the word "closed" for use in a more traditional sense later on.

Now denote by $\mathcal{S}$ the set of R-submodules of F and by $\mathcal{I}$ the set of complete right ideals of Λ. Then we have the correspondences

$$\Phi : \mathcal{S} \longrightarrow \mathcal{I} \quad \text{and} \quad \Psi \mathcal{I} \longrightarrow \mathcal{S}$$

given by $M \longmapsto \Phi(M) := \mathrm{Hom}\,(F, M)$ for $M \in \mathcal{S}$ and $I \mapsto \Psi(I) := IF$ for $I \in \mathcal{I}$.

Proposition 1. *The correspondences Φ and Ψ are inverse bijections between the sets $\mathcal{S}$ of submodules of F and $\mathcal{I}$ of complete right ideals* *of Λ. We have $\Phi(\sigma F) = \sigma\Lambda$ for $\sigma \in \Lambda$ so principal right ideals are complete and are matched with the endomorphic images of F.*

Proof. For $I \in \mathcal{I}$ we have $\Phi\Psi(I) = \Phi(IF) = \mathrm{Hom}\,(F, IF) = I$ by completeness of I.

Now let $M \in \mathcal{S}$ and let $\{x_\nu\}$ be an R-basis of F. For $w \in M$, define $\rho = \rho_w : F \to F$ by $\rho(x_\nu) = w$ for all x_ν. Since F is free on $\{x_\nu\}$ this extends to an R homomorphism of F into $Rw \subseteq M$, that is, to an element $\rho \in \Lambda$. Clearly in fact $\rho \in \Phi(M)$. Then we have

$$w \in \rho F \subseteq \mathrm{Hom}\,(F, M)F = \Phi(M)F.$$

Since w is arbitrary in M, we conclude that $M \subseteq \Phi(M)F \subseteq M$. This gives $M = \Phi(M)F = \Psi\Phi(M)$ and establishes the first part of the proposition.

Now let $\sigma \in \Lambda$. Then σF is an R submodule of F. Consider $\Phi(\sigma F) = \mathrm{Hom}\,(F, \sigma F)$. If $\lambda F \subseteq \sigma F$, i.e., $\lambda \in \Phi(\sigma F)$, write $\lambda x_\nu = y_\nu$ for the basis $\{x_\nu\}$ of F. Then $y_\nu \in \sigma F$ by hypothesis on λ so $y_\nu = \sigma w_\nu$ for some w_ν in F. Define $\delta : F \to F$ by $\delta x_\nu = w_\nu$. Then $\lambda x_\nu = \sigma\delta x_\nu$ for all ν so $\lambda = \sigma\delta \in \sigma\Lambda$. This shows that $\Phi(\sigma F) \subseteq \sigma\Lambda$, and obviously $\sigma\Lambda$ is contained in $\Phi(\sigma F)$, so we have equality. In particular, $\sigma\Lambda$ is complete as is any ideal $\Phi(M)$, and we see that principal ideals and endomorphic images correspond as asserted. This completes the proof. $\square$

For emphasis, we point out the following corollary.

Corollary 1. *Principal right ideals are complete and every complete right ideal of Λ is principal if and only if every R-submodule of F is an endomorphic image.*

3. On density. We will say in what follows that an endomorphism λ of a module is of *finite type* if its image is contained in a finitely generated submodule. For example, every endomorphism of a finitely generated module is of finite type. The following lemma is well known but we state it in order to establish some useful notation.

Lemma 1. *Let F be the free R module on the basis $\{x_\nu \mid \nu \in N\}$ and let Λ be the endomorphism ring of F. For any $\nu \in N$ the Λ-exact sequence*

$$0 \longrightarrow L_\nu \longrightarrow \Lambda \longrightarrow F \longrightarrow 0, \quad \lambda \rightsquigarrow \lambda x_\nu$$

is split exact. If $\{\varepsilon_\nu\}$ are the projections of F onto the Rx_ν, then L_ν is the left annihilator $\Lambda(1-\varepsilon_\nu)$ of ε_ν, $F \cong \Lambda\varepsilon_\nu$ as left Λ-module and $F = \Lambda x_\nu$ is cyclic projective over Λ with each x_ν as generator.

The next result is basic to our discussion. It is given, with another application, in [**7**] but a proof is included here for the reader's convenience.

Proposition 2. *If I is a right ideal of Λ, then $IF = F$ if and only if I contains all the endomorphisms of F of finite type.*

Proof. Assume that $IF = F$ for the right ideal I of Λ. Then in the notation of the lemma, $F = IF = I\Lambda x_\nu$ so the map $\lambda \rightsquigarrow \lambda x_\nu$ in the lemma takes I onto F. Therefore, $L_\nu + I = \Lambda$ so $\varepsilon_\nu = \alpha_\nu - \lambda_\nu$ for some $\alpha_\nu \in i$ and $\lambda_\nu \in L_\nu$. Then

$$\lambda_\nu = \lambda_\nu(1-\varepsilon_\nu) = (\varepsilon_\nu + \lambda_\nu)(1-\varepsilon_\nu) = \alpha_\nu(1-\varepsilon_\nu) \in I.$$

We conclude that $\varepsilon_\nu = \alpha_\nu - \lambda_\nu \in I$ and this is true for all ν. Now if λ is an endomorphism of finite type so λF is contained in $\sum_{i=1}^m Ry_i$ say, we may write

$$y_i = \sum_{\nu \in S} r_{i\nu} x_\nu$$

for all i and some fixed finite set S of indices. Since $\sum_{\nu\in S} \varepsilon_\nu$ is the identity map on $\oplus_{\nu \in S} Rx_\nu$ and $\lambda F \subseteq \oplus_{\nu\in S} Rx_\nu$, we have $\lambda = (\sum_{\nu\in S}\varepsilon_\nu)\lambda = \sum_{\nu\in S}\varepsilon_\nu\lambda \in I$. Therefore, I contains all endomorphisms of finite type.

The converse, that if I contains all endomorphisms of finite type, then $IF = F$, is clear. □

If F is given the discrete topology and the set of functions F^F is given the product topology, then $\Lambda = \mathrm{End}_R(F)$ is a closed subset of F^F. We recall that the induced topology of Λ is called the *finite topology.* A base for the system of neighborhoods of $\lambda \in \Lambda$ is given by the collection of finite subsets X of F by $N_X(\lambda) = \{\mu \in \Lambda \mid \mu(x) = \lambda(x)$ for all $x \in X\}$. Thus, for example, a base for the system of neighborhoods of zero is the set of all annihilators of finite subsets of F. As is well known, Λ is a topological ring relative to this structure and the ideal of endomorphisms of finite type is dense in Λ. As a corollary to the proposition we have

Corollary 2. *If I is a right ideal in Λ and $IF = F$, then I is dense in the finite topology. In particular, a closed ideal I satisfies $IF = F$ if and only if $I = \Lambda$.*

It is easy to see that finitely generated right ideals of Λ are closed in the finite topology, so $IF = F$ if and only if $I = \Lambda$ for such an ideal. Observe too that if F is finitely generated over R, then the finite topology is discrete so $IF = F$ implies that $I = \Lambda$ always.

4. Principal, complete and closed ideals. If R is viewed as a free module of rank 1 over itself, then every ideal is closed, but of course not every such need be principal. On the other hand, for F free of infinite rank, the ideal of endomorphisms of finite type is dense, not closed. We attempt to straighten some of this out as follows.

Proposition 3. *Over any ring R, principal right ideals of Λ are complete and complete right ideals are closed.*

Proof. We have already seen that the principal ideal $\sigma\Lambda$ is complete since $\sigma\Lambda = \Phi(\sigma F)$ (Proposition 1). Now let J be any complete right ideal, and suppose that $\lambda \in \Lambda$ satisfies $N_X(\lambda) \cap J \neq \varnothing$ for every finite subset X of F. Taking X to be the singleton $\{x_\nu\}$ for each basis element of F in turn, we see that there exists $\rho_\nu \in J$ such that $\rho_\nu x_\nu = \lambda x_\nu$. Since $\rho_\nu x_\nu \in JF$, we conclude that $\lambda x_\nu \in JF$ for all ν. Therefore $\lambda F \subseteq JF$, i.e., $\lambda \in \operatorname{Hom}(F, JF)$ and, since J is complete, we have $\lambda \in J$. Thus J is closed. $\square$

So far we have not imposed any restrictions on the base ring R.

However, our main interest concerns conditions under which the right ideals of $\lambda = \mathrm{End}_R(F)$ are principal. For example, if the free module F has rank 1, so F is the left module R over itself, then Λ is the ring of right multiplications by elements of R, i.e., the opposite ring R^0 to R and right ideals of Λ correspond to left ideals of R. These are principal if and only if R is a left principal ideal ring. Some conditions are therefore necessary and Corollary 1 suggests condition $(*)$ below. While this may appear to be giving away the store in some sense, there are many natural situations in which the condition is satisfied.

Therefore, we now assume of the free module F:

$(*)$ Every R-submodule of F is an endomorphic image of F.

Proposition 4. *Under $(*)$ every complete right ideal of Λ is principal.*

Proof. Suppose that I is complete so that $I = \mathrm{Hom}\,(F, IF)$. By $(*)$, $IF = \sigma F$ for some $\sigma \in \Lambda$. Proposition 1 now gives

$$I = \mathrm{Hom}\,(F, IF) = \mathrm{Hom}\,(F, \sigma F) = \Phi(\sigma F) = \sigma\Lambda$$

so I is principal. $\square$

Proposition 5. *Under $(*)$ closed right ideals I of Λ are principal.*

Proof. Let I be a closed ideal and form its "completion"

$$I' = \{\lambda \in \Lambda \mid \lambda F \subseteq IF\} = \mathrm{Hom}\,(F, IF) = \Phi\Psi(I).$$

Then $I' = \delta\Lambda$ for some δ by Proposition 4 because I' is complete. Observe also that

$$IF = \Psi(I) = \Psi\Phi\Psi(I) = \Psi(I') = I'F.$$

Put $P = \{\lambda \mid \delta\lambda \in I\}$. Then δ_l, left multiplication by δ on Λ, takes Λ onto I' and takes P onto I.

$$\begin{array}{ccccc} \Delta & \longrightarrow & I' & \longrightarrow & 0 \\ \Big\downarrow & \delta_l & \Big\downarrow & & \\ P & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

Moreover, the kernel of δ_l is contained in PF for, if $\delta x = 0$, define σ on the basis $\{x_\nu\}$ of F by $\sigma x_\nu = x$ for all ν and extend to F. Then $\delta\sigma = 0$, so certainly $\sigma \in P$ and $x = \sigma x_\nu \in \sigma F \subseteq PF$ as asserted.

Now δ_l takes F onto $\delta F = \delta\Lambda F = I'F$ and takes PF onto $\delta PF = IF$. But $IF = I'F$ and, since $\ker \delta_l \subseteq PF$, we conclude that $PF = F$. Then, by Corollary 2, P is dense in Λ. However, $P = \delta_l^{-1}(I)$ with I closed and δ_l continuous, so P is also closed. But then P must be equal to Λ so I equals I' and is principal. $\square$

We summarize these remarks in

Theorem 1. *Let R be any ring and let F be a free module over R in which every submodule is an endomorphic image. Then a right ideal of* $\mathrm{End}_R(F)$ *is principal if and only if it is closed in the finite topology.*

We close with a few comments on the scope of this theorem. First of all, for any ring R a free module F which has a basis of cardinality at least as large as the cardinality of F itself, satisfies $(*)$. Indeed, then every submodule has a set of generators (e.g., the submodule itself) of cardinality less than or equal to that of the basis of F so there is a homomorphism of F onto the submodule. For example, this is true for a free module of infinite rank over a finite or countably infinite ring.

On the other hand there are various kinds of rings for which condition $(*)$ is satisfied for any free module. Thus, if R is a field or a division ring, or more generally a semi-simple ring, then every module is completely reducible so the condition holds. Similarly, for principal ideal domains R, it follows from the theory of elementary divisors, for example, that $(*)$ holds in any free module of finite rank. And more generally, submodules of free modules of any rank r over a principal ideal domain are themselves free of rank not exceeding r, so the condition holds here too.

For free modules of finite rank the finite topology on Λ is discrete so all right ideals are closed. Therefore our results contain the classical case of (finite) matrices over a principal ideal domain, the motivating example.

Finally we note that the hypothesis $(*)$ on F is inescapable in any case, if we want to conclude that the principal ideals are exactly the closed ideals (Corollary 1).

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