ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 1, Spring 2003

ON POINCARÉ'S FOURTH AND FIFTH EXAMPLES OF LIMIT CYCLES AT INFINITY

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ABSTRACT. Errors are found in example problems from Henri Poincaré's paper, "Mémoire sur les courbes définies par une équation différentielle." Examples Four and Five from Chapter Seven and Examples One, Two and Three from Chapter Nine do not have the limit cycles at infinity predicted by Poincaré. Instead they have fixed points at every point at infinity. In order to understand the errors made by Poincaré, Examples Four and Five are studied at length. Replacement equations for the fourth and fifth Examples are suggested based on the supposition that terms were omitted from Poincaré's equations.

1. Introduction. In the late nineteenth and early twentieth century, Henri Poincaré began to study the qualitative aspects of systems of differential equations. This analysis was a breakthrough in the field because one no longer had to obtain a specific solution of the equation in question to understand its general behavior. This manner of analysis was introduced in 1881 by Poincaré in his paper, "Mémoire sur les courbes définies par une équation différentielle" [**8**]. Poincaré worked primarily with systems of two variables and played a key role in identifying the existence of limit cycles with the Poincaré-Bendixson theorem. Poincaré also searched for a complete global analysis of a system of two variables; to do so, he introduced analysis at infinity by means of the Poincaré Sphere. These two aspects of his analysis join in a behavior known as a limit cycle at infinity.

To illustrate the application of these techniques, Poincaré presents example problems in chapters seven and nine. In Chapter Seven, Poincaré presents five example problems. He states that Examples Three, Four and Five have limit cycles at infinity. (Examples Three, Four and Five throughout the paper will be referred to as P3, P4 and P5, respectively.) However, when one checks Poincaré's assertion, one finds that Examples P4 and P5, in fact, do not have limit cycles at

Received by the editors on February 22, 2000, and in revised form on January 22, 2001.

infinity. The errors remain in the collected works [**7**] and the error in P4 remains in a modern work that includes translations of Examples P3 and P4 [**3**, p. 155].

In Chapter Nine, Poincaré presents three additional example problems. He states that each of these examples has limit cycles at infinity. However, he makes the same error analyzing these equations as he did for P4 and P5.

Throughout this paper, in an effort to understand why Poincaré made these errors, we focus on studying Examples P4 and P5 from Chapter Seven. In the conclusion we will attempt to apply what we have learned about the errors in P4 and P5 to the examples from Chapter Nine.

Because the analysis at infinity is algebraically simple for each of these problems, one becomes perplexed at this finding. Surely Poincaré could not have made such an error. With this belief, one might suppose that there was a simple omission of some sort. If one were to find equations that have the same behavior as Poincaré predicts for P4 and P5 and that, were one to drop certain terms, could become his printed equations, this would serve as a plausible explanation. We look at P4 in the context of P3 and consider a plausible modification. Such an analysis is encouraged by the strong geometric similarities between the two systems. The modification restores the limit cycle at infinity and hence supports the hypothesis of an error of omission, becoming a key candidate for the intended fourth example. We will call this Example R4. Example P5 does not fit in the same genre of geometric system, and hence no such modification suggests itself.

When one investigates the uniqueness of these "solutions" to the problem of finding modifications of Poincaré's equation that match his analysis for P4, one can find a classification of all modifications of the original equation, of a certain type, that result in the correct behavior. A great number of such modifications exist; however, this remarkable nonuniquenss in "solutions" does not cause skepticism that R4 is the intended equation for the fourth example. It is seen that the modification to obtain R4 is the simplest of the modifications found under the classification and, hence, through application of the philosophical principle known as Occam's razor, R4 remains the most plausible equation for the fourth example.

Based on the success of the technique applied to P4, a similar technique is applied to P5 to try to determine a possible modification of P5 which yields the correct behavior. This technique provides a number of modifications which agree with Poincaré's analysis for P5, thus illustrating the power of the technique. One of these, which we shall call R5, is chosen as a suggested replacement to P5. (No modifications are suggested for the examples from Chapter Nine.)

To aid the reader with the analysis of differential equations at infinity, a section is included presenting the standard techniques for analysis of the behavior at infinity for a polynomial system in rectangular coordinates. A more geometric analysis is also presented that is quite valuable in many cases in which such a system simplifies in polar coordinates.

2. Differential equations at infinity. Differential equations that are defined on the plane can be analyzed at infinity by extending the plane to a representation of the plane that includes the "points at infinity." This analysis began with the work of Poincaré and Bendixson [**8**], [**2**]. Throughout this paper we use Poincaré's method of analysis. More modern and readable accounts on the Poincaré's method are available in books by Perko [**6**], Lefschetz [**4**] and Minorsky [**5**]. This section begins with a brief summary of the standard results as in Perko [**6**, pp. 264–268]. A theorem is presented, consistent with the standard techniques, based on polar coordinates, that demonstrates the geometric nature of these systems at infinity. This theorem often makes it possible for one to determine by inspection whether a system has a limit cycle at infinity.

Through the use of projective geometry, the complete behavior of a two-dimensional system of differential equations can be seen in its behavior on a sphere of finite radius known as the Poincaré sphere. To do this, one places the phase plane tangent to the sphere and makes points on the plane correspond with the points on the sphere by central projection (a point on the plane corresponds with an antipodal pair on the sphere). One must note that the line intersects the sphere at two points; to remove this nonuniqueness, antipodal points are identified on the Poincaré sphere. The points that were at infinity on the original plane become points on the equator of the sphere.

From the behavior on the Poincaré sphere, one can construct the "global phase portrait." To consider the global phase portrait of a system, one projects the trajectories from the upper hemisphere orthogonally down onto the plane that goes through the center of the sphere and is parallel to the original plane. In this way the complete behavior on the original plane becomes the behavior on the finite disk of this new plane. Points at infinity become the points at which the sphere intersects the new plane, the boundary of the disk.

The first analysis that one would do for a system in the finite plane is to find the location of the fixed points. With this as motivation, the primary problem of analyzing the behavior of a system at infinity is to determine the location of fixed points, if there are any. One considers the system:

$$\frac{dx}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y),$$

where P and Q are polynomials of degree m in x and y. Denote by P_m and Q_m the homogeneous polynomials consisting of terms of degree exactly m.

Theorem 1. *The critical points at infinity for the system above occur at the points $(X, Y, 0)$ on the equator of the Poincaré sphere where $X^2 + Y^2 = 1$ and*

$$XQ_m(X, Y) - YP_m(X, Y) = 0 \tag{2.1}$$

or equivalently at the polar angles θ and $\theta + \pi$ which are solutions of

$$G_{m+1}(\theta) \equiv \cos\theta Q_m(\cos\theta, \sin\theta) - \sin\theta P_m(\cos\theta, \sin\theta) = 0. \tag{2.2}$$

This equation has at most $m + 1$ pairs of roots θ and $\theta + \pi$ unless $G_{m+1}(\theta)$ is identically zero. If $G_{m+1}(\theta)$ is not identically zero, then the flow on the equator of the Poincaré sphere is counterclockwise at points corresponding to polar angles θ where $G_{m+1}(\theta) > 0$ and it is clockwise at points corresponding to polar angles where $G_{m+1}(\theta) < 0$.

See Perko [**6**, Section 3.10].

An important consequence of Theorem 1 is that any polynomial system in rectangular coordinates can be extended to the Poincaré sphere.

One should be aware that, in the proof of Theorem 1, a new time scale is defined to study the trajectories in the proximity of the equator. There are two opposing conventions used in the degenerate case where (2.1) is identically zero. The most common one, used by Perko [**6**] and Minorsky [**5**] is to reparameterize time in such a way that the equator is forced to consist of trajectories and fixed points. The second convention, followed by Lefschetz [**4**] and Poincaré [**8**] uses a different time scale, allowing trajectories to cross the equator (implying that trajectories in one of the hemispheres will flow in a direction inconsistent with the flow on the plane). Because the former method is more common, we will adhere to it, even though Poincaré used the latter. (A brief comment will be made in the analysis of Example P4 about why this does not invalidate our results.)

If there are no fixed points at infinity, there is a cycle at infinity. If trajectories in the proximity of this cycle at infinity approach (or recede from) it, it is a limit cycle at infinity. A corollary immediately follows:

Corollary 1. *Let r denote radial distance in polar coordinates. If the system considered above has no fixed points at infinity and $dr/dt \neq 0$ for $r \geq R$, for some R, then there exists a limit cycle at infinity.*

While this theorem makes it relatively easy to determine the behavior of a system at infinity, there is a more geometric analysis that, for certain systems, will make the behavior at infinity particularly easy to determine.

Theorem 2. *Consider a polynomial system*

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y) \tag{2.3}$$

with expressions for dr/dt and $d\theta/dt$ of order I and J in r as $r \to \infty$, respectively. Let $k = I - J$. Then if $k \geq 2$, the equator of the Poincaré sphere consists entirely of fixed points. If $k \leq 1$, then: if $G_{m+1}(\theta) \neq 0$ for all θ, there is a cycle at infinity. Furthermore, if dr/dt satisfies *the conditions of the above corollary, it is a limit cycle. Otherwise, the equator of the Poincaré sphere has finitely many fixed points located at* θ *such that* $G_{m+1}(\theta) = 0$*, as in Theorem* 1.

Proof of Theorem 2. Suppose that P and Q are polynomials of degree m in x and y. We can express this system in polar coordinates for $r \neq 0$ as follows

$$\begin{aligned}
\frac{dr}{dt} &= (\cos\theta P_0 + \sin\theta Q_0) \\
&\quad + r(\cos\theta P_1(\cos\theta, \sin\theta) + \sin\theta Q_1(\cos\theta, \sin\theta)) \\
&\quad + \cdots + r^m(\cos\theta P_m(\cos\theta, \sin\theta) + \sin\theta Q_m(\cos\theta, \sin\theta)). \\
\frac{d\theta}{dt} &= r^{-1}(\cos\theta Q_0 - \sin\theta P_0) \\
&\quad + (\cos\theta Q_1(\cos\theta, \sin\theta) - \sin\theta P_1(\cos\theta, \sin\theta)) \\
&\quad + \cdots + r^{m-1}(\cos\theta Q_m(\cos\theta, \sin\theta) - \sin\theta P_m(\cos\theta, \sin\theta)).
\end{aligned}$$

Throughout the remainder of the proof, we require $r \neq 0$. The following definitions will greatly simplify notation:

$$\begin{aligned}
\eta_0(\theta) &= (\cos\theta P_0 + \sin\theta Q_0), \\
&\vdots \\
\eta_I(\theta) &= (\cos\theta P_I(\cos\theta, \sin\theta) + \sin\theta Q_I(\cos\theta, \sin\theta)),
\end{aligned}$$

and

$$\begin{aligned}
\xi_{-1}(\theta) &= (\cos\theta Q_0 - \sin\theta P_0), \\
&\vdots \\
\xi_J(\theta) &= (\cos\theta Q_{J+1}(\cos\theta, \sin\theta) - \sin\theta P_{J+1}(\cos\theta, \sin\theta)),
\end{aligned}$$

where I and J are the highest degree terms, in r, in the equations for dr/dt and $d\theta/dt$, respectively. With this notation, we obtain

$$\begin{aligned}
\frac{dr}{dt} &= \eta_0(\theta) + \cdots + r^I \eta_I(\theta), \\
\frac{d\theta}{dt} &= r^{-1}\xi_{-1}(\theta) + \xi_0(\theta) + \cdots + r^J \xi_J(\theta).
\end{aligned}$$

By the definition of k we have $k = I - J$. If we put this system in differential form, we obtain

$$(r^{-1}\xi_{-1}(\theta) + \cdots + r^J \xi_J(\theta))\, dr - (\eta_0(\theta) + \cdots + r^I \eta_I(\theta))\, d\theta = 0. \tag{2.4}$$

To determine the behavior at infinity, one projects a differential equation onto the Poincaré sphere. To understand the details of the behavior at infinity, project the upper hemisphere of the Poincaré sphere onto the cylinder of radius 1 with axis orthogonal to the phase plane that is tangent to the Poincaré sphere at the equator. (We can restrict our attention to the upper hemisphere because we follow the standard convention which does not allow trajectories to cross the equator.) The "equator" of the cylinder is the part of the cylinder that touches the sphere. By geometric analysis of the projection from the plane directly to the cylinder, $s = 1/r$ and correspondingly $dr = (-1/s^2)\, ds$. This projection leaves θ unchanged.

Projecting (2.4) from the plane directly onto the cylinder gives:

$$\left(s\xi_{-1}(\theta) + \cdots + \frac{1}{s^J}\xi_J(\theta)\right)\left(-\frac{1}{s^2}\, ds\right) - \left(\eta_0(\theta) + \cdots + \frac{1}{s^I}\eta_I(\theta)\right) d\theta = 0,$$

which simplifies to

$$\left(s\xi_{-1}(\theta) + \cdots + \frac{1}{s^J}\xi_J(\theta)\right) ds + s^2\left(\eta_0(\theta) + \cdots + \frac{1}{s^I}\eta_I(\theta)\right) d\theta = 0. \tag{2.5}$$

Case 1. $k \geq 2$. In this case we have $I - J \geq 2$, so $I - 2 \geq J$. We multiply (2.5) by s^{I-2} to clear the denominators obtaining:

(2.6)

$$(s^{I-1}\xi_{-1}(\theta) + \cdots + s^{I-J-2}\xi_J(\theta))\, ds + (s^I \eta_0(\theta) + \cdots + \eta_I(\theta))\, d\theta = 0.$$

However, this equation would indicate trajectories crossing the equator (from one side or the other) for all points $(\theta, 0)$ where $\eta_I(\theta) \neq 0$. Because we wish to maintain the same convention as in Theorem 1, the equator is required to consist only of fixed points and trajectories moving along the equator. As a result, the trajectories crossing the equator would lead to a violation of the uniqueness of solutions to

(2.6) at these points $(\theta, 0)$. To resolve this, we must multiply (2.6) by an additional value of s, a reparameterization of time that causes these trajectories to slow down so that they do not cross the equator. We obtain:

(2.7)
$$(s^I\xi_{-1}(\theta) + \cdots + s^{I-1-J}\xi_J(\theta))\, ds + (s^{I+1}\eta_0(\theta) + \cdots + s\eta_I(\theta))\, d\theta = 0.$$

With regards to this new time scale, τ, (2.7) can be expressed as the system:

$$\begin{aligned} \frac{ds}{d\tau} &= -(s^{I+1}\eta_0(\theta) + \cdots + s\eta_I(\theta)), \\ \frac{d\theta}{d\tau} &= (s^I\xi_{-1}(\theta) + \cdots + s^{I-J-1}\xi_J(\theta)). \end{aligned} \tag{2.8}$$

Because $k = I - J \geq 2$, at $s = 0$, we have

$$\frac{ds}{d\tau} = 0, \qquad \frac{d\theta}{d\tau} = 0.$$

We conclude that every point on the equation is a fixed point.

Case 2. $k \leq 1$. We have $I - J \leq 1$ and $J \leq I - 1$. We multiply (2.5) by s^J to clear the denominators. In this case, no trajectories approach the equator in a finite time, hence there is no need for the reparameterization done in Case 1.

We obtain

$$\begin{aligned} &(s^{J+1}\xi_{-1}(\theta) + \cdots + s\xi_{J-1}(\theta) + \xi_J(\theta))\, ds \\ &\qquad + (s^{J+2}\eta_0(\theta) + \cdots + s^{2-(I-J)}\eta_I(\theta))\, d\theta = 0. \end{aligned}$$

Expressing this in the form of a system, with respect to the new time scale τ, we obtain

$$\begin{aligned} \frac{ds}{d\tau} &= -s^{2-(I-J)}(s^I(\eta_0(\theta) + \cdots + \eta_I(\theta)), \\ \frac{d\theta}{d\tau} &= (s^{J+1}\xi_{-1}(\theta) + \cdots + \xi_J(\theta)). \end{aligned}$$

So, on the equator, we have $s = 0$ and

$$\frac{ds}{d\tau} = 0, \qquad \frac{d\theta}{d\tau} = \xi_J(\theta).$$

So the fixed points at infinity are for θ such that $\xi_J(\theta) = 0$. It is easy to see that, in this case, $\xi_J(\theta) = G_{m+1}(\theta)$, as defined in Theorem 1. Hence, the fixed points at infinity are at θ such that

$$G_{m+1}(\theta) = \cos\theta Q_m(\cos\theta, \sin\theta) - \sin\theta P_m(\cos\theta, \sin\theta) = 0.$$

We conclude that if $G_{m+1}(\theta) \neq 0$ for all θ, then there is a cycle at infinity, and that, if nearby trajectories approach that cycle (or recede from it), as in the corollary to Theorem 1, then it is a limit cycle.

This concludes a proof of Theorem 2.

One should notice that in the proof of this theorem one finds that, for the case $k \leq 1$, one has $G_{m+1}(\theta) = \xi_J(\theta)$, the highest order term in r of the $d\theta/dt$ equation. This makes this theorem particularly useful; one can often tell whether a system has a limit cycle at infinity by merely expressing the system in polar coordinates.

One should also note that, in the proof case 1, if $k > 2$, equation (2.8) gives that a trajectory approaching (or receding from) any point on the equator does so orthogonally to the equator.

3. The Poincaré examples. To enable the reader to understand Poincaré's fourth and fifth examples, three of Poincaré's examples from Chapter Seven of his paper [**8**, pp. 274–281] are presented. Poincaré's third example, P3, works as an introduction to his fourth and fifth examples and serves as an example of a system that has a limit cycle at infinity, the behavior that Poincaré claims for P4 and P5. Examples P4 and P5 are then presented with demonstration of the error in his analysis at infinity. At the end of this section, the error in Poincaré's analysis of his examples from Chapter Nine is briefly discussed.

Example P3. Poincaré considers the equation:

$$\frac{dx}{x(x^2+y^2-1) - y(x^2+y^2+1)} = \frac{dy}{y(x^2+y^2-1) + x(x^2+y^2+1)}.$$

This differential form of the equation is equivalent to the more familiar form of a system as noted in [**6**, p. 266].

$$\begin{aligned}\frac{dx}{dt} &= x(x^2+y^2-1) - y(x^2+y^2+1),\\ \frac{dy}{dt} &= y(x^2+y^2-1) + x(x^2+y^2+1).\end{aligned}$$

The geometric aspects of this system become far more obvious in polar form:

$$\begin{aligned}\frac{dr}{dt} &= r(r^2-1),\\ \frac{d\theta}{dt} &= r^2+1.\end{aligned}$$

Poincaré presents the key qualitative features of this system:

- The origin: $(x,y) = (0,0)$ is a fixed point. One has a stable spiral at the origin.
- A repelling limit cycle at radius $r = 1$ centered about the origin.
- An attracting limit cycle at infinity.

All of the features in the finite plane are easily established, so only the features of this system at infinity are analyzed in detail. Poincaré establishes that there is a limit cycle at infinity because he claims that there are no fixed points at infinity: "Il n'y a aucun point singulier sur l'équateur, qui est une caractéristique et qui est par conséquent un cycle limite" [**8**, p. 278]. He omits this calculation, so we verify it here.

We apply Theorem 2 to show that there are no fixed points at infinity. From the expression for the P3 in polar coordinates, we find $I = 3$ and $J = 2$. Further, $\xi_2(\theta) = 1$ for all θ, demonstrating the existence of a cycle at infinity. Because $dr/dt > 0$ when $r \geq 2$, this cycle is a limit cycle.

Example P4. Poincaré's fourth example can be considered a more complicated version of his third example. Poincaré writes the equation as:

$$\frac{dx}{x(x^2+y^2-1)(x^2+y^2-9)-y(x^2+y^2-2x-8)}$$
$$= \frac{dy}{y(x^2+y^2-1)(x^2+y^2-9)+x(x^2+y^2-2x-8)},$$

which he transforms into the system

$$\begin{aligned}\frac{dx}{dt} &= x(x^2+y^2-1)(x^2+y^2-9)-y(x^2+y^2-2x-8),\\ \frac{dy}{dt} &= y(x^2+y^2-1)(x^2+y^2-9)+x(x^2+y^2-2x-8).\end{aligned}$$

Poincaré states the key features of the system:

- The origin: $(x,y) = (0,0)$ is a fixed point. One has an unstable spiral at the origin.
- A: $(x,y) = (1/2, \sqrt{35}/2)$ and B: $(x,y) = (1/2, -\sqrt{35}/2)$ are fixed points: an unstable node and a saddle, respectively, at the intersection of the circles defined by $x^2+y^2-9=0$ and $x^2+y^2-2x-8=0$.
- An attracting limit cycle of radius $r=1$ centered about the origin.
- Two heteroclinic orbits connecting fixed points A to fixed point B along the circle $x^2+y^2-9=0$.
- An attracting limit cycle at infinity.

All of the stated features in the finite plane can be easily established. (A similar analysis is applied to a slightly more complicated system in the following section five.) The geometric nature of the system becomes clear in polar coordinates. The system becomes:

$$\begin{aligned}\frac{dr}{dt} &= r(r^2-1)(r^2-9),\\ \frac{d\theta}{dt} &= r^2-2r\cos\theta-8.\end{aligned}$$

The phase portrait corresponding to Poincaré's analysis is included in Figure 3.1. He claims that, as in P3, there are no fixed points at infinity,

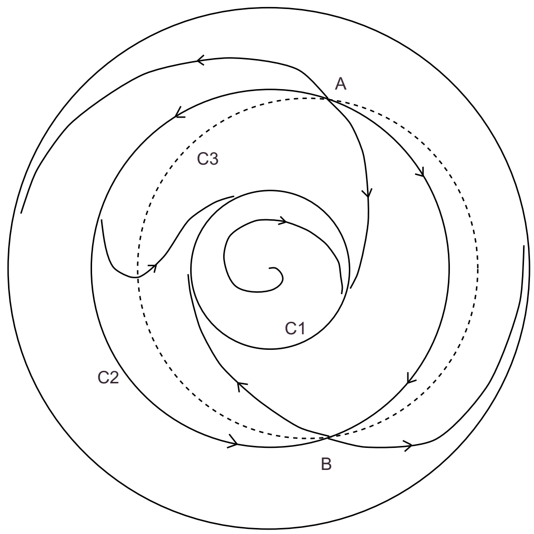

FIGURE 3.1. Sketch of the global phase portrait corresponding to Poincaré's analysis of example P4. Circles C_1, C_2 and C_3, which are referred to later in this paper, are labeled. (The dashed line for C_3 indicates that it is a nullcline and not a trajectory.)

and hence a limit cycle. However, when one does the calculations to check Poincaré's assertion, one finds that this is not the case.

To check the assertion, one applies Theorem 2 to the system. By looking at the expression for P4 in polar coordinates, one finds that $I = 5$ and $J = 2$, hence $k = 3$ and every point at infinity is a fixed point. There is no limit cycle at infinity because every point on the equator is a fixed point. (If one had used the second convention which was used by Poincaré, one would have found trajectories crossing the equator. With this convention, the trajectories crossing the equator eliminate the possibility of a cycle on the equator. All reference to this second convention will be dropped for all of the following examples.) The correct global phase portrait for the system is presented in Figure 3.2.

Example P5. Poincaré's fifth example is not as clear of an extension of his third and fourth examples as it does not quite fit the same "genre"

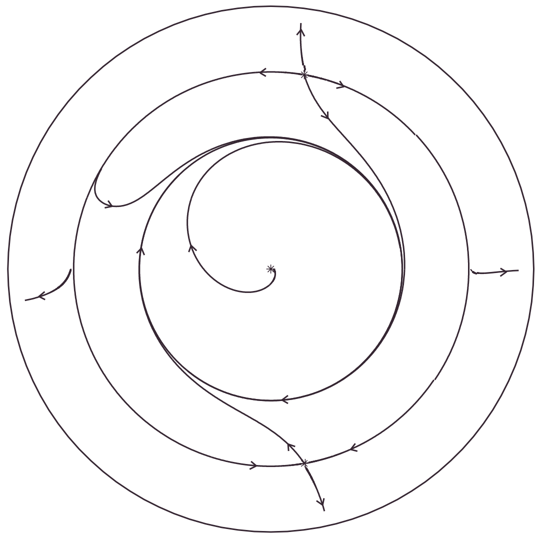

FIGURE 3.2. The correct global phase portrait corresponding to Example P4. (Figures 3.2 and 4.1 were calculated numerically using a Runge Kutta method.)

of system. The main difference is that, whereas P3 and P4 are more naturally understood in a polar coordinate system, P5 is more easily understood in a coordinate system based on lemniscates. Example P5 is also more complicated because it depends upon the parameter c. He considers the equation:

$$\frac{dx}{x(2x^2+2y^2+1)((x^2+y^2)^2+x^2-y^2-c)-y(2x^2+2y^2-1)}$$
$$=\frac{dy}{y(2x^2+2y^2-1)((x^2+y^2)^2+x^2-y^2-c)+x(2x^2+2y^2+1)}.$$

Poincaré asserts that P5 has the following behavior, which depends on the parameter c.

1. For $c \leq -1/4$, the system has:
   - A saddle point at the origin.
   - Two unstable spirals at $(0, \pm 1/\sqrt{2})$.

- An attracting limit cycle at infinity.

2. For $-1/4 < c < 0$, the system has:
   - A saddle point at the origin.
   - Two stable spirals at $(0, \pm 1/\sqrt{2})$.
   - Two repelling limit cycles surrounding each of the two above spirals, respectively.
   - An attracting limit cycle at infinity.
3. For $c = 0$, the system has:
   - A saddle point at the origin.
   - Two stable spirals at $(0, \pm 1/\sqrt{2})$.
   - Two homoclinic orbits which branch from the saddle and surround each of the spirals, respectively.
   - An attracting limit cycle at infinity.
4. For $c > 0$, the system has:
   - A saddle point at the origin.
   - Two stable spirals at $(0, \pm 1/\sqrt{2})$.
   - One single repelling limit cycle centered at the origin which surrounds all three fixed points.
   - An attracting limit cycle at infinity.

The four qualitatively different phase portraits are sketched in Figure 3.3. As in the example P4, all of these features in the finite plane can be easily verified. Because Poincaré does not include his analysis of the behavior at infinity, we check it here. Example P5 does not simplify in polar coordinates, so it is easiest to use Theorem 1 to determine the behavior at infinity. The highest order terms are of degree seven, so we have:

$$P_7 = 2X(X^2 + Y^2)^3,$$
$$Q_7 = 2Y(X^2 + Y^2)^3.$$

So to find the fixed points at infinity, according to Theorem 1, one must solve the following system:

$$XQ_7 - YP_7 = 2XY(X^2 + Y^2)^3 - 2YX(X^2 + Y^2)^3 = 0,$$
$$X^2 + Y^2 = 1.$$

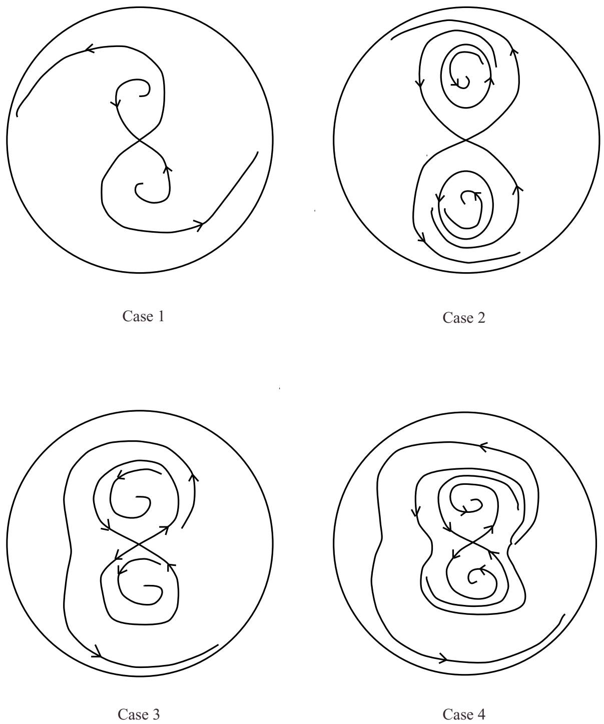

FIGURE 3.3. Global phase portraits corresponding to Poincaré's analysis to the four different cases of P5.

The first equation is satisfied for all X, Y so every point at infinity is a fixed point. Hence there can be no limit cycle at infinity. (This error was first discovered by Professor Lawrence Perko of Northern Arizona University when he read a preliminary draft of this paper.)

Examples from Chapter Nine. Poincaré presents three additional examples in Chapter Nine [**8**, pp. 292–296] of his paper to illustrate the more general techniques that he developed in Chapter Eight. Poincaré claims that each of these examples has a limit cycle at infinity. However, one can easily check his claim using Theorem 1 and Theorem 2, as was done above for P3, P4 and P5, to see that each of these examples has fixed points at every point on the equator. Because it is easy to check this, the verification is left to the reader. Furthermore, throughout this paper, we focus on a discussion of examples P4 and P5.

The fact that these errors were so easily detected immediately makes one wonder whether Poincaré used a similar form of analysis for finding fixed points at infinity as was presented in Section 2. When one looks at his first and second examples [**8**, pp. 274–278] one finds that he uses the equations from Theorem 1 to find fixed points at infinity. Because the calculations for finding fixed points at infinity for each of these examples are so simple (using Theorem 1 or Theorem 2) one is led to speculate as to the cause of Poincaré's error.

4. Error of omission. The facts that Poincaré's errors in finding the fixed points at infinity for his examples P4 and P5 were so easily detected, and that he used the same type of criterion that is used here, suggest that, perhaps, his analysis was of different equations than the ones printed in the paper. Were there to be an error of omission, terms which would lead to an unsolvable system for finding the fixed points at infinity could be missing which would not change the qualitative nature of his system on the finite plane, where his analysis matched the printed equation. If this were the case, there most probably was an error of omission in the production of his manuscript, not in the mathematics. After lengthy investigation, a system which could very plausibly become the system printed in the paper as P4, were one to omit certain terms, becomes convincing candidates for the intended fourth example.

In terms of the qualitative features predicted, Poincaré's fourth example appears to be an augmented version of his third example; however, the equation listed is not an algebraically augmented version of the equation in his third example. Algebraically, when in polar coordinates, the dr/dt equation is augmented with another nullcline at $r = 3$. To create the new fixed points A and B, the $d\theta/dt$ equation has a new nullcline which crosses the dr/dt nullcline at $r = 3$. However, the $d\theta/dt$ equation is missing the factor $(r^2 + 1)$, which would not create any nullclines, but which was necessary for P3 to have a limit cycle at infinity (see calculation). A plausible attempt at finding Poincaré's actual equation is to include this term in the polar version of example P4. This attempt would fit with the intuition that P4 is merely an augmented version of P3. (Example P5 is not considered "in the context of examples P3 and P4" because the limit sets for P5 are based on lemniscates instead of circles.)

Example R4. The system becomes:

$$\frac{dr}{dt} = r(r^2 - 1)(r^2 - 9),$$
$$\frac{d\theta}{dt} = (r^2 - 2r\cos(\theta) - 8)(r^2 + 1).$$

In Cartesian coordinates, the system becomes:

$$\frac{dx}{dt} = x(x^2 + y^2 - 1)(x^2 + y^2 - 9) - y(x^2 + y^2 - 2x - 8)(x^2 + y^2 + 1),$$
$$\frac{dy}{dt} = y(x^2 + y^2 - 1)(x^2 + y^2 - 9) + x(x^2 + y^2 - 2x - 8)(x^2 + y^2 + 1).$$

This new version of Poincaré's fourth example has all of the qualitative features that his printed system has on the finite plane, and has the additional feature of a limit cycle instead of fixed points at infinity. Hence, this system exactly matches Poincaré's analysis of P4. The global phase portrait corresponding to example P4 is included in Figure 4.1. Verification that this system has the same qualitative features as Poincaré claims for example P4 will not be presented here because it is a result of Theorem 3 in the following section. Because this system algebraically follows from Poincaré's third example, according to the hypothesis that Poincaré merely augmented his third example to obtain his fourth example, this compels the author to believe that this was Poincaré's intended system.

5. A method of classification. The results of the last section raise the question: how can one change the algebraic statement of a differential equation and have the qualitative aspects in the finite plane remain the same, while changing the behavior at infinity? Rather than using ad hoc methods for changing an equation, can there be a more systematic way of finding a certain modification of an equation with the desired consequences?

To begin to answer these questions, this section contains a systematic way for finding which, among a certain class of changes, cause the

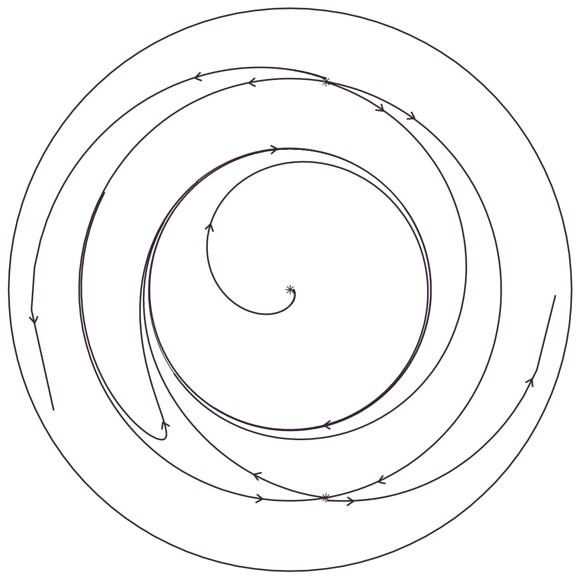

FIGURE 4.1. The global phase portrait corresponding to example R4.

equation listed in Poincaré's paper as P4 to be qualitatively equivalent to his analysis listed for P4. Based on the success of this technique with P4, it is applied to P5 leading to a recommended modification of P5 which we will call R5. These results not only serve as a demonstration of such techniques, but also as a statement about the remarkable nonuniqueness of equations with this specific phase portrait.

This is the modification of P4:

Theorem 3 (Classification Theorem for P4). *Let $W(x,y)$ be a polynomial in x and y of degree N, and let $W_N(x,y)$ be the homogeneous portion of W which is of degree N. If*

1. $W(x,y) > 0$ *for all* $(x,y) \in \mathbf{R}^2$,
2. $N \geq 2$ *and*
3. $W_N(x,y)$ *is positive definite*,

then the phase portrait of the system given by

$$\frac{dx}{dt} = x(x^2+y^2-1)(x^2+y^2-9) - y(x^2+y^2-2x-8)W(x,y),$$
$$\frac{dy}{dt} = y(x^2+y^2-1)(x^2+y^2-9) + x(x^2+y^2-2x-8)W(x,y),$$

gives a phase portrait qualitatively equivalent to the analysis Poincaré gives in example P4.

First, one must verify that the system has the same number, location and type of fixed points. To do this, first convert the system to polar coordinates, obtaining:

$$\frac{dr}{dt} = r(r^2-1)(r^2-9),$$
$$\frac{d\theta}{dt} = (r^2 - 2r\cos\theta - 8)W(r\cos\theta, r\sin\theta).$$

Because the new factor in the $d\theta/dt$ equation is never zero, it does not create any new nullclines. Hence, because there are no new nullclines, only the fixed points that existed in the equation listed in the paper (A, B and O) exist in these modified examples.

To verify that these fixed points have the same local behavior as presented by Poincaré, one must calculate the trace, τ, and determinant, Δ, of the Jacobian matrix of the system (in rectangular coordinates) symbolically at arbitrary (x_0, y_0). To simplify the calculation consider the following definitions:

$$C_1(x,y) = x^2+y^2-1,$$
$$C_2(x,y) = x^2+y^2-9,$$
$$C_3(x,y) = x^2+y^2-2x-8.$$

With these definitions, our system becomes

$$\frac{dx}{dt} = xC_1(x,y)C_2(x,y) - yC_3(x,y)W(x,y),$$
$$\frac{dy}{dt} = yC_1(x,y)C_2(x,y) + xC_3(x,y)W(x,y).$$

The expressions are calculated as follows (the arguments of circles C_1, C_2 and C_3 are dropped):

$$\tau = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y},$$
$$\Delta = \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}.$$

One obtains:

$$\tau = 2C_1C_2 + 2(x^2+y^2)(C_1+C_2) + 2yW(x,y) + C_3\left(x\frac{\partial W(x,y)}{\partial y} - y\frac{\partial W(x,y)}{\partial x}\right),$$

$$\begin{aligned}\Delta = &\left(C_1C_2 + 2x^2(C_1+C_2) - y(2x-2)W(x,y) - yC_3\frac{\partial W(x,y)}{\partial x}\right)\cdot\\ &\left(C_1C_2 + 2y^2(C_1+C_2) + 2xyW(x,y) + xC_3\frac{\partial W(x,y)}{\partial y}\right)\\ &-\left(2xy(C_1+C_2) - C_3W(x,y) - 2y^2W(x,y) - yC_3\frac{\partial W(x,y)}{\partial y}\right)\cdot\\ &\left(2xy(C_1+C_2) + C_3W(x,y) + x(2x-2)W(x,y) + xC_3\frac{\partial W(x,y)}{\partial x}\right).\end{aligned}$$

When one evaluates these quantities at one of the fixed points, to check the linearization, one is confronted with the unknown function $W(x,y)$. One must make restrictions on $W(x,y)$ so that the linearizations satisfy Poincaré's analysis for P4.

At fixed point O all terms multiplied by x or y vanish, so one obtains $\tau = 18$, $\Delta = 81 + 64W(0,0)^2$ and $\tau^2 - 4\Delta = -256W(0,0)^2$. Because $\tau > 0$ and because $\tau^2 - 4\Delta < 0$ (because $W(0,0) > 0$) we have an unstable spiral.

At fixed point B, we have $\Delta = -144\sqrt{35}W(1/2, -\sqrt{35}/2)$. Once again, because $W(x,y) > 0$ for all (x,y), we have $\Delta < 0$, hence a saddle.

At fixed point A we have $\tau = 144 + \sqrt{35}W(1/2, \sqrt{35}/2)$ and $\Delta = 144\sqrt{35}W(1/2, \sqrt{35}/2)$. Because $W(x,y) > 0$ we have $\tau > 0$. Furthermore,

$$\tau^2 - 4\Delta = (144 - \sqrt{35}W(1/2, \sqrt{35/2}))^2 \geq 0.$$

Because $\tau^2 - 4\Delta \geq 0$ and $\tau > 0$, we find that fixed point A is an unstable node.

Hence, for any W satisfying the conditions that $W(x,y) > 0$ for all (x,y), then the fixed points of the modified system have the same behavior as listed in Poincaré's analysis of P4.

It is easy to verify that there is a stable limit cycle at $r = 1$. One must notice that $dr/dt = r(r^2-1)(r^2-9) = 0$ for $r = 1$, $dr/dt > 0$ for $0 < r < 1$, $dr/dt < 0$ for $1 < r < 3$ and that $d\theta/dt = (r^2 - 2r\cos\theta - 8)W(r\cos\theta, r\sin\theta) < 0$ for $0 < r < 2$. Hence, $r = 1$ is a stable limit cycle.

To establish the existence of the heteroclinic orbits, notice that for $r = 3$, $dr/dt = 0$. On this circle of radius three, to the left of A and B, $d\theta/dt > 0$, and to the right of A and B, $d\theta/dt < 0$. The sign of $d\theta/dt$ remains the same in these regions as in the printed equation because $W(x,y) > 0$ for all (x,y) in $\mathbf{R}^2$. Hence there are trajectories which maintain $r = 3$, connecting A to B in both directions along the circle.

Finally we apply Theorem 2 to establish the existence of a limit cycle at infinity. We can express $W(x,y)$ in polar coordinates in the following way:

$$\begin{aligned}&W(r\cos\theta, r\sin\theta)\\&\quad = W_0(\cos\theta,\sin\theta) + rW_1(\cos\theta,\sin\theta) + \cdots + r^N W_N(\cos\theta,\sin\theta).\end{aligned}$$

With $W_N(\cos\theta,\sin\theta) > 0$ for all θ because $W_N(x,y)$ is positive definite. Hence, using the notation of Theorem 2, $I = 5$ and $J = 2+N$, where the degree of $W(x,y)$, $N \geq 2$. Furthermore, $\xi_J(\theta) = W_N(\cos\theta,\sin\theta) \neq 0$ for all θ. Hence, there is a cycle at infinity. This cycle is a limit cycle because, for $r \geq 4$, $dr/dt > 0$. This concludes the proof of Theorem 3.

Based on the success of the above method, one may wish to see how well it works for another system; we use P5.

Theorem 4 (Classification Theorem for P5). *Let $W(x,y)$ be a polynomial in x and y of degree N, and let $W_N(x,y)$ be the homogeneous portion of W which is of degree N. If*

(i) $W(x,y) > 0$ *for all* $(x,y) \in \mathbf{R}^2$,

(ii) $N \geq 4$*, and*

(iii) $W_N(x,y)$ *is positive definite*,

then the phase portrait of the system is given by

$$\frac{dx}{dt} = x(2x^2+2y^2+1)((x^2+y^2)^2+x^2-y^2-c) - y(2x^2+2y^2-1)W(x,y),$$
$$\frac{dy}{dt} = y(2x^2+2y^2-1)((x^2+y^2)^2+x^2-y^2-c) + x(2x^2+2y^2+1)W(x,y),$$

gives a phase portrait qualitatively equivalent to the analysis Poincaré gives in Example P5.

We follow the same technique as in the previous theorem; first we verify that the modification results in the same fixed point behavior. Clearly, $W(x,y)$ is never zero, so it does not lead to any new fixed points. Hence, we must only check that the existing fixed points have the same behavior. To make the analysis easier, we make the following definitions:

$$A(x,y) = x(2x^2+2y^2+1),$$
$$B(x,y) = y(2x^2+2y^2-1),$$
$$C(x,y) = (x^2+y^2)^2 + x^2 - y^2 - c.$$

With these definitions, the system becomes:

$$\frac{dx}{dt} = A(x,y)C(x,y) - B(x,y)W(x,y),$$
$$\frac{dy}{dt} = B(x,y)C(x,y) + A(x,y)W(x,y).$$

At the fixed points $A(x,y) = B(x,y) = 0$ and $x = 0$ so in the expressions for the trace and the determinant we have

$$\tau = 8y^2C(x,y),$$
$$\Delta = (2y^2+1)(6y^2-1)((C(x,y))^2 + (W(x,y))^2).$$

At (0,0) we have $\Delta = -(c^2 + W(0,0)^2)$ which is negative for all (x,y) because $W(0,0) > 0$. Hence we have a saddle at the origin, independent of the parameter value c and the modification W.

At $(0, \pm 1/\sqrt{2})$ we have $\tau = 4C(0,\pm 1/\sqrt{2}) = -4(1/4+c)$ and $\Delta = 4C(0,\pm 1/\sqrt{2})^2 + 4W(0,\pm 1/\sqrt{2})^2$. Hence, $\tau^2 - 4\Delta = -16W(0,\pm 1\sqrt{2})^2 <$

0 resulting in spirals for any choice of W. The stability depends upon τ; here $\tau > 0$ for $c < -1/4$ and $\tau < 0$ for $c > -1/4$. This matches Poincaré's assertion of unstable spirals for $c < -1/4$ and stable spirals for $c > -1/4$. Hence, as long as $W(x, y) > 0$ for all (x, y), the behavior of the fixed points is unchanged by W.

Next we demonstrate that $C(x, y) = 0$ is a conserved quantity. This results in the limit cycle(s) of cases 2 and 4 and the homoclinic orbits of case 3. Consider the rate of change in $C(x, y)$ with respect to time:

$$\begin{aligned}\frac{\partial C(x,y)}{\partial t} &= \frac{\partial C(x,y)}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial C(x,y)}{\partial y}\frac{\partial y}{\partial t} \\ &= \frac{\partial C(x,y)}{\partial x}(AC - BW) + \frac{\partial C(x,y)}{\partial y}(BC + AW).\end{aligned}$$

Now one makes the observation that $(\partial C/\partial x) = 2A$ and $(\partial C/\partial y) = 2B$, so that

$$\frac{\partial C}{\partial t} = 2(A^2 + B^2)C.$$

So clearly $C(x, y) = 0$ is a conserved quantity. At this point we see that the homoclinic orbits for case 4 have been verified because, for $c = 0$, the algebraic curve $C(x, y) = 0$ is a leminescate based at the origin. Further, for $C < 0$, we have $(\partial C/\partial t) < 0$ and for $C > 0$ we have $(\partial C/\partial t) > 0$. Based on these results, one may find the necessary "trapping regions" to prove the existence of the limit cycles using the Poincaré-Bendixson theorem.

Finally, Poincaré asserted that, for all of these values of c, there is a limit cycle at infinity. To verify this, we determine whether there are fixed points at infinity by using Theorem 1. Using that $N \geq 4$, we obtain:

$$\begin{aligned}XQ_m - YP_m = 2(X^2 + Y^2)^2 W_N(X, Y) = 0,\\ X^2 + Y^2 = 1,\end{aligned}$$

which is clearly inconsistent because $W_N(X, Y)$ is positive definite by hypothesis.

Now that it has been verified that there are no fixed points at infinity, to prove that there is a limit cycle at infinity it must be shown that trajectories near infinity do, in fact, approach the cycle at infinity. In this case we must apply slightly different techniques than Corollary 1 because dr/dt is not single signed for all values of θ. However, we only must show that the system's behavior near infinity does approach the cycle at infinity. To do this, we note that $dC/dt = 2(A^2 + B^2)C > 0$ for all points in the plane which are outside of the limit cycle(s) or homoclinic orbits defined by the algebraic equation $C(x, y) = 0$. Based on this, one can rule out other limiting sets in the proximity of infinity. Because there are no other limiting sets within a certain distance from the equator, the trajectories near the equator must approach the cycle at the equator–the limit cycle at infinity.

This concludes the proof of Theorem 4.

These theorems demonstrate the incredible algebraic nonuniqueness of the equations which have the qualitative features outlined by Poincaré for P4 and P5. This nonuniquenss might be of concern in speculating which modifications are the right ones. However, were one to speculate, without insight into Poincaré's other examples, as to which of these possible modifications to P4 might be his equation, one might first try the most simple of all, W, $W(x, y) = (x^2 + y^2 + 1)$. This gives example R4 exactly. It is interesting that through choice by simplicity (application of Occam's Razor) one obtains the same result which was found through analysis of the context of the problem. This interesting philosophical observation continues to persuade the author that R4 really was the system intended by Poincaré for P4.

Based on all of the above success, the author would like to suggest an adequate system to replace P5:

Example R5. The system becomes

$$\begin{aligned}
\frac{dx}{dt} &= x(2x^2 + 2y^2 + 1)((x^2 + y^2)^2 + x^2 - y^2 - c) \\
&\quad - y(2x^2 + 2y^2 - 1)(x^4 + y^4 + 1), \\
\frac{dy}{dt} &= y(2x^2 + 2y^2 - 1)((x^2 + y^2)^2 + x^2 - y^2 - c) \\
&\quad + x(2x^2 + 2y^2 + 1)(x^4 + y^4 + 1).
\end{aligned}$$

The fact that R5 matches Poincaré's analysis of P5 is a clear application of Theorem 4.

6. Discussion. The reader may wonder why these problems were investigated. This section briefly answers this question and attempts to bring closure to the problem.

Verification of Poincaré's assertions about his fourth example were given as part of the take-home project for a differential equations class. We were given the English translation as a guide [**3**]. One aspect of the assignment was to attempt, through the use of a computer, to obtain an actual global phase portrait. When this was done, the trajectories were found not to cycle around near the equator, but to come in virtually orthogonal to it! (See Figure 3.2.) It is by this means that the error was first suspected. Much speculation took place before all of the analysis of P4 was done.

After a complete analysis of P4 came about, the error in example P5 was discovered by Perko. With example P5 in need of a modification and with faith in the classification technique used for P4 (despite the fact that it only considers one type of modification), the author set out to do a similar analysis for P5. It is by means of this type of analysis that R5 was quickly obtained.

The errors found from Chapter Nine were found significantly later. Because the errors made with these equations are consistent with the previous errors, little analysis has been done other than identifying the errors.

The author finds it convincing that Poincaré could have made an error of omission on his fourth example. The relation between the third and fourth examples strongly supports this belief and the fact that of many plausible modifications of Poincaré's fourth example, R4 is the simplest makes it convincing. However, the fifth example does not fit within this genre of system and it is not clear that the error was one of omission. The further errors in Chapter Nine make it convincing that the remaining errors were merely standard mathematical errors.

The author would like to conclude with a quotation of Poincaré:

> How is an error possible in mathematics? A sane mind should not be guilty of a logical fallacy, yet there are some very fine minds incapable of following mathematical demonstrations. Need we add that mathematicians themselves are not infallible? [**1**].

Acknowledgments. I would like to thank Professor Jay P. Fillmore of University of California at San Diego (UCSD) for providing insight, guidance, and most importantly, inspiration to me for this project. If it weren't for his efforts, this paper would not exist. I would like to thank Professor Donald Smith of UCSD for providing many helpful suggestions about this paper. I would also like to thank the teaching assistant Greg Leibon for helping me accept that there actually could be an error in Poincaré's example. I would like to thank Professor Lawrence Perko of Northern Arizona University for his suggestions and especially for finding the error in P5. Finally, I would like to thank both Professor John Guckenheimer and Professor Richard Rand from Cornell University for the discussions that I had with each of them regarding the analysis of differential equations at infinity.

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