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REMARKS ABOUT THE HISTORY OF ABELIAN GROUPS IN ENGLAND AND GERMANY

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1. Introduction and precautions. A historian told me that history does not end in the past but goes on into the present time; moreover, it tries to predict the future. Accordingly, my report will end with some unpublished papers and forthcoming PhD theses (completed in 2002), but I will not speculate about the direction into which abelian groups in the UK and Germany might develop. Moreover, the reader of these notes should be aware of the fact that this paper reflects my very personal point of view. I will naturally be more explicit in areas close to my own (research) interests, this because I know them better than others. I intend to cover the other areas of the now wide range of abelian groups but, despite the desire to be fairly conclusive, this turns out to be a hopeless task. Fortunately, abelian groups of the last decades have grown fascinatingly in depth and breadth, thus I must and will be brief whenever I have the opportunity to refer to a monograph or to a survey article. Moreover, it seems impossible to isolate developments of abelian groups from international activities; this is particularly obvious from progress over the last 50 years. I will try to stay loosely in the requested coordinates and rarely mention the tremendous input from outside, most notably from the USA, Russia and other European countries; thus, this report should be seen as part of the other reports on the history of abelian groups published in this volume.

The sections of this paper are dictated, as I see this, by the developments around the world and essentially cut down to activities in the UK and Germany. Another restriction is necessary. I will only discuss (some of) those papers related to more general module theory which at the same time extend visibly our knowledge on abelian groups; otherwise, as is well known, I would have to report endlessly about the activities in the UK and Germany. A good excuse for these additional

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restrictions is the contributions in the 1985 monograph and the most recent one by Fuchs, Salce [**48**], [**49**] as well as various chapters in the book by Eklof and Mekler [**38**]. Finally I would like to apologize to those friends and colleagues who feel that their favored playground does not appear in the right light, which is not my intention.

2. The finite age. At the beginning of linear algebra we teach or study the Gauß algorithm in order to diagonalize $n \times m$-matrices, and we notice that this algorithm also works for matrices M over a principal ideal domain R, provided we divide with care, thus getting the elementary divisor theorem:

$$PMQ = \mathrm{Diag}\,(r_1, \dots, r_k, 0, \dots, 0) =: D$$

with D a diagonal matrix, $r_i \mid r_{i+1}$ in R and P, Q two regular matrices. The stacked basis theorem is an easy consequence: Express the generators of a submodule U of a free R-module $F = \oplus_{i \le m} \mathrm{Re}_i$ by the e_i's to get a coefficient matrix M, then apply the Gauß algorithm above to get the stacked bases

$$\{P^{-1}e_1, \dots, P^{-1}e_m\}, \{r_1 P^{-1}e_1, \dots, r_k P^{-1}e_k\}.$$

Now obviously the quotient F/U is a direct sum of cyclics. The fundamental theorem for finite(ly generated) abelian groups, as we say Gauß's theorem for abelian groups, is shown. Thus it is immediate from the Gauß algorithm, found in the *Disquisitiones arithmeticae* in 1801, see Gauß [**50**, Vol. 4]. See also Fuchs [**44**, p. 34] who indicates that a version of primary decomposition theorem is described in Gauß [**50**].

Therefore I would consider the year 1801 as the beginning of abelian groups, which is 200 years ago – despite the fact that groups were not yet defined and that it needed over half a century before these creatures got a name. It was Kronecker in 1870 who introduced in [**75**] the notion of an abstract abelian group. Working with radical field extensions it must have been natural to him to name commutative groups after the Norwegian mathematician Nils Henrik Abel whose 200th birthday commemoration was celebrated at Oslo University this year. Kronecker [**75**] also re-proved Gauß's theorem for finite abelian groups and provided a primary decomposition for these groups, see van der Waerden [**95**, p. 149]. Van der Waerden [**95**, p. 149] also mentions that another proof of Gauß's theorem appears in Abel's work [**1**] and Schering, a pupil of Gauß, published a paper [**88**] about binary quadratic forms in which he proves Gauß's theorem for finite abelian groups. Another proof from 1878 by Frobenius and Stickelberger [**42**] arrives directly at the stacked basis theorem. Moreover, they can show that the elementary divisors are invariants of the group. Their point of view and Steinitz's [**94**] extension to modules over Dedekind domains led to recent work on stacked basis theorems, see Cohen, Gluck [**12**] and Section 6.2.

From a certain point of view everything was now known about abelian groups – an impression which, kept alive over a generation, even reached Saharon Shelah and made him believe this until 1973, see Eklof and Mekler [**38**]. Thus abelian groups remained dormant for almost half a century.

3. The countable phase. Only the influence of the Göttingen school around Hilbert, Emmy Noether and others who taught us to look at infinite algebraic structures quite rigorously, brought new life to research on abelian groups. Levi, who attended lectures by Hilbert, Weber, Landau and Toeplitz, in 1914 submitted his Habilitationsschrift on *Abelsche Gruppen mit abzählbaren Elementen* (Leipzig 1919). This was the beginning of the "countable age" of abelian groups. I can be brief on this until the more recent developments because all this is nicely treated in the monographs by Fuchs [**44**], [**45**], [**43**] as well as in Kaplansky's little red book [**74**]. Details about Levi's life and contributions to abelian groups can be found in Fuchs and Göbel [**103**, pp. 1–14]. Partly influenced by an error about decompositions of countable abelian p-groups in one of Levi's papers, Prüfer developed the foundation of the fascinating classification of abelian p-groups which finally was achieved in the mid 1960's – the totally projective p-groups. He produced its building blocks which are now called (generalized) Prüfer groups. For Prüfer's publications and work, see the article by Mader [**102**, pp. 1–8] which provides all the details about Prüfer's impact on abelian groups. Surely the other component needed for the classification theorem for totally projective p-groups is Ulm's paper *Zur Theorie der abzählbar-unendlichen abelschen Gruppen* from 1933 arriving from matrix theory and solving the countable case for classifying abelian p-groups. Ulm's results include Zippin's later existence theorem for p-groups with prescribed Ulm-Kaplansky invariants. For references, recent progress on Ulm's theorem and Ulm's vitae, see Göbel [**104**, pp. 1–10].

4. The first steps beyond countability. Despite the fact that certain results on abelian p-groups automatically also hold for uncountable structures, only Baer's paper [**4**] is a first systematic study of not necessarily countable (torsion-free) abelian groups. Like his papers on abelian p-groups it was written after he had left Nazi Deutschland. This paper is part of a discussion of his work on abelian groups by Fuchs, [**102**, pp. xv–xxi] and also appears in detail in [**44**], [**45**]. Baer's questions on "uncountable" abelian groups and his early work on the Ext-functor, his first examples of indecomposable abelian groups up to the rank $2^{\aleph_0}$ certainly influenced the later developments. Also Corner's [**13**] theorem realizing rings as endomorphism rings of torsion-free abelian groups of size $< 2^{\aleph_0}$ rests on these early observations about pure subgroups of the p-adic integers. Leptin's [**78**] theorem that for $p \geq 5$, two abelian p-groups with isomorphic automorphism groups are isomorphic (extended by Liebert [**102**, pp. 5–31] to $p = 3$, and by Schultz [**108**, pp. 373–379] to $p = 2$) is a natural, but clever and complicated step after the Baer-Kaplansky theorem was shown (see [**45**, p. 224]). Leptin was a pupil of Ulm and the theorem on automorphism groups of p-groups can be seen as an extension of Ulm's classification for countable abelian p-groups. In contrast to these structure results on p-groups, we have Corner's [**14**] theorems (not included in [**45**]) from 1969 which realize rings with additive group torsion-free, pure-injective of p-adic rank $< 2^{\aleph_0}$ as endomorphism rings of separable p-groups modulo small endomorphisms. Together with the existence of suitable rings many tempting conjectures for p-groups (like the Krull-Schmidt properties) can be discarded simultaneously with the torsion-free case. A more recent strange torsion-free group, which is very decomposable, can be seen in Corner [**100**, pp. 354–357]. Many other authors and results should be mentioned here; again I simply refer to the green and red monographs above.

5. The transition phase. In order to overcome the lack of tools to study structures beyond countability, additional methods offered help by themselves: homology and topology, strangely enough not set theory. This also was a general trend in algebra of the time in the 50's and 60's in the last century. Early results on Ext, like Stein's theorem, see [**45**], duality arguments inspired by Nunke [**45**] paved the road of research. Faltings [**39**] considered automorphism groups of p-groups from the "geometry of their lattice." This was also used by Liebert [**102**, pp. 5–31] to extend Leptin's theorem to $p = 3$. Liebert had worked earlier for his PhD on endomorphism rings of p-groups characterizing them ring theoretically (see [**45**]), so their units belong to his favored mathematical playground; see also his more recent paper, [**100**, pp. 384–398]. Grosse and Gräbe in their PhD theses and subsequent papers studied the (iterated) Ext- and Hom-functor (see [**45**]), and Hausen (also supervised by Reinhold Baer at Frankfurt University) wrote a PhD thesis on automorphism groups of certain classes of abelian groups. Many of the results can be found in the proceedings of the Montpellier conference [**97**] in 1967.

A very important input to abelian groups at that time came from Michael Butler, which was inspired by representation theory. He investigated what he calls the diagrammatic groups, now known as Butler groups, see [**9**] and Arnold's lecture notes [**2**]. Obviously, I must come back to this topic in Section 6.1 because it opened one of the most important new directions of abelian group theory.

6. Recent history of abelian groups in the UK and Germany. It is only natural to begin this report in 1973 after Fuchs's [**44**], [**45**] monographs appeared. There are various directions in which abelian groups have now intensively developed. They have also been applied to other areas in algebra in order to answer their problems, see the short Section 6.6. I will begin with the very active area of:

6.1 Butler groups, Warfield modules, acd- and bcd-groups. After Baer's [**4**] classification of completely decomposable torsion-free groups (we say cd-groups), the next major step toward classification theorems for torsion-free abelian groups followed only decades later in 1965 by Butler [**9**] from the University of Liverpool. He established a nice characterization of torsion-free images of cd-groups of finite rank.

Following Lady, these groups are called Butler groups. Butler groups are also closely connected to representations of posets (see Butler [**102**, pp. 291–301]), which provides a large supply of tools to tackle them. Using Jónsson's quasi-isomorphism, also decompositions can be controlled, see also the survey in Loth's book [**79**]. Early work on Butler groups is the PhD thesis under the direction of Corner by Nongxa, *A problem in abelian group theory*, (Oxford 1982), see also Nongxa [**83**] and the references on Nongxa's other papers related to his thesis in this paper. In the last two decades the group around Mutzbauer, who began his mathematical research in non-commutative groups with Heineken at Erlangen University, has become a center in Germany for investigating Butler groups. From his over 40 publications on this topic, I can only mention a few. Starting from his particular knowledge on torsion-free groups of rank 2 and 3 – which he studied in his Habilitationsschrift from 1979 at Würzburg University – (rank 1 was settled by Baer [**4**] as well as by Levi (see [**103**, pp. 1–14] or [**80**]), he and his co-authors investigated the regulating subgroups of Butler groups and were particularly successful in the study of acd-groups. This class of Butler groups came to life by an important paper of Lady [**77**], in which he defined the regulating subgroup. As acd-groups are finite extensions of cd-groups, they are close enough to the classified cd-groups, but complicated enough to make a search for classification theorems interesting. Note that Lady's regulating subgroups are subgroups obtained by summing up the homogeneous summands for critical type sets. A key for further investigation is Burkhardt's lemma, that the regulator (the intersection of regulating subgroups) is a cd-group. (Burkhardt received his PhD under supervision of Huppert at Mainz on Suzuki groups and then moved to Würzburg University.) The regulator, which is a canonical subgroup of an acd-group, obviously leads to (Burkhard-) invariants. Mutzbauer and his co-authors (Burkhardt, Krapf, Mader and Vinsonhaler) used these invariants to derive a nice structure theory for acd-groups. (For Butler groups in general regulating subgroups do not behave as nicely as for acd-groups, see Mutzbauer, [**103**, pp. 209–217]. Müller, Jarisch, Dittman and Nahler, respectively, wrote their PhD theses on abelian groups (*Gemischte abelsche Gruppen endlichen Ranges* – 1992; *Characterizing a class of Warfield modules* – 1998, *Generating cosets and tight subgroups in almost completely decomposable groups* – 2001, *Isomorphism invariants of almost completely decomposable groups within a near-isomorphism class* – 2001) supervised by Mutzbauer. Some more results classifying acd-groups can be found in Mutzbauer and Krapf [**101**, pp. 151–161], jointly by Mader and Mutzbauer [**103**, pp. 257–275] or for endomorphism rings of Butler groups of finite rank, see Mutzbauer [**107**, pp. 319–343] and [**106**, pp. 373–383]. Mader's book [**80**] on acd-groups is a good source for many of the results from the Würzburg group and the 'regular' visitors Fomin, Mader and Toubassi. A recent survey on Butler groups of finite rank can be found in Arnold and Vinsonhaler [**103**, pp. 17–41].

After considering Butler groups of finite rank and acd-groups I want to pass on to their relatives of infinite rank. The class of acd-groups naturally extends to what we called bcd-groups, that is, the class of bounded extensions of cd-groups. There is some hope for good results because bounded abelian groups are direct sums of cyclics. A pioneering paper by Mader, Mutzbauer and Rangaswamy [**105**, pp. 257–272] provides interesting examples of bcd-groups, which are interesting on their own and also studied in a PhD thesis, *Torsionsfreie Gruppen mit linear geordneter Typenmenge*, by Elter at Essen University (1996). Her results, based on the *MaMuRa-group* were investigated further in Mader and Strüngmann [**81**].

The breakthrough for extending Butler groups to infinite rank, due to Bican and Salce [**100**, pp. 171–189], opened further research and intensive activity worldwide. The fact that Butler groups of even very special type cannot be classified by any reasonable invariants follows from a realization theorem for all rings with free additive groups as endomorphism rings of Butler (B_2)-groups, see Dugas and Göbel [**29**]. The open problem that B_2-groups, those defined by a smooth ascending chain of pure subgroups, just adding on a finite rank Butler group from one to the next, and the B_1-groups defined in Bican and Salce [**100**, pp. 171–189] by the Bext-functor need not coincide, was solved only recently by Shelah and Strüngmann [**92**]. There are models of ZFC with $2^{\aleph_0} = \aleph_4$ in which the obvious inclusion is proper. Recall that these classes coincide for countable groups and, assuming the $\square$-principle (or $V = L$) this was shown in general by Fuchs and Magidor [**47**]. I would like to suggest the survey on infinite rank Butler groups by Fuchs [**105**, pp. 121–139] for further reading.

Extending totally projective groups by torsion-free groups (or better extending torsion-free by torsion) with the goal to prove classification theorems, we arrive at certain mixed groups, Warfield modules. As they constitute a canonical class which can be classified by Ulm-Kaplansky-Warfield invariants, it is only nature to relate them to other classes of abelian groups. Mutzbauer, Jarisch and Toubassi [**82**] relate them to cd-groups. Opdenhoevel showed in his PhD thesis at Essen (1999), *Über Summen zweier Automorphismen von Moduln*, that a theorem of Hill extends: the endomorphisms of local Warfield modules of finite torsion-free rank are sums of two automorphisms, see also [**55**]. The same problem for certain classes of torsion-free groups was investigated after preliminary work by Freedman, Goldsmith etc. in a paper by Goldsmith, Pabst and Scott [**70**]. Moreover, after the solution of the finite rank case (in a diploma paper by Wans at Essen University) finally is shown in a very recent PhD thesis by Meehan (DIT Dublin 2001, under supervision of Goldsmith) that any endomorphism of a free abelian group of rank > 1 is a sum of two automorphisms. The parallel case for rational rank-1 groups is amazingly complicated and needs substantial number theory; it was also investigated by Opdenhoevel and Meehan.

6.2 Toward representation theory. This section is closely connected to Section 6.1 because results on representation theory of posets, rep-poset for short, as discussed in Simson's book [**93**], give rise to torsion-free abelian groups as first noticed by Brenner, Butler and Corner in the 60's of the last century; see [**7**], [**16**] and the references in Arnold [**2**], [**3**]. This connection is elaborated and discussed in all details in Arnold [**2**, Section 3], see also the references in [**2**]. An early paper by Cruddis [**22**], published much later than written for his PhD (around 1961 at University of Liverpool under supervision of Butler) deals with the existence of indecomposable abelian groups which are modules over subrings of Q with fewer than 5 primes. From [**9**] follows that, for rings with 4 primes, there are indecomposables of all even ranks, while for less primes, due to finite representation type, their ranks are at most 3. Butler's second PhD student on these topics was Shahzamanian (Liverpool, 1979) with a thesis on *Representations of Dynkin graphs by Abelian p-groups* dealing with finite representation type, so the groups are of exponents p^n with $n < 4$, see also [**11**]. Some of these results were extended a few years later by Arnold; they are also included in [**2**]. More importantly, in his monograph [**3**] Arnold also discusses the recent results on representation theory over valuation domains, which adopt the work of Crawley-Boevey (from University of Leeds) on modules of finite length over their endomorphism rings. Other publications by Brenner and Butler can also be found in [**2**], [**3**], [**93**]. Thus I can skip rep-poset application to finite rank abelian groups, most notably to acd-groups.

This brings me directly to the applications of representation theory to torsion-free abelian groups of infinite rank. (I will neglect applications to p-groups.) Work by Corner, Brenner and Butler mentioned above and Brenner and Ringel [**8**] are early references for this. A joint paper with Franzen on *The Brenner-Butler-Corner-theorem and its application to modules*, [**102**, pp. 209–227] adds combinatorial arguments by Shelah and gives applications for constructing complicated torsion-free abelian groups of infinite rank. This was the starting point for the PhD thesis by Böttinger, *Representations of partially ordered sets over commutative rings for application in abelian groups* at Essen 1989. These results were 'upgraded' further (looking at the critical posets due to Kleiner, e.g., to an antichain of four elements or to Simson's 115 other critical diagrams) in [**53**], [**54**], [**66**], [**67**]. If the application of rep-poset is done with care, then the groups turn out to be Butler groups, also in the case of infinite rank. Thus one might worry to find no extension of classification results for classes of Butler groups of finite rank which hold for infinite rank. Even the case rep-2 of free groups with two distinguished subgroups is wild in general as follows from a realization theorem for rings with free additive group as seen in [**31**]. Fortunately in rep-2 with suitable restriction on the two distinguished subgroups permits an extension of an interesting result first shown in Lewis's PhD thesis (Honolulu, 1992) for finite rank. There is a stacked basis theorem for such rep-2 groups, thus one can classify them as direct sum of indecomposables of rank 1 or 2, and derive a corresponding result for Butler groups, see [**41**], [**30**] and references given there. Another direction of representation theory for investigating direct summands of $B^{(1)}$-groups, inspired by Metelli, was performed by Höfling, a former PhD student of Amberg at Mainz University, see Arnold [**3**, p. 229]. Moreover, Salce and Strüngmann [**86**] worked on extension of the stacked basis theorem recently.

6.3. Set theoretic and model theoretic methods in abelian groups. As Eklof and Mekler [**38**] already indicate, this begins in 1973 with Shelah's paper [**89**] which decisively opened the use of set theoretic models, of various prediction principles coming from his solution of the Whitehead problem as well as his proof of the existence of a class of indecomposable torsion-free abelian groups. We would also like to draw attention to Eklof's survey [**35**] for explaining this direction and to further survey articles by Eklof [**100**, pp. 275–283], Göbel [**111**, pp. 107–127] or [**27**] for complementary reading.

Results realizing rings as endomorphism rings over various classes of abelian groups can be looked up in Corner and Göbel [**19**]. The combinatorics of those results come from Shelah (e.g., [**90**]), and the results on abelian groups often extend earlier joint work with Dugas from the period when he was at Essen University, see the references in [**19**]. Some other papers are devoted to stronger realization theorems by using additional set theoretic axioms like $\diamondsuit$ or weak diamond, see, for example, the joint papers with Dugas [**24**] and Goldsmith [**51**].

Another combinatorial method is hidden in Shelah [**89**] which we call the 'Shelah elevator' [**54**] because it allows us to lift small rigid systems of torsion-free abelian groups (having no homomorphisms $\neq 0$ between them) to rigid systems of any size – even including groups of singular cardinality cofinal to $\aleph_0$. The papers Franzen and Göbel (in [**102**]), Corner [**17**] and Göbel and May [**53**] deal with application of this elevator. This again was used by Fuchs [**46**] to construct rigid families of divisible modules over certain valuation domains. Other applications can be found for Kronecker modules in [**66**], [**67**].

Surprisingly, the investigation of products $\mathbf{Z}^\kappa$ caused first, but isolated, conflict with set theory; see Eklof's [**35**] report in this volume on the Łos theorem and his slender groups. Interesting aspects of the decent looking Baer-Specker group $\mathbf{Z}^{\aleph_0}$ of sequences of integers are considered in a recent paper by Corner and Goldsmith [**21**]. The group $\mathbf{Z}^{\aleph_0}$ contains highly complicated subgroups studied in [**21**] by their endomorphisms and automorphisms. Even the easily defined subgroups described by the growth type of these sequences of integers are amazingly complicated such that the full power of set theory between $\aleph_0$ and $2^{\aleph_0}$ comes to play. The study of Specker's growth types of the Baer-Specker group $\mathbf{Z}^{\aleph_0}$ and a refinement of slenderness with respect to growth types in a joint paper with Wald [**65**] thus turned out to be of special interest. This, because the four slender classes defined by growth types we found in ZFC is also best possible in the sense that Blass and Shelah (see the references in [**35**]) construct a set theoretic universe having no more classes of slenderness. Contrary to this, there are $2^{2^{\aleph_0}}$ such classes if we assume Martin's axiom, see [**65**]; see also other papers by Wald [**102**, pp. 229–239] or [**100**, pp. 362–370]. These papers (including Wald's Habilitationsschrift at the Freie Universität Berlin from 1986) deal with interesting phenomena on abelian groups depending on properties of their cardinality. An application of one of these is the construction of radicals R_κ related to any (necessarily) regular cardinal below the first weakly compact cardinal such that R_κ commutes with $\prod$ in cartesian products $\prod_{i<\lambda} G_i$ of groups $G_i \neq 0$ if and only if $\lambda < \kappa$; see [**20**]. The assumption GCH under which this holds seems also necessary as Shelah indicated (in yet unpublished notes). In [**34**], Dugas and Vergohsen investigated under $V = L$ the question when a separable, abelian p-group is determined by its socle. The principal result of the paper [**37**] by Eklof, Huber and Mekler is a relative of the last one; it is shown that for countable limit ordinals λ, the statement: "every totally Crawley group of length λ is a direct sum of countable groups," is undecidable in ZFC. Huber received his Habilitation from Freiburg University in 1982 with the Habilitationsschrift, *On reflexive modules and abelian groups*, see also [**72**]. He has written a number of (elegant) papers on abelian groups, mostly connected with set theoretic principles or methods. Particularly remarkable also is [**73**]. Huber supervised a PhD thesis by Unseld on *Untermodulverbände und Endomorphismenringe* submitted at Freiburg in 1986. Like the preceding PhD thesis *Untermodulverbände torsionsfreier Moduln* by Brehm (Freiburg 1983) this is an intensive study of the relationship of the endomorphism rings and the lattice of submodules; it ends with an investigation of the classical geometric aspects of semi-linear maps known from projective geometry which was also used recently for characterizing automorphism groups of modular lattices.

In 1978 Salce developed the foundation of cotorsion-theories and got a nice description for the rational cotorsion-theories. Recall that cotorsion-theories can be obtained by looking at torsion-theories and replacing the defining Hom-functor by Ext. This is to say that they deal with pairs $(\mathcal{C}, \mathcal{F})$ of maximal classes of abelian groups such that $\operatorname{Ext}(\mathcal{C}, \mathcal{F}) = 0$, see Salce [**99**, pp. 11–32]. This also immediately leads to splitters, which are groups G with $\operatorname{Ext}(G, G) = 0$. The existence of 'nonclassical' splitters in [**59**] fortunately led to further results by

Bican, Eklof, El Bahir, Trlifaj, Enochs and others, finally proving the flat cover conjecture, see [**5**], as well as to the existence of an abundance of cotorsion-theories in [**63**]; variations on the last topic are considered in Strüngmann and Wallutis [**110**, 269–283]. Some of these new results will be reported in a book [**64**].

The question of the existence of reflexive groups G with $G \oplus \mathbf{Z} \ncong G$ (a problem in [**38**]) is answered under ZFC+MA (thus this holds under $ZFC + CH$ and also in models without CH) in a joint paper with Shelah [**110**, pp. 145–158]. However, paper [**61**] is easier to read because assuming $V = L$ makes the algebraic part transparent.

6.4. More on abelian groups. The splitting-mixed problem raised by Baer, also after Griffith's solution (that Baer groups are free) attracted further attention. Stratton (Exeter) considered extensions towards module theory which also give new insight in case of abelian groups (see [**98**, pp. 119–125]). Sands (Dundee), who recently wrote a widely recognized article on factorization of *finite* abelian groups into subsets [**87**], is to me more known for his work and interest in radicals, which he also considered in [**101**, pp. 305–314], in case of the endomorphism ring of abelian p-groups. Papers [**14**], [**19**] and those in their references, also deal with those endomorphism rings, however only for groups that have no elements of infinite height. Thus it is very natural to ask whether similar realization theorems of rings modulo small homomorphisms carry over to p-groups of arbitrary length. A first step is Corner's paper [**15**]. His PhD student, Peterken, wrote a thesis, *Some problems in abelian groups* (Oxford 1971), which concentrates on simply presented p-groups of arbitrary length, which is the class of groups where one does not expect such a realization theorem. On the other hand, a first affirmative answer was given for p-groups of length $\omega + n$, n a natural number, in a paper by Corner's PhD student, Goldsmith [**69**], several years later: The above theorem extends if "small" is replaced by its natural analogue "thin" for nonseparable p-groups. Similarly, one can deal with p-groups of arbitrary Ulm-length as shown in Behler's PhD thesis, *Abelian groups of arbitrary length*, Essen 1990. This immediately leads to the question how the automorphism group acts on those Ulm factors, a question investigated by Corner [**15**] and related to Kaplansky's transitivity and full transitivity. This topic was followed up by Hennecke in his PhD thesis, *Transitivity* *and full transitivity over subgroups of abelian p-groups* at Essen 1999, see also Hennecke [**109**, pp. 43–53]. Hennecke considers maximal subgroups with transitive and fully transitive action respectively of the automorphism group/endomorphism ring and so he can deal quite rigorously with these transitivity problems. The paper by Hennecke and Strüngmann [**71**] is a continuation of this approach. Droste, in a joint paper [**23**] gives a unified proof for results on semantics of programming languages and Ulm's theorem in view of universal objects. The PhD theses, *Kaplansky's Testprobleme in der Modultheorie über kommutativen Ringen*, by Pabst (now Wallutis) Essen (1994) and Kowalski, *Über die Idempotent- und die Co-Idempotenteigenschaft von Endomorphismenringen*, Essen (1990) deal with decompositions of modules, particular attention is given to Kaplansky's test problems as stated in [**74**]. A part of F.-V. Kuhlmann's Habilitationsschrift on *Henselian function fields and tame fields* at Heidelberg (1990) deals with particular questions on abelian groups, see his paper [**105**, pp. 217–241], or his manuscript for a book, and also S. Kuhlmann [**76**]. In her Habilitationsschrift *Direct products of modules and algebraic compactness* (Universität München 1978) Huisgen-Zimmermann investigated the relationship and splitting of direct sums in direct products of modules. Her main result on the (George) Reid classes of iterated products and direct sums of copies of $\mathbf{Z}$ and the positive answer to the Reid problem that all Reid classes are different, was extended (by generalizing the Chase lemma on products of groups (see, e.g., [**38**]) and completed in a joint paper with Dugas [**102**, pp. 179–193]. This also inspired variations on this topic for torsion groups for instance. The splitting of sums in products initiated in the early 60's of the last century by Chase led to intensive investigation of pure injectivity and its module theoretic extension; for early work by Huisgen-Zimmermann, (W.) Zimmermann and others, and later progress I can refer to a survey article by Huisgen-Zimmermann [**111**, pp. 331–367]. Further results on the related topic can be looked up in [**38**, Chapters X, XI]. Pure-injective modules were also studied from the model theoretic point of view, see the important work by Ziegler [**96**] (cf. Prest [**85**]), see also Felgner [**40**], who characterized those pure injective abelian groups which naturally relate to ultrafilters.

Several publications by Hirsch (QMC, London) and various coauthors (Hallett and Zassenhaus, the references are in [**45**]) deal with the problem which finite groups are automorphism groups of torsion-free abelian groups. The results were completed and corrected in an unpublished but widely circulated manuscript by Corner. This problem is better posed as a question about the group of units of endomorphism rings of cotorsion-free abelian groups G. For finite groups of units, it is enough to assume that G is countable. Because of [**13**] or [**19**] the problem becomes a question on orders in $\mathbf{Q}$-algebras, and it is answered in Corner [**18**].

6.5. Applications to topological abelian groups, Pontryagin duality. A problem posed in Fuchs [**45**] on the Pontryagin dual of totally projective groups initiated research on the question to see what the group invariants look like on the topological side. The same question is also interesting for Warfield modules. Kiefer from Erlangen answered the first problem in [**100**, pp. 297–304] and Loth, a former student from Erlangen, investigated the second problem in his PhD thesis at Wesleyan University (Connecticut). He devoted a last section of his book on Warfield modules for a discussion of this particular Pontryagin duality. Leopold wrote a PhD thesis under the direction of Plaumann on *Die kompakt-offene Topologie auf Ext und End* at Erlangen (1999), which deals with the functors Ext and End in the category of locally compact abelian groups. What makes this topology Hausdorff and locally compact respectively? For answers, see also Leopold in [**109**, pp. 293–299].

6.6. Application to group, ring and field theory. Finally we summarize some applications of results mentioned in the earlier sections to other areas of algebra. An intensive study of E-rings by Pierce and Vinsonhaler initiated by Schultz led to the question of the existence of large E-rings. This was answered affirmatively by Dugas, Mader, Vinsonhaler [**32**] using methods and ideas from [**90**], [**24**], [**19**]. For prediction principles, see also Göbel [**111**, pp. 107–127]. In models of $V = L$ the proof of the main theorem [**32**] can be simplified and strengthened; see the PhD thesis by Strüngmann, *Almost-free* $E(R)$-*algebras over commutative domains* (Essen 1998). The existence of generalized E-rings R will be shown in [**62**], i.e., there is a non-commutative ring R with $R = \operatorname{End} R^+$ which cannot be an E-ring; see a question by Feigelstock, Hausen and Raphael in [**109**, pp. 231–239] and Vinsonhaler.

The existence of non-trivial locally finite complete p-groups (with no outer automorphisms and trivial center) in [**26**] answers a problem of P. Hall. The proof uses the existence of essentially indecomposable abelian p-groups from [**19**]. In fact, it was shown that any countable group is the outer automorphism group of such a locally finite p-group.

Countability was removed only recently in [**6**] by application of a characterization of direct sums of cyclic p-groups due to Kulikov (see Fuchs [**44**]). A parallel result can be shown for the class of metabelian groups, which uses techniques from [**13**] in the case of outer automorphism groups of cardinality less than $2^{\aleph_0}$, see [**109**, pp. 309–314] and methods from [**19**] for automorphism groups of size larger than the continuum (see [**56**]). The result is in sharp contrast to the abelian case, where only a few groups are the (outer) automorphism groups, for example, $\mathbf{Z}/7\mathbf{Z}$ is not as follows from the characterization [**18**]. Paras spent several years in Essen while this joint work was done. The existence of non-commutative splitters, that are groups G where every self-extension of G by G splits into a direct sum, implies the answer to a problem by Hall on the existence of groups $G \neq 1$ where every self-extension is isomorphic to G. In answering this question, fully rigid systems of torsion-free abelian groups are used (see [**60**]). The existence of fully rigid systems of torsion-free abelian groups was shown in [**19**]. The construction in [**60**] is a by-product of embedding finite simple groups H into large simple groups G such that the only endomorphisms of G are inner automorphisms which act uniquely transitive on the copies of H in G. The latter result is used to answer a question by E. Dror Farjoun on homotopy theory (see [**57**]). The well-known fact that homomorphisms of cartesian products (below the first measurable cardinal) into slender groups are uniquely determined on their restricted direct sum, also holds for countable complete free products as shown by Higman (see reference in [**91**]), but only recently it was shown that this so-called Specker phenomenon fails already, if the number of factors G in the complete free product is uncountable and $G = \mathbf{Z}$ (see [**91**]).

It has been my intention throughout this paper to comply with the request to depict the developments of abelian groups in England and Germany. However, European and international activities in abelian groups can no longer be regarded separately as they were until the first decades of the last century. Researchers in this field have truly become a big international family, cross-border alliances in the mathematical and political sense have made the field fairly strong.

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