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PURE PROJECTIVES AND INJECTIVES

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ABSTRACT. A module over a ring R is pure projective or pure injective if it has the projective or injective property relative to all pure short exact sequences. Here a more restrictive concept of purity is introduced which singles out a certain subset of the pure short exact sequences. The modules which have the projective and injective property relative only to this smaller subset are studied.

1. Introduction. For infinite cardinals $\mu, \aleph$, $(\mu^<, \aleph^<)$-pure submodules, and their derivatives, $(\mu^<, \aleph^<)$-pure exact sequences (see Definitions 1.2 and 1.3), were first introduced in [**2**]. The $(\mu^<, \aleph^<)$-pure projective modules are new and appear here for the first time in this full generality. The special case when $\mu = \aleph_0$ and $\aleph = \aleph_0$ is the usual well-known case of pure submodules, pure exact sequences, pure injectives and pure projectives. In this special case the pure projectives appear in [**7**]. A module that has a presentation with fewer than μ generators and fewer than $\aleph$ relations is said to be $(\mu^<, \aleph^<)$-presented. Here Section 2 ends with a satisfactory characterization (Theorem 2.4); a module M is $(\mu^<, \aleph^<)$-pure projective if and only if it is a direct summand of a direct sum of $(\mu^<, \aleph^<)$-presented modules.

A module D is $(\mu^<, \aleph^<)$-pure injective by definition if D has the injective property relative to all $(\mu^<, \aleph^<)$-pure short exact sequences. There is a satisfactory theory in the finite $\mu = \aleph = \aleph_0$ case for (ordinary) pure injectives [**5**, Vol. 1, pp. 158–174] and/or [**4**, pp. 118–122]. However, in contrast to the projective case, so far the author has been unable to develop a theory of $(\mu^<, \aleph^<)$-pure injective modules. Here Section 3 gives all that can be said in the special case $\mu = \aleph_0 \cdot \aleph$ in which case an $(\aleph_0 \cdot \aleph^<, \aleph^<)$-pure injective module is simply called $\aleph^<$-pure injective. One of the main results here is Theorem 3.1. Its proof is very different from the known finite case $\aleph = \aleph_0$. The author is unable to prove it for the general $(\mu^<, \aleph^<)$-case, and it is not clear

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whether it is even true in this case. Unlike the $(\mu^{<}, \aleph^{<})$-pure projective, $\aleph^{<}$-pure injective modules have been studied before (e.g., [**6**]). Here Theorem 3.1 and the resulting criterion 3.2(3) for $\aleph^{<}$-pure injectivity seem to be new. Ordinary pure projective modules are $(\mu^{<}, \aleph^{<})$-pure projective for $\aleph_0 \leq \mu$, $\aleph$. Also pure-injective modules are $(\mu^{<}, \aleph^{<})$-pure injective.

1. Preliminaries. Some concepts dependent on the solvability of equations are defined. Throughout $\aleph \geq \aleph_0$ will be an infinite cardinal. However, 1.1, 1.2 and 1.3 give well-defined concepts even if $\aleph < \aleph_0$.

Definition 1.1. Modules M are right unital over an arbitrary ring, R. A *system* $\mathcal{S}$ of equations over M consists of

$$\mathcal{S} : \sum_{i \in I} X_i r_{ij} = d_j \in M, \quad j \in J, \quad r_{ij} \in R, \quad \|r_{ij}\| \text{ is column finite.}$$

The cardinalities $|I|$ and $|J|$ of I and J are arbitrary, except that $|I| \leq \aleph_0 \cdot |J|$ and, moreover, if J is finite then so also is I. For definitions of when $\mathcal{S}$ is *solvable*, or *consistent* (= *compatible*), see [**1**, pp. 368–367] or [**2**, 2.1]. The system is $\aleph^{<}$-*solvable* if, for any subset $J(1) \subseteq J$ with $|J(1)| < \aleph$, the equations indexed by $J(1)$ only are solvable in M. Then this subset of equations indexed by $J(1)$ is called an $\aleph^{<}$-*subsystem*.

For any cardinals $2 \leq \mu$, $\aleph_0 \leq \aleph$, a submodule $M < N$ is $(\mu^{<}, \aleph^{<})$-*pure*, if whenever is given any system $\mathcal{S}$ over M as above but with $|J| < \aleph$ and $|I| < \mu$ which is solvable in N, then it is also solvable in the submodule M. A system $\mathcal{S}$ with $|J| < \aleph$ will be called an $\aleph^{<}$-*system*.

Note that either $\aleph < \aleph_0$, in which case also $\mu < \aleph_0$ or $\aleph_0 \leq \aleph$, in which case $\mu \leq \aleph_0 \cdot \aleph = \aleph$. In the latter case when $\aleph_0 \leq \aleph$, an $(\aleph^{<}, \aleph^{<})$-pure submodule will be called $\aleph^{<}$-*pure*.

Definition 1.2. A short exact sequence of modules $0 \to A \to B \to C \to 0$ is $(\mu^{<}, \aleph^{<})$-*pure short exact* if the image of A in B is. A module P is $(\mu^{<}, \aleph^{<})$-*pure projective* if it has the projective property relative to all $(\mu^{<}, \aleph^{<})$-pure short exact sequences.

An $(\aleph_0 \cdot \aleph^{<}, \aleph^{<})$-pure short exact sequence is called simply $\aleph^{<}$-*pure short exact* and a module D is $\aleph^{<}$-*pure injective* if D has the injective property relative to all $\aleph^{<}$-pure short exact sequences.

Notation 1.3. The submodule generated by a subset $X \subseteq M$ is denoted by $\langle X\rangle$. A *presentation* p of a module M is a pair $p = \langle \{y_i\}_{i\in I} \mid \|r_{ij}\|\rangle$ and is denoted by $M = \langle y_i, i \in I \mid \sum_{i\in I} y_i r_{ij} = 0,\ j \in J\rangle$. (See [**2**, 2.6] or [**3**, 1.2].) For any cardinals $\mu, \aleph, M$ is $(\mu^<, \aleph^<)$-presented if it has a presentation with $|I| < \mu$ and $|J| < \aleph$.

Note that if $K \subseteq I$ and $\bigoplus\{y_k R \mid k \in K\}$ is a free direct summand of M, that then for all $k \in K$ and $j \in J$, $r_{kj} = 0$, i.e., the rows indexed by K are zero.

A *subrepresentation* of p is determined by any subset $F = I(1) \times J(1) \subseteq I \times J$ as $M_F = \langle y_i, i \in I(1) \mid \sum_{i\in I(1)} y_i r_{ij} = 0, j \in J(1)\rangle$. Note that it is possible that $j(1) \in J(1)$, $r_{i,j(1)} \neq 0$ but $i \notin I(1)$.

2. Pure projectivity. This section outlines the connection between $(\mu^<, \aleph^<)$-pure short exact sequences and $(\mu^<, \aleph^<)$-projective modules and concludes with a structure theorem characterizing the latter.

Lemma 2.1. *For $\aleph_0 \leq \aleph$, $\aleph_0 \leq \mu \leq \aleph$, and any module M, there exists a $(\mu^<, \aleph^<)$-pure short exact sequence of modules $0 \to K \to P \to M \to 0$. Furthermore, P is a direct sum of $(\mu^<, \aleph^<)$-presented modules.*

Proof. Let $\mathcal{P} = \{p\}$ be the set of all presentations p of $M = \langle \{y_i\}_{i\in I} \mid \sum_{i\in I} y_i r_{ij} = 0, j \in J\rangle$. Presentations p are viewed as ordered pairs as in 1.3, and two presentations with the same generators but different relations matrices are regarded as different. For a fixed $p \in \mathcal{P}$ of the above form with index set $I \times J$, take all possible $(\mu^<, \aleph^<)$-subpresentations $F = I(1) \times J(1) \subseteq I \times J$, where $|J(1)| < \aleph$, $|I(1)| < \mu$. Each F determines a module M_F with the following generators and relations

$$M_F = \langle x_i^F, i \in I(1) \mid \sum_{i\in I(1)} x_i^F r_{ij} = 0, j \in J(1)\rangle,$$

and an R-map $\eta_F : M_F \to M$ by $\eta_F x_i^F = y_i$. Let $\Gamma(p)$ be the set of all such $F = I(i) \times J(1) \subseteq I \times J$ as above.

There is a short exact sequence

$$0 \longrightarrow K \longrightarrow P = \bigoplus_{p\in\mathcal{P}} \Big[\bigoplus_{F\in\Gamma(p)} M_F\Big] \xrightarrow{f} M \longrightarrow 0,$$

where $K < P$ and $f|M_F = \eta_F$ for any $p \in \mathcal{P}$ and $F \in \Gamma(p)$.

Suppose that $\mathcal{S}(0) : \sum_{i\in I(0)} x_i r_{ij} = k_j \in K$, $j \in J(0)$ is a $(\mu^<, \aleph^<)$-system that has a solution $x_i = a_i \in P$. Then there exists a presentation $p \in \mathcal{P}$ of M of the previously described form 1.3, where $\{y_i = fa_i \mid i \in I(0)\} \subseteq \{y_i \mid i \in I\}$, $I(0) \subseteq I$ are among the generators, and $\sum_{i\in I(0)}(fa_i)r_{ij} = \sum_{i\in I(0)} y_i r_{ij} = 0$, $j \in J(0) \subseteq J$ among the relations. Hence $F = I(0) \times J(0) \subseteq I \times J$ gives a $(\mu^<, \aleph^<)$-subrepresentation as in 1.3. Then M_F is a direct summand of P with $\sum_{i\in I(0)} x_i^F r_{ij} = 0$, $j \in J(0)$, and $fx_i^F = \eta_F x_i^F = fa_i$. Consequently, $f(a_i - x_i^F) = 0$ and $a_i - x_i^F \in K$, $i \in I(0)$, is a solution of $\mathcal{S}(0)$ in K. Hence $K < P$ is $(\mu^<, \aleph^<)$-pure and $M \cong P/K$. $\square$

In the finite case $\mu = \aleph = \aleph_0$, the next lemma appears in [**7**, p. 702], although the proof there being based on tensor products does not generalize.

The proof of the next lemma is omitted; it is similar to the proof in the finite $(\aleph_0^<, \aleph_0^<)$-case in [**1**, pp. 373–374, 371].

Lemma 2.2. *Let $\aleph_0 \le \aleph$ and $\mu \le \aleph$. A short exact sequence of modules $0 \to A \to B \xrightarrow{\pi} C \to 0$ is $(\mu^<, \aleph^<)$-pure $\Leftrightarrow$ it has the projective property relative to all $(\mu^<, \aleph^<)$-presented modules.*

Corollary 2.3. *For μ and $\aleph$ as above, an extension $A < B$ of modules is $(\mu^<, \aleph^<)$-pure if and only if for every $(\mu^<, \aleph^<)$-presented module M, the map $\pi^* : \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, B/A)$ is onto.*

The special finite case $\mu = \aleph = \aleph_0$ of the next theorem appears in [**7**, pp. 700, 703], but its proof does not generalize.

Theorem 2.4. *For any cardinals $\aleph_0 \le \mu$ and $\aleph_0 \le \aleph$, and any module M over a ring R, M is $(\mu^<, \aleph^<)$-pure projective $\Leftrightarrow$ M is a direct summand of a direct sum of $(\mu^<, \aleph^<)$-presented modules.*

Proof. $\Leftarrow$. The proof is easy by 2.2 and 2.3 and is omitted.

$\Rightarrow$. By 2.1, there is a $(\mu^<, \aleph^<)$-pure short exact sequence of modules

$0 \to K \to P \xrightarrow{\pi} M \to 0$, where P is a direct sum of $(\mu^<, \aleph^<)$-presented modules. Since each of the summands of P is $(\mu^<, \aleph^<)$-projective by 2.1, and since it can be shown that $(\mu^<, \aleph^<)$-projective modules are closed under direct sums, P is $(\mu^<, \aleph^<)$-projective. By Lemma 2.2, the identity map $1_M : M \to M$ lifts to $g : M \to P$ such that $\pi g = 1_M$. Thus $M \cong gM$ and $P = gM \oplus (1_M - g\pi)P$. □

3. Pure injectivity. The next theorem gives a concrete and simple description of a module H and the solution of the system of equations in H.

Theorem 3.1. *For any ring R and R-module D and cardinal $\aleph_0 \le \aleph$, let $\mathcal{S}$ be an $\aleph^<$-solvable system in D where $\mathcal{S}$ is*

$$\mathcal{S} : \sum_{i \in I} X_i r_{ij} = d_j \in D, \quad j \in J; \quad |J| \text{ arbitrary.}$$

Let $F = \oplus\{y_i R \mid i \in I\}$ be a free module on the free generators $\{y_i \mid i \in I\}$ and $G < F \oplus D$ the submodule generated by $\langle \sum_{i \in I} y_i r_{ij} - d_j \mid j \in J \rangle$. Define $\overline{D} = (G + D)/G$ and $H = (F \oplus D)/G$. Then

(i) $\overline{D} \cong D$;

(ii) $\overline{D} < H$ is $\aleph^<$-*pure*;

(iii) $X_i = y_i + G$, $i \in I$, *is a solution of $\mathcal{S}$ in H.*

Proof. (i) Since $\mathcal{S}$ is finitely solvable, it is consistent. If S is consistent, then $G \cap [(0) \oplus D] = 0$ (see [**2**, 2.1] and [**1**, p. 384]). Thus $\overline{D} \cong D$. (iii) This is clear from the way H is defined.

(ii) Let $\mathcal{S}(2)$ be an $\aleph^<$-system (as in Definition 1.1) over $\overline{D}$ which is solvable in H. It has to be shown that it is also solvable in $\overline{D}$, where

$$\mathcal{S}(2) : \sum_{k \in \mathcal{K}} X_k s_{kp} = h_p + G \in \overline{D};$$
$$h_p \in D, \quad p \in \mathcal{P}; \quad |\mathcal{P}| < \aleph, \quad |\mathcal{K}| < \aleph_0 \cdot \aleph;$$

and, given is a solution of $\mathcal{S}(2)$ in H,

$$X_k = \sum_{i \in I} y_i c_{i_k} + v_k + G, \quad c_{i_k} \in R, \quad v_k \in D, \quad k \in \mathcal{K}.$$

Define $I(\mathcal{K}) = \{i \in I \mid \exists k \in \mathcal{K}, c_{i_k} \neq 0\}$. In the finite case $\aleph = \aleph_0$, both $\mathcal{K}$ and $I(\mathcal{K})$ are finite, and if $\aleph_0 < \aleph$, then $|\mathcal{K}| < \aleph$ and $|I(\mathcal{K})| \leq \aleph_0 \cdot |\mathcal{K}| < \aleph$.

For each $p \in \mathcal{P}$, there are a finite number of s_{kp}, $t_{jp} \in R$ with

(1)

$$\sum_{k \in \mathcal{K}} \Bigg(\sum_{i \in I(\mathcal{K})} y_i c_{i_k} + v_k \Bigg) s_{kp} - h_p = \sum_{j \in J(\mathcal{P})} \Bigg(\sum_{i \in I(\mathcal{P})} y_i r_{ij} - d_j \Bigg) t_{jp}; \quad p \in \mathcal{P};$$

where $J(\mathcal{P}) \subseteq J$ refers to only those $j \in J$ which appear in the above equation with a $0 \neq t_{jp} \in R$ and $I(\mathcal{P}) \subseteq I$ refers to those i only with $0 \neq r_{ij} t_{jp} \in R$ for at least one p. Note that all of the four matrices $\|r_{ij}\|$, $\|s_{kp}\|$, $\|c_{ik}\|$ and $\|t_{jp}\|$ are column finite. Thus $J(\mathcal{P}) = \{j(0) \in J \mid \exists p \in \mathcal{P}, t_{j(0)p} \neq 0\}$.

As before, if $|\mathcal{P}| < \aleph_0$ is finite, so is $J(\mathcal{P})$, and if $\aleph_0 < \aleph$, then $|J(\mathcal{P})| \leq \aleph_0 \cdot |\mathcal{P}| < \aleph$. Next $I(\mathcal{P}) = \{i(0) \in I \mid \exists j(0) \in J(\mathcal{P}), r_{i(0)j(0)} \neq 0\}$. Again, when the total number of equations $|\mathcal{P}| < \aleph_0$ is finite, then so is $|I(\mathcal{P})| < \aleph_0 \leq \aleph$. If $\aleph_0 < \aleph$, then $|I(\mathcal{P})| \leq \aleph_0 \cdot |\mathcal{P}| < \aleph$.

Upon interchanging some orders of summation and rearranging equation (1), we obtain the following element in $F \cap D = 0$,

$$(2) \qquad \sum_{i \in I(\mathcal{K})} y_i \sum_{k \in \mathcal{K}} c_{i_k} s_{kp} - \sum_{i \in I(\mathcal{P})} y_i \sum_{j \in J(\mathcal{P})} r_{ij} t_{jp} = - \sum_{k \in \mathcal{K}} v_k s_{kp} + h_p - \sum_{j \in J(\mathcal{P})} d_j t_{jp}, \quad p \in \mathcal{P}.$$

Since F is free on the y_i's, there are three kinds of i's and three different forms of the above equation:

$$\begin{aligned} i \in I(\mathcal{K}) \setminus I(\mathcal{P}) :& \qquad \sum_{k \in \mathcal{K}} c_{i_k} s_{kp} = 0; \\ i \in I(\mathcal{K}) \cap I(\mathcal{P}) :& \quad \sum_{k \in \mathcal{K}} c_{i_k} s_{kp} - \sum_{j \in J(\mathcal{P})} r_{ij} t_{jp} = 0; \\ i \in I(\mathcal{P}) \setminus I(\mathcal{K}) :& \qquad - \sum_{j \in J(\mathcal{P})} r_{ij} t_{jp} = 0. \end{aligned}$$

First if $i \notin I(\mathcal{P}) = \{i \mid \exists j(0) \in J(\mathcal{P}),\ r_{ij(0)} \neq 0\}$, then for every $j(0) \in J(\mathcal{P})$, automatically $r_{ij(0)} = 0$. Secondly, if $i \notin I(\mathcal{K})$, then

$c_{i_k} = 0$ for all $k \in \mathcal{K}$. As a consequence, in the first and third cases the equations can be written as in the second. Thus in all three cases we have for all $p \in \mathcal{P}$ that

$$i \in I(\mathcal{K}) \cup I(\mathcal{P}) : \sum_{k\in\mathcal{K}} c_{i_k} s_{kp} - \sum_{j\in J(\mathcal{P})} r_{ij} t_{jp} = 0, \tag{3}$$

$$-\sum_{k\in\mathcal{K}} v_k s_{kp} + h_p - \sum_{j\in J(\mathcal{P})} d_j t_{jp} = 0. \tag{4}$$

Set $I^* = I(\mathcal{K}) \cup I(\mathcal{P})$. Then $|I^*| \leq |I(\mathcal{K})| + |I(\mathcal{P})| < \aleph + \aleph = \aleph$.

The subsystem $\mathcal{S}(3)$ of equations indexed by $J(\mathcal{P})$ can be indexed by $I(\mathcal{P}) \times J(\mathcal{P})$ because, for every $j(0) \in J(\mathcal{P})$, all i for which $r_{ij}(0) \neq 0$ are already in $I(\mathcal{P})$. By the $\aleph^<$-solvability hypothesis of $\mathcal{S}$, the subset $\mathcal{S}(3) \subset \mathcal{S}$ of $\mathcal{S}$ has a solution

$$\{a_i \mid i \in I(\mathcal{P})\} \subset D, \qquad \sum_{i\in I(\mathcal{P})} a_i r_{ij} = d_j, \quad j \in J(\mathcal{P}).$$

Define $a_i = 0$ for $i \in I(\mathcal{K}) \setminus I(\mathcal{P})$. We will now show that $X_k = \sum\{a_i c_{i_k} + v_k \mid i \in I^*\}$ is a solution of $\mathcal{S}(2)$ now viewed as being over $F \oplus D$, i.e., $\sum\{X_k s_{kp} \mid k \in \mathcal{K}\} = h_p$, $p \in \mathcal{P}$. We substitute this X_k into the latter and use the first equation (3) and then the fact that the a_i's solve $\mathcal{S}(3)$ to obtain

(5)

$$\begin{aligned}\sum_{i\in I^*} a_i\Big(\sum_{k\in\mathcal{K}} c_{i_k} s_{kp}\Big) + \sum_{k\in\mathcal{K}} v_k s_{kp} &= \sum_{i\in I^*} a_i\Big(\sum_{j\in J(\mathcal{P})} r_{ij} t_{jp}\Big) + \sum_{k\in\mathcal{K}} v_k s_{kp} \\ &= \sum_{j\in J(\mathcal{P})} d_j t_{jp} + \sum_{k\in\mathcal{K}} v_k s_{kp}, \quad p \in \mathcal{P}.\end{aligned}$$

Finally, equation (4) shows that the right side of (5) is simply equal to h_p, i.e., our X_k's solve the equations $\sum\{X_k s_{kp} \mid k \in \mathcal{K}\} = h_p$, $p \in \mathcal{P}$. Hence $D \cong \overline{D} < H$ is $\aleph^<$-pure. $\square$

The equivalence (1) $\Leftrightarrow$ (2) below appears in [**5**]; the criterion (3) seems to be new.

Theorem 3.2. *For an infinite cardinal $\aleph$ and any ring R and any R-module D, the following are all equivalent*:

(1) *D is $\aleph^<$-pure injective.*

(2) *D is a direct summand in every module in which it is contained as an $\aleph^<$-pure submodule.*

(3) *Any arbitrary $\aleph^<$-solvable system of equations over D has a global solution in D.*

Proof. (1) $\Rightarrow$ (2). If $0 \to D \xrightarrow{i} B \to B/D \to 0$ is $\aleph^<$-pure short exact, then the identity map $1_D : D \to D$ extends to $g : B \to D$ such that $gi = 1_D$ by hypothesis (1). Hence $B = iD \oplus (1_B - ig)B$.

(2) $\Rightarrow$ (3). Let $\mathcal{S} : \sum\{X_i r_{ij} \mid i \in I\} = d_j \in D$, $j \in J$, be an arbitrary $\aleph^<$-solvable system over D. Form the module $H = (F \oplus D)/G$ exactly as in the last theorem. Since $\mathcal{S}$ is finitely solvable, it is consistent. Consequently, as in the proof of 3.1 (i), $G \cap [(0) \oplus D] = 0$. Thus, as in the last theorem, embed $D \cong \overline{D} = (D \oplus G)/G < H = (F \oplus D)/G$ as an $\aleph^<$-pure submodule of H, where now the isomorphically transferred system $\overline{\mathcal{S}}$:

$$\sum\{X_i r_{ij} \mid i \in I\} = \bar{d}_j = d_j + G \in \overline{D}, \quad j \in J$$

has the global solution $X_i = y_i + G = \bar{y}_i$, $i \in I$, by (iii) of the last theorem. Now, by hypothesis (2), $H = \overline{D} \oplus Q$ for some $Q < H$. Hence, $\bar{y}_i = \bar{z}_i + q_i$, $\bar{z}_i \in \overline{D}$, $q_i \in Q$, $i \in I$. Hence,

$$\sum\{\bar{z}_i r_{ij} \mid i \in I\} - \bar{d}_j = -\sum\{q_i r_{ij} \mid i \in I\} \in \overline{D} \cap Q = 0$$

for all $j \in J$. Then if $\bar{z}_i = z_i + G$, $z_i \in D$, in view of $D \cap G = 0$, we get that $\sum\{z_i r_{ij} \mid i \in I\} = d_j$, $j \in J$ is a global solution of $\mathcal{S}$ in D.

(3) $\Rightarrow$ (1). Let $0 \to A \to B \to C \to 0$ be an $\aleph^<$-pure short exact sequence. Suppose $\varphi : A \to D$ is an R-homomorphism. Take any presentation of $B/A = \langle b_i + A, i \in I \mid \sum\{(b_i + A)r_{ij} \mid i \in I\} = 0, j \in J\rangle$ in terms of generators and relations. Then define $a_j = \sum\{b_i r_{ij} \mid j \in J\}$.

The system $\mathcal{S} : \sum\{X_i r_{ij} \mid i \in I\} = a_j$, $j \in J$, over A has a global solution $X_i = b_i \in B$. By the $\aleph^<$-purity of A in B, the system $\mathcal{S}$ is $\aleph^<$-solvable in A. Now form the system $\varphi(\mathcal{S}) : \sum\{X_i r_{ij} \mid i \in I\} = \varphi a_j$. Any $\aleph^<$-subsystem of $\varphi(S)$ is a set of equations as above indexed by a subset $J(1) \subseteq J$ with $|J(1)| < \aleph$. It determines an $\aleph^<$-subsystem of

$\mathcal{S}$, indexed by the same subset $J(1) \subseteq J$. The latter has a solution $X_i = c_i \in A$, $\sum\{c_i r_{ij} \mid i \in I\} = a_j$, $j \in J(1)$. Hence, also, $\sum\{(\varphi c_i) r_{ij} \mid i \in I\} = \varphi a_j$, $j \in J(1)$. But then, by hypothesis, (3) applied to $\varphi(S)$, there exists a global solution $X_i = d_i \in D$, $i \in I$, of all of $\varphi(\mathcal{S})$ in D. Define $g : A \cup \{b_i \mid i \in I\} \to D$ by $g \mid A = \varphi$ and $gb_i = d_i$ and then extend g by R-linearity to all of B.

To show that g is well defined, it remains to show that if $\sum\{b_i u_i \mid i \in I\} = a \in A$, $u_i \in R$, then $\sum\{d_i u_i \mid i \in I\} = \varphi a$ also. But, for a finite number of $t_j \in R$, and for all i, by definition of a presentation for B/A, $u_i = \sum\{r_{ij} t_j \mid j \in J\}$. Consequently, we conclude that

$$\sum_{i \in I} b_i u_i = \sum_{j \in J} \left(\sum_{i \in I} b_i r_{ij} \right) t_j = \sum_{j \in J} a_j t_j = a \in A.$$

But then

$$\sum_{i \in I} d_i u_i = \sum_{j \in J} \left(\sum_{i \in I} d_i r_{ij} \right) t_j = \sum_{j \in J} (\varphi a_j) t_j = \varphi a.$$

Thus D is $\aleph^<$-pure injective. □

Remarks 3.3. (1) Some information about how far $\aleph^<$-pure injective modules are from either injective or ordinary pure injective ones is given in [**6**, p. 141].

(2) So far, the author has not been able to construct $\aleph^<$-pure injective hulls of modules. In the case of the ordinary pure-injective hull, Zorn's lemma can be used since finitely solvable systems of equations are used. For $\aleph \geq \aleph_1$, Zorn's lemma is no longer available.

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