

SEQUENTIAL DEFINITIONS OF CONTINUITY FOR REAL FUNCTIONS

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ABSTRACT. A function $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous at a point u if, given a sequence $\mathbf{x} = (x_n)$, $\lim \mathbf{x} = u$ implies that $\lim f(\mathbf{x}) = f(u)$. This definition can be modified by replacing \lim with an arbitrary linear functional G . Generalizing several definitions that have appeared in the literature, we say that $f : \mathbf{R} \rightarrow \mathbf{R}$ is G -continuous at u if $G(\mathbf{x}) = u$ implies that $G(f(\mathbf{x})) = f(u)$. When $G(\mathbf{x}) = \lim_n n^{-1} \sum_{k=1}^n x_k$, Buck showed that if a function f is G -continuous at a single point then f is linear, that is, $f(u) = au + b$ for fixed a and b . Other authors have replaced convergence in arithmetic mean with A -summability, almost convergence and statistical convergence. The results in this paper include a sufficient condition for G -continuity to imply linearity and a necessary condition for continuous functions to be G -continuous, thereby generalizing several known results in the literature. It is also shown that, in many situations, the G -continuous functions must be either precisely the linear functions or precisely the continuous functions. However, examples are found where this dichotomy fails, which, in particular, leads to a counterexample to a conjecture of Spigel and Krupnik.

1. Introduction. The typical ‘advanced calculus’ student is often relieved to find that the standard $\varepsilon - \delta$ definition of continuity for real-valued functions of a real variable can be replaced by a sequential definition of continuity. That many of the properties of continuous functions can be easily derived using sequential arguments has also been, no doubt, a source of relief to the occasional advanced calculus instructor.

In this paper we investigate the impact of changing the definition of the convergence of sequences on the structure of the set of continuous functions. This continues a line of research initiated with a 1946 American Mathematical Monthly problem. Robbins [24] asked readers

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to show that a function $f : \mathbf{R} \rightarrow \mathbf{R}$ which exhibits the property

$$(1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_k = u \quad \implies \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(x_k) = f(u),$$

even at a single point $u = u_0$, has to be linear, that is, of the form $f(u) = au + b$ for all u , where a, b are fixed real numbers. In other words, when in the usual definition of continuity sequential convergence is replaced by convergence in arithmetic mean then only the linear functions remain continuous in the new sense. In the same problem, Robbins coined the term ‘Cesàro continuous’ for functions satisfying (1). Buck’s solution was published in 1948 [7]; the problem was also solved by five others.

Since then, there have been a number of similar investigations that replace the usual definition of sequential convergence with one of a variety of other definitions that are typically related to matrix summability, almost convergence or statistical convergence (these will be discussed below). In all of these investigations, the resulting continuous functions were either precisely the linear functions or precisely the functions which are continuous in the ordinary sense. This paper shows that, as long as the new definition of convergence satisfies certain restrictions, this is always the case. But we will also find examples where this dichotomy does not hold. In particular, we will give a counterexample to a conjecture of Spigel and Krupnik [28].

Before we can begin, it will be necessary to introduce some definitions and notation. We will use boldface letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$ for sequences $\mathbf{x} = (x_n)_{n=1}^{\infty}, \dots$ of real numbers. If f is a real-valued function of a real variable then we define $f(\mathbf{x}) = (f(x_n))_n$. By a *method of sequential convergence*, or briefly a *method*, we mean a linear functional G defined on a linear subspace c_G of the vector space of all real-valued sequences. A sequence $\mathbf{x} = (x_n)$ is said to be *G-convergent* to l if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = l$. In particular, \lim denotes the limit functional $\lim \mathbf{x} = \lim_n x_n$ on the space c of convergent sequences. A method G is called *regular* if every convergent sequence $\mathbf{x} = (x_n)$ is G -convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. Throughout this paper we shall denote by I a nondegenerate interval in \mathbf{R} . We are now ready to define the G -continuity of a function.

Definition. Let G be a method of sequential convergence and $f : I \rightarrow \mathbf{R}$ a function. Then f is G -continuous at $u \in I$ provided that whenever an I -valued sequence $\mathbf{x} = (x_n)$ is G -convergent to u then the sequence $f(\mathbf{x}) = (f(x_n))$ is G -convergent to $f(u)$. For a subset D of I , f is called G -continuous on D if it is G -continuous at every $u \in D$, and f is G -continuous if it is G -continuous on its domain I .

The G -continuity of f (on its domain) can also be expressed briefly as follows: If \mathbf{x} is an I -valued G -convergent sequence then

$$G(f(\mathbf{x})) = f(G(\mathbf{x})).$$

It is important to distinguish between the G -continuity of f on a subinterval J and the G -continuity of $f|_J$. Clearly, if f is G -continuous on J then $f|_J$ is G -continuous, but the converse is not necessarily true because in the latter case the sequences \mathbf{x} are restricted to J . We demonstrate this by an example.

Example 1. A regular method G and a function $f : \mathbf{R} \rightarrow \mathbf{R}$ so that $f|_{[a,b]}$ is G -continuous for every interval $[a,b]$ but f itself is not G -continuous. It suffices to consider $c_G = c + \text{span}\{(2^n)\}$ with $G(\mathbf{x} + \lambda(2^n)) = \lim \mathbf{x}$ for $\lambda \in \mathbf{R}$. Then the function $f : u \mapsto u^2$ clearly has the desired properties.

Another phenomenon that is unknown for ordinary continuity is that G -continuity need not be a local property. Antoni and Šalát [2] have given an example of a regular method G and a function $f : \mathbf{R} \rightarrow \mathbf{R}$ that is G -continuous only at 0 although f coincides on $[-1, 1]$ with a linear and hence G -continuous function.

We now discuss some special classes of methods of sequential convergence that have been studied in the literature. Probably the most important class is the class of matrix methods. Consider an infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ of real numbers. Then, for any sequence $\mathbf{x} = (x_n)$ the sequence $A\mathbf{x}$ is defined as

$$A\mathbf{x} = \left(\sum_{k=1}^{\infty} a_{nk} x_k \right)_n,$$

provided that each of the series exists. A sequence \mathbf{x} is *A-convergent* (or *A-summable*) to l if $A\mathbf{x}$ exists and is convergent with

$$\lim A\mathbf{x} = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} x_k = l.$$

Then l is called the *A-limit* of \mathbf{x} . We have thus defined a method of sequential convergence, called a *matrix method* (or an *ordinary matrix method* to distinguish it from the strong matrix methods considered below), and A is called a *summability matrix*. For matrix methods the notion of regularity introduced above coincides with the classical notion of regularity for matrices. See [5], [16], [22], [20] and [31] for an introduction to regular summability matrices.

A number of authors (Posner [23], Iwiński [17], Srinivasan [29], Antoni and Šalát [2], [1], Spigel and Krupnik [28]) have studied *G*-continuity defined by a regular summability matrix A . In this case *G*-continuity is usually called *A-continuity*. Note that Buck's original result is for the Cesàro matrix $C_1 = (a_{nk})$ with $a_{nk} = 1/n$ if $k \leq n$ and 0 otherwise.

The Hahn-Banach theorem can be used to define methods which are not generated by a regular summability matrix. Banach used this theorem to show that the limit functional can be extended from the convergent sequences to the bounded sequences while preserving linearity, positivity and translation invariance [3]; these extensions have come to be known as Banach limits. If a bounded sequence is assigned the same value l by each Banach limit, the sequence is said to be *almost convergent to l*. Lorentz [18] proved that a sequence $\mathbf{x} = (x_n)$ is almost convergent to l if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_{k+j} = l \quad \text{uniformly in } j.$$

Observe that a sequence is almost convergent if and only if it is 'uniformly Cesàro convergent'; replacing the Cesàro matrix by a regular summability matrix A (or even a family of general regular methods) leads to another collection of methods of sequential convergence. For an introduction to the theory of almost convergence (with respect to the Cesàro matrix), see [18], [22] and [4].

Some authors (Öztürk [21], Savaş and Das [25], [26], Borsík and Šalát [6]) have studied G -continuity for the method of almost convergence or for related methods. In particular, Borsík and Šalát have obtained the analogue of Buck's result for almost convergence.

Next we consider a class of methods that is unrelated to the preceding two classes. Fast [13] introduced the definition of *statistical convergence* in 1951. Recall that for subsets A of \mathbf{N} the *asymptotic density* of A , denoted $\delta(A)$, is given by

$$\delta(A) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in A\}|,$$

if this limit exists, where $|B|$ denotes the cardinality of the set B . A sequence $\mathbf{x} = (x_n)$ is *statistically convergent to l* if

$$\delta(\{n : |x_n - l| > \varepsilon\}) = 0 \quad \text{for every } \varepsilon > 0.$$

In this case l is called the *statistical limit of \mathbf{x}* . The statistically convergent sequences form a linear subspace of the space of all real-valued sequences and the statistical limit is a linear functional on this space. Note that convergent sequences are statistically convergent to the same limit and that if a sequence is statistically convergent to l , then the sequence has a subsequence which converges to l in the ordinary sense, cf. [9].

The notion of statistical convergence can be generalized to μ -*statistical convergence* by replacing the asymptotic density δ with an arbitrary *density* μ , that is, a finitely additive set function taking values in $[0, 1]$ defined on a field of subsets of \mathbf{N} with $\mu(\mathbf{N}) = 1$ such that if $|A| < \infty$ then $\mu(A) = 0$ and if $A \subset B$ with $\mu(B) = 0$ then $\mu(A) = 0$, cf. [10] and [11]. This notion covers several other variants of statistical convergence that have been considered in the literature.

For an introduction to statistical convergence, see [14], [9], [15], and [11].

The G -continuity for methods of statistical convergence has been considered in two papers (Schoenberg [27], Demirci [12]). The behavior here is markedly different. While G -continuity in the case of regular matrix summability or almost convergence typically leads to Buck-type results, Schoenberg showed that *every* continuous function is 'statistically continuous,' see also Proposition 4 below.

The final class of methods that we want to discuss is given by strong matrix summability. Let $A = (a_{nk})_{n,k=1}^{\infty}$ be a nonnegative matrix with $\lim_n a_{nk} = 0$ for all $k \in \mathbf{N}$, $\sup_n \sum_{k=1}^{\infty} a_{nk} < \infty$ and $\sum_{k=1}^{\infty} a_{nk} \not\rightarrow 0$ as $n \rightarrow \infty$. Then a sequence \mathbf{x} is *strongly A -convergent* (or *strongly A -summable*) to l if $\sum_{k=1}^{\infty} a_{nk}|x_k - l| < \infty$ for all n and

$$\sum_{k=1}^{\infty} a_{nk}|x_k - l| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In this case l is called the *strong A -limit* of \mathbf{x} . Under the stated assumptions this limit is unique and strong A -convergence defines a regular method, called a *strong matrix method*. The assumptions on A are satisfied in particular if A is a nonnegative regular matrix. For an introduction to strong matrix summability, see [20] and [31].

It seems that G -continuity for strong matrix methods G will be studied here for the first time, see Theorems 4 and 6, among others.

In the following we let

$$\mathcal{L}(I), \mathcal{G}(I) \quad \text{and} \quad \mathcal{C}(I)$$

denote the collections of linear, G -continuous, and continuous functions $f : I \rightarrow \mathbf{R}$, and we set $\mathcal{L} = \mathcal{L}(\mathbf{R})$, $\mathcal{G} = \mathcal{G}(\mathbf{R})$ and $\mathcal{C} = \mathcal{C}(\mathbf{R})$; note that a function f will be called linear if it is of the form $f(u) = au + b$ with constants a and b . We turn our attention to investigating the relationships between these three sets, which is the emphasis of this paper.

2. A sufficient condition for $\mathcal{G} = \mathcal{L}$. We begin with the following simple observation.

Proposition 1. *If G is a regular method, then every linear function $f : I \rightarrow \mathbf{R}$ is G -continuous, that is, $\mathcal{L}(I) \subset \mathcal{G}(I)$.*

This shows that \mathcal{L} is the smallest possible space of G -continuous functions, if G is regular. As noted above, Buck [7] showed that every Cesàro-continuous function is a linear function, that is, $\mathcal{G} = \mathcal{L}$ for the method G of Cesàro-summability. In [2], Antoni and Šalát isolate a

property that is shared by the Cesàro matrix and that implies $\mathcal{G} = \mathcal{L}$ for an arbitrary regular matrix method G . When trying to extend this result to general regular methods one faces the problem that the proof of Antoni and Šalát requires that G -continuity implies continuity, which is no longer necessarily true for non-matrix methods, see Example 3. The following lemma will help us to overcome this problem.

Lemma 1. *Let $\alpha \neq 0, 1$. If $f : I \rightarrow \mathbf{R}$ is a function such that*

$$(2) \quad f(\alpha u + (1 - \alpha)v) = \alpha f(u) + (1 - \alpha)f(v)$$

whenever u, v and $\alpha u + (1 - \alpha)v$ belong to I , then there is a dense subset of I on which f is linear.

Proof. First we note that we can assume that $\alpha \in (0, 1)$; for if $\alpha > 1$ then we replace α by $(1/\alpha)$ and $\alpha u + (1 - \alpha)v$ by u , and if $\alpha < 0$ then we replace α by $(\alpha/(\alpha - 1))$ and $\alpha u + (1 - \alpha)v$ by v .

We now fix distinct points u_0 and v_0 in I . Then there are $a, b \in \mathbf{R}$ with $f(u) = au + b$ for $u = u_0$ and $u = v_0$. Let $D = \{u \in I : f(u) = au + b\}$. We claim that the closure \overline{D} of D is an interval. Otherwise there is an interval $J = (u_1, v_1)$ with $u_1, v_1 \in \overline{D}$ that meets no points of \overline{D} . But by (2), $u_1, v_1 \in \overline{D}$ implies that $\alpha u_1 + (1 - \alpha)v_1 \in \overline{D} \cap J$, which is a contradiction. Hence \overline{D} is an interval, and a similar reasoning shows that $\overline{D} = I$. \square

Using this lemma we obtain the generalization of the Antoni-Šalát theorem to general methods. With Antoni and Šalát [2] we consider the following property for methods G :

(L_1) There exists a G -convergent 0-1-sequence \mathbf{z} such that $G(\mathbf{z}) = \alpha$ with $\alpha \neq 0, 1$.

Theorem 1. *Let G be a regular method with property (L_1) . Then every G -continuous function $f : I \rightarrow \mathbf{R}$ is linear, that is, $\mathcal{G}(I) = \mathcal{L}(I)$.*

Proof. The first part of the proof is the same as that of Antoni and Šalát. We give it here for the sake of completeness. Let $f : I \rightarrow \mathbf{R}$ be a G -continuous function, and let $u, v \in I$. Consider a G -convergent

0-1-sequence $\mathbf{z} = (z_n)$ and a scalar α as in property (L_1) , and define

$$x_n = z_n u + (1 - z_n)v,$$

which equals u if $z_n = 1$, and v otherwise. By linearity and regularity of G we have that $\mathbf{x} = (x_n) \in c_G$ with

$$G(\mathbf{x}) = \alpha u + (1 - \alpha)v.$$

On the other hand, $f(x_n)$ equals $f(u)$ if $z_n = 1$, and $f(v)$ otherwise. Hence

$$f(x_n) = z_n f(u) + (1 - z_n)f(v),$$

so that we also have

$$G(f(\mathbf{x})) = \alpha f(u) + (1 - \alpha)f(v).$$

Since f is G -continuous we then have

$$f(\alpha u + (1 - \alpha)v) = f(G(\mathbf{x})) = G(f(\mathbf{x})) = \alpha f(u) + (1 - \alpha)f(v)$$

whenever u, v and $\alpha u + (1 - \alpha)v$ belong to I .

At this point Lemma 1 implies that there is a dense subset D of I and $a, b \in \mathbf{R}$ such that

$$f(u) = au + b \quad \text{for all } u \in D.$$

Now let $u \in I$, and let $x_n \in D$ such that $x_n \rightarrow u$. Then also $u = G(\mathbf{x})$, so that $G(f(\mathbf{x})) = G((ax_n + b)_n) = au + b$ by linearity and regularity of G . Since f is G -continuous, we finally have

$$f(u) = G(f(\mathbf{x})) = au + b,$$

which had to be shown. \square

Apart from the Antoni-Šalát theorem this result also covers Theorem 1 of Savaş and Das [26] who consider methods that generalize the method of almost convergence.

The example in [2] shows that the theorem does not remain true if f is only required to be G -continuous at a single point. In Section 6 we will obtain corresponding results under this weaker assumption.

Example 2. The converse of Theorem 1 is not true. There is a regular method G that does not have property (L_1) for which $\mathcal{G}(I) = \mathcal{L}(I)$ for any interval I .

To construct G let $\mathbf{z} = (1, 0, -1, 1, 0, -1, 1, \dots)$ and $c_G = c + \text{span}\{\mathbf{z}\}$ with $G(\mathbf{x}) = \lim_n x_{3n+2}$ for $\mathbf{x} \in c_G$. Then G is a regular method that does not have property (L_1) because, if $G(\mathbf{x}) = l$, then l is a subsequential limit of \mathbf{x} .

Let $f : I \rightarrow \mathbf{R}$ be a G -continuous function, and let $\mathbf{x} = \mathbf{y} + \lambda \mathbf{z}$ be an I -valued sequence with $\lim \mathbf{y} = u_0$ and $\lambda \in \mathbf{R}$, hence also $G(\mathbf{x}) = u_0$. Then by the G -continuity of f we have that

$$f(\mathbf{x}) = (f(y_1 + \lambda), f(y_2), f(y_3 - \lambda), \dots) \in c_G$$

with

$$G(f(\mathbf{x})) = f(u_0).$$

This implies that there is some $\kappa \in \mathbf{R}$ with

$$\begin{aligned} f(y_{3n+2}) &\longrightarrow f(u_0), \\ f(y_{3n+1} + \lambda) &\longrightarrow f(u_0) + \kappa, \\ f(y_{3n+3} - \lambda) &\longrightarrow f(u_0) - \kappa \end{aligned}$$

as $n \rightarrow \infty$. If $\lambda = 0$ and \mathbf{y} is an arbitrary I -valued sequence with $\lim \mathbf{y} = u_0$, then the first limit relation implies that f is continuous at u_0 . Taking $\lambda \in \mathbf{R}$ such that $u_0 + \lambda$ and $u_0 - \lambda$ belong to I and letting $y_n = u_0$ for all n the second and third limit relations imply that

$$\frac{f(u_0 + \lambda) + f(u_0 - \lambda)}{2} = f(u_0).$$

By Lemma 1 the linearity of f follows. \square

3. A topological view of G -continuity, and a necessary condition for $\mathcal{G} = \mathcal{C}$. In this section we study the connection between G being a subsequential method and the comparison of \mathcal{G} with \mathcal{C} .

Definition. A method is called *subsequential* if whenever \mathbf{x} is G -convergent with $G(\mathbf{x}) = l$ then there is a subsequence (x_{n_k}) of \mathbf{x} with $\lim_k x_{n_k} = l$.

For a fixed subsequence (n_k) , define the method $G = I_{(n_k)}$ by letting \mathbf{x} be G -convergent if and only if $G(\mathbf{x}) := \lim_k x_{n_k}$ exists. The method $I_{(n_k)}$ is obviously subsequential. The same is true for the method considered in Example 2. Another example is statistical convergence, see [9, Corollary 2.4].

In order to link subsequentiality with the size of \mathcal{G} we introduce the following ‘topological’ notions.

Definition. Let $U \subset \mathbf{R}$ and $l \in \mathbf{R}$. Then l is in the G -hull of U if there is a sequence $\mathbf{x} = (x_n)$ of points in U such that $G(\mathbf{x}) = l$. A set is G -closed if it contains all of the points in its G -hull.

We let \overline{U}^G denote the G -hull of a set U . If G is a regular method, then $U \subset \overline{U} \subset \overline{U}^G$, and hence U is G -closed if and only if $\overline{U}^G = U$. Note that, depending upon G , one can have either $\overline{U} = \overline{U}^G$ or \overline{U} a proper subset of \overline{U}^G ; observe that if G is Cesàro summability then the G -hull of $\{0, 1\}$ is $[0, 1]$, while if G is statistical convergence then the G -hull of $\{0, 1\}$ is $\{0, 1\}$. Also note that it need not be the case that the G -hull of \overline{U}^G is equal to \overline{U}^G ; for example, if \mathbf{x} is G -convergent if and only if $G(\mathbf{x}) := \lim_n (x_n + x_{n+1})/2$ exists then for $U = \{0, 1\}$ we have $\overline{U}^G = \{0, (1/2), 1\}$, and the G -hull of \overline{U}^G is $\{0, (1/4), (1/2), (3/4), 1\}$.

Proposition 2. Let G be a regular method. Then $\overline{U} = \overline{U}^G$ for every subset U of \mathbf{R} if and only if G is a subsequential method.

Proof. Suppose that G is a subsequential method and that $l \in \overline{U}^G$. Then there is a sequence $\mathbf{x} = (x_n)$ in U such that $G(\mathbf{x}) = l$. As G is subsequential, there is a subsequence (x_{n_k}) of \mathbf{x} such that $\lim_k x_{n_k} = l$ and hence $l \in \overline{U}$. As G is regular, it follows that $\overline{U} = \overline{U}^G$.

Now suppose that $\overline{U} = \overline{U}^G$ for every subset U of \mathbf{R} . Let $\mathbf{x} = (x_n)$ be a G -convergent sequence with $G(\mathbf{x}) = l$. Observe that, since G is regular, $G(\mathbf{x})$ depends only upon the ‘tail’ of \mathbf{x} and hence $l \in \overline{\{x_n : n \geq N\}}^G$ for each $N \in \mathbf{N}$. As $\overline{\{x_n : n \geq N\}}^G = \overline{\{x_n : n \geq N\}}$ we obtain that $l \in \cap_N \overline{\{x_n : n \geq N\}}$. Hence there is a subsequence (x_{n_k}) of \mathbf{x} such that $\lim_k x_{n_k} = l$. \square

Lemma 2. *Suppose that $f : \mathbf{R} \rightarrow \mathbf{R}$ is G -continuous and U is G -closed. Then $f^{-1}(U)$ is G -closed.*

Proof. Let $V = f^{-1}(U)$ and suppose that $l \in \overline{V}^G$. Then there is a V -valued sequence \mathbf{x} such that $G(\mathbf{x}) = l$. Now, since $G(f(\mathbf{x})) = f(l)$, $f(\mathbf{x})$ is U -valued and U is G -closed we obtain that $f(l) \in U$. But now $l \in V$ and hence $\overline{V}^G \subset V$. \square

With this lemma we obtain the following. Using a different method we will obtain a stronger result in the next section, see the Corollary to Theorem 5.

Proposition 3. *If G is a regular subsequential method, then every G -continuous function is continuous, that is, $\mathcal{G} \subset \mathcal{C}$.*

Proof. We show that the inverse image of any closed set is closed. Let $U \subset \mathbf{R}$ be closed. As G is subsequential, U is also G -closed. Since f is G -continuous, it follows from Lemma 2 that $f^{-1}(U)$ is G -closed and hence closed. \square

Theorem 2. *Let G be a regular method. If every continuous function is G -continuous, that is, if $\mathcal{C} \subset \mathcal{G}$, then G is a subsequential method.*

Proof. We suppose that G is not subsequential and produce a function $f : \mathbf{R} \rightarrow \mathbf{R}$ which is continuous but not G -continuous. As G is not subsequential it follows from Proposition 2 that there is a subset U of \mathbf{R} whose closure is properly contained in its G -hull. Let $l \in \overline{U}^G \setminus \overline{U}$. Then there is a continuous function f such that $f(l) = 0$ and $f|_{\overline{U}} = 1$.

We claim that f is not G -continuous. Since $l \in \overline{U}^G$ there is a U -valued sequence $\mathbf{x} = (x_n)$ such that $G(\mathbf{x}) = l$. Now $f(x_n) = 1$ for all n and $f(l) = 0$. As

$$G(f(\mathbf{x})) = 1 \neq 0 = f(l)$$

we have established the claim. \square

In Example 2 we have given a regular subsequential method such that the only G -continuous functions are linear. Thus the converse of the theorem is not true.

We see from the previous two results that for a regular method G we have

$$\mathcal{C} \subset \mathcal{G} \iff \mathcal{G} = \mathcal{C}.$$

This shows that \mathcal{G} can never properly contain \mathcal{C} . If there is a method G for which \mathcal{G} contains functions that are not in \mathcal{C} then there also have to be continuous functions that are not G -continuous.

For special methods it is possible to characterize when the case $\mathcal{G} = \mathcal{C}$ occurs. For ordinary matrix methods such a result follows immediately from a theorem of Iwiński [17, Theorem 3] and its proof (note, however, that the notion of G -continuity used in [17] is weaker than ours). Recall the definition of the method $I_{(n_k)}$ given at the beginning of this section; it is clearly a regular matrix method that satisfies $\mathcal{C} \subset \mathcal{G}$, hence $\mathcal{G} = \mathcal{C}$.

Theorem 3 (Iwiński). *If G is a regular matrix method, then the following assertions are equivalent:*

- (i) *the function $u \mapsto u^2$ is G -continuous,*
- (ii) $\mathcal{C} \subset \mathcal{G}$,
- (iii) $\mathcal{G} = \mathcal{C}$,
- (iv) $G = I_{(n_k)}$ *for some sequence (n_k) .*

In fact, by [17, Theorem 3], the function $u \mapsto u^2$ may be replaced by many other functions.

We add that the method $I_{(2n)}$ obviously gives an affirmative answer to the question posed by Iwiński at the end of his paper [17].

It turns out that Theorem 3 also holds for strong matrix summability. Note that each method $I_{(n_k)}$ coincides with a method of strong A -summability for a suitable matrix A .

Theorem 4. *If G is a strong matrix method, then the following assertions are equivalent:*

- (i) *the function $u \mapsto u^2$ is G -continuous,*

- (ii) $\mathcal{C} \subset \mathcal{G}$,
- (iii) $\mathcal{G} = \mathcal{C}$,
- (iv) $G = I_{(n_k)}$ for some sequence (n_k) .

Proof. By the discussion preceding Theorem 3 we need only show that (i) implies (iv). Let G be a method of strong A -summability, where A is a nonnegative matrix that satisfies the assumptions stated in the introduction and assume that (i) holds. Let $B = (b_{nk})$ be the matrix obtained from A by deleting all zero-columns. By the assumptions on A , B is also an infinite matrix, and it will suffice to show that strong B -summability coincides with ordinary convergence.

Following the proof of [17, Theorem 3] we let $\alpha_k = \sup_n b_{nk}$ for $k \in \mathbf{N}$ and $\alpha = \inf_k \alpha_k$. Suppose that $\alpha = 0$. Then there is an increasing sequence (k_m) with $\sum_{m=1}^{\infty} \alpha_{k_m}^{1/3} < \infty$. It follows easily that the sequence \mathbf{x} given by

$$x_k = \begin{cases} \alpha_{k_m}^{-2/3} & \text{if } k = k_m \\ 0 & \text{otherwise} \end{cases}$$

is strongly B -convergent to 0. On the other hand, choosing n_m such that $b_{n_m k_m} \geq (\alpha_{k_m}/2)$ for all m we consider the sequence \mathbf{y} with

$$y_k = \begin{cases} \alpha_{k_m}^{-4/3} & \text{if } k = k_m \\ 0 & \text{otherwise.} \end{cases}$$

Then it also follows easily that $\sum_{k=1}^{\infty} b_{n_m k} |y_k| \rightarrow \infty$ as $m \rightarrow \infty$ so that \mathbf{y} is not strongly B -convergent to 0. Since $y_k = x_k^2$ for all k , this shows that the function $u \mapsto u^2$ is not G -continuous at 0, which is a contradiction to (i).

Hence we have that $\alpha > 0$. This implies that for every $k \in \mathbf{N}$ there is some n_k with $b_{n_k k} \geq \frac{\alpha}{2}$ for all k . Since $\sum_{k=1}^{\infty} b_{nk} < \infty$ for all n we must have that $n_k \rightarrow \infty$ as $k \rightarrow \infty$. Hence if \mathbf{x} is strongly B -convergent to l then we have

$$\frac{\alpha}{2} |x_k - l| \leq b_{n_k k} |x_k - l| \leq \sum_{j=1}^{\infty} b_{n_k j} |x_j - l| \rightarrow 0$$

as $k \rightarrow \infty$, so that \mathbf{x} is convergent to l . This shows that strong B -summability coincides with ordinary convergence. \square

4. A sufficient condition for $\mathcal{G} \subset \mathcal{C}$. We have seen that the space \mathcal{L} of linear functions is the smallest possible space of G -continuous functions if G is regular, and that the case $\mathcal{G} = \mathcal{L}$ can occur. We now ask how large \mathcal{G} can be. It was first noted by Posner [23], see also [17], [29], that for regular matrix methods G we always have $\mathcal{G} \subset \mathcal{C}$, that is, that every G -continuous function is continuous in the ordinary sense; his proof even shows that if a function is G -continuous at a point then it is continuous at this point. In this section Posner's result is extended to a wide class of methods; but we also show by a counterexample that it does not hold for all methods. We consider the following property for methods G :

(S) There is no sequence \mathbf{x} with $x_n \rightarrow \infty$ for which each subsequence (x_{n_k}) is G -convergent.

Theorem 5. *Let G be a regular method with property (S). Then every function $f : I \rightarrow \mathbf{R}$ that is G -continuous at $u_0 \in I$ is also continuous there.*

Proof. Suppose that f is G -continuous at u_0 but not continuous there. Then there is a sequence \mathbf{x} in I with $x_n \rightarrow u_0$ but $f(x_n) \not\rightarrow f(u_0)$. We choose a subsequence \mathbf{y} of \mathbf{x} with $|f(y_n) - f(u_0)| \geq \delta$ for some $\delta > 0$. We may assume that $f(y_n) \geq f(u_0) + \delta$ for all n . Now, if $f(\mathbf{y})$ is a bounded sequence then there is a subsequence \mathbf{w} of \mathbf{y} with $f(w_n) \rightarrow a \geq f(u_0) + \delta$. This implies that $G(\mathbf{w}) = u_0$ and $G(f(\mathbf{w})) = a \neq f(u_0)$, which contradicts the G -continuity of f at u_0 . Hence $f(\mathbf{y})$ is unbounded, so that there is a subsequence \mathbf{w}' of \mathbf{y} such that $f(w'_n) \rightarrow \infty$. Since every subsequence of \mathbf{w}' converges to u_0 we see that by G -continuity the sequence $f(\mathbf{w}')$ and all its subsequences are G -convergent, in contradiction to our assumption that G has property (S). \square

Corollary. *Let G be a regular method. If*

- (a) (Posner) G is an ordinary matrix method, or
- (b) G is a strong matrix method, or
- (c) G is totally regular, that is, no sequence \mathbf{x} with $x_n \rightarrow \infty$ is G -convergent, or

(d) G is subsequential,

then every function $f : I \rightarrow \mathbf{R}$ that is G -continuous at $u_0 \in I$ is also continuous there.

Proof. Assertion (a) follows immediately from a well-known theorem of Buck [8] by which every regular matrix method has property (S). Case (c) is trivial. Case (b) follows from (c) since by the assumptions on the nonnegative matrices that define strong matrix methods it follows that every such method is totally regular. Finally, (d) is a special case of (c). \square

Apart from regular ordinary and strong matrix methods also the methods of almost convergence in the sense of Lorentz and μ -statistical convergence have property (S), so that all the special methods considered in the introduction have this property, provided they are regular.

In fact, in the case of almost convergence a function that is G -continuous at a point u_0 has to be a linear function, see [6] or Section 6.

For statistical convergence, Schoenberg [27, Lemma 5] has shown that if a function is continuous at a point then it is also ‘statistically continuous’ there, see also [27, Lemma 1]. More generally we have the following result for μ -statistical convergence as defined in the introduction.

Proposition 4. *Let G be a method of μ -statistical convergence. Then a function $f : I \rightarrow \mathbf{R}$ is G -continuous at $u_0 \in I$ if and only if it is continuous there.*

Proof. Clearly μ -statistical convergence is a totally regular method G for any density μ . In view of the corollary we need only show that if a function $f : I \rightarrow \mathbf{R}$ is continuous at u_0 then it is G -continuous at this point. Thus let $x_n \rightarrow u_0$ μ -statistically, and let $\varepsilon > 0$. By continuity of f at u_0 there is an $\eta > 0$ such that $|v - u_0| \leq \eta$ implies that $|f(v) - f(u_0)| \leq \varepsilon$. Hence, by the definition of μ -statistical convergence, we have that

$$\mu(\{n : |f(x_n) - f(u_0)| > \varepsilon\}) \leq \mu(\{n : |x_n - u_0| > \eta\}) = 0,$$

so that $f(x_n) \rightarrow f(u_0)$ μ -statistically, and f is G -continuous at u_0 . \square

We will next see that Theorem 5 is not true for all regular methods.

Example 3. A regular method G and a function $f : \mathbf{R} \rightarrow \mathbf{R}$ which is G -continuous, but not continuous, at 0.

Proof. First we define c_G and G . Let Y be the set of all sequences $\mathbf{y} = (y_n)$ with $y_n \in \{0\} \cup \{j^j : j \in \mathbf{N}\}$ for each n such that the nonzero y_n tend to infinity. Let W be the linear span of Y . Then we define $c_G = c + W$ and $G : c_G \rightarrow \mathbf{R}$ by $G(\mathbf{x}) = \lim \mathbf{z}$ if $\mathbf{x} = \mathbf{z} + \mathbf{w}$ with $\mathbf{z} \in c$ and $\mathbf{w} \in W$.

We need to establish that G is well-defined. The key step is to show that for each sequence $\mathbf{w} \in W$ with $\mathbf{w} \neq 0$ the nonzero elements tend to infinity. To see this let $\mathbf{w} = \sum_{\nu=1}^N a_\nu \mathbf{y}^\nu \neq 0$ with $a_\nu \in \mathbf{R}$ and $\mathbf{y}^\nu \in Y$. Let $M = \max_\nu |a_\nu|$ and

$$m = \min \left\{ \left| \sum_{\nu \in H} a_\nu \right| : H \subset \{1, \dots, N\}, \sum_{\nu \in H} a_\nu \neq 0 \right\};$$

note that $m > 0$. By the definition of Y there is a sequence (J_n) of positive integers with $J_n \rightarrow \infty$ such that, for $\nu = 1, \dots, N$ and $n \in \mathbf{N}$,

$$y_n^\nu = 0 \quad \text{or} \quad y_n^\nu \geq (J_n)^{J_n}.$$

Now, for $n \in \mathbf{N}$ we can write

$$w_n = \sum_{j=1}^{\infty} \left(\sum_{y_n^\nu = j^j} a_\nu \right) j^j,$$

where only finitely many terms are nonzero. If $w_n \neq 0$ we have that

$$\sum_{y_n^\nu = j^j} a_\nu \neq 0$$

for some j , and we let $K = K(n)$ be the largest such j . Then $K \geq J_n$ and hence

$$\begin{aligned} |w_n| &= \left| \left(\sum_{y_n^\nu = K^K} a_\nu \right) K^K - \sum_{y_n^\nu < K^K} a_\nu y_n^\nu \right| \\ &\geq mK^K - MN(K-1)^{K-1} \\ &\geq (mK - MN)(K-1)^{K-1} \\ &\geq (mJ_n - MN)(J_n - 1)^{J_n-1} \longrightarrow \infty \end{aligned}$$

as $n \rightarrow \infty$, which had to be shown.

As a consequence we see that the sum $c_G = c + W$ is a direct sum so that G is well-defined. It is clear that G is linear and regular.

Now define $f : \mathbf{R} \rightarrow \mathbf{R}$ by

$$f(u) = \begin{cases} j^j & \text{if } u = 1/j \\ 0 & \text{otherwise.} \end{cases}$$

Observe that f is clearly not continuous at 0. To see that it is G -continuous at 0 let $\mathbf{x} = \mathbf{z} + \mathbf{w} \in c_G$ with $G(\mathbf{x}) = \lim \mathbf{z} = 0$. Then $f(x_n) \in \{0\} \cup \{j^j : j \in \mathbf{N}\}$ for all n . Also, $f(x_n) \neq 0$ only if $z_n + w_n \in \{(1/j) : j \in \mathbf{N}\}$, which for sufficiently large n implies that $w_n = 0$ and $z_n \in \{(1/j) : j \in \mathbf{N}\}$. Consequently, $f(\mathbf{x})$ either has only finitely many nonzero terms or its nonzero terms tend to infinity. In both cases we have $f(\mathbf{x}) \in c_G$ and $G(f(\mathbf{x})) = 0$, which implies that f is G -continuous at 0. \square

Note, however, that the function f in this example is not G -continuous at any point $u = (1/j)$. We do not know if a regular method exists for which there is a G -continuous function on \mathbf{R} that is not continuous.

5. The dichotomy. From the various results and examples in the literature, and supported also by the results in the paper so far, it seems that for a regular method G there are only two possibilities: *either* only the linear functions are G -continuous on \mathbf{R} *or* every continuous function is G -continuous on \mathbf{R} , that is, *either* $\mathcal{G} = \mathcal{L}$ *or* $\mathcal{G} = \mathcal{C}$.

No regular method was known until now that allowed a nonlinear G -continuous function without admitting all continuous functions to be G -continuous. In fact, the dichotomy was conjectured by Spigel and Krupnik [28, p. 147] to hold when G is restricted to methods that are generated by matrices. We will show here that the dichotomy fails, even for regular matrix methods, thus providing a counterexample to the Spigel-Krupnik conjecture.

We first show that essentially every method that is defined by strong matrix summability leads to a counterexample to the dichotomy. Recall our earlier result, Theorem 4, that for the methods $G = I_{(n_k)}$, which are special strong matrix methods, we have $\mathcal{G} = \mathcal{C}$.

Theorem 6. *Let G be a strong matrix method. If $G \neq I_{(n_k)}$ for all sequences (n_k) , then*

$$\mathcal{L} \subsetneq \mathcal{G} \subsetneq \mathcal{C}.$$

Proof. By Proposition 1 and the corollary to Theorem 5 the inclusions $\mathcal{L} \subset \mathcal{G} \subset \mathcal{C}$ always hold. Since for every nonnegative matrix $A = (a_{nk})$ we have

$$\sum_{k=1}^{\infty} a_{nk} ||x_k| - |u|| \leq \sum_{k=1}^{\infty} a_{nk} |x_k - u|$$

for all n , we see that the nonlinear function $u \mapsto |u|$ is G -continuous at every $u \in \mathbf{R}$, so that $\mathcal{L} \neq \mathcal{G}$. Finally, $\mathcal{G} \neq \mathcal{C}$ follows from Theorem 4. \square

Zeller [30] has shown that strong Cesàro summability is a matrix method, that is, a (regular) matrix A exists so that the method of strong Cesàro summability is identical with the method of A -summability; by [19] the same is indeed true for strong matrix summability with respect to an arbitrary row-finite regular matrix. Since strong Cesàro summability clearly differs from each method $I_{(n_k)}$, the last result leads to a counterexample to the conjecture of Spigel and Krupnik.

Corollary. *There is a regular matrix method G for which*

$$\mathcal{L} \subsetneq \mathcal{G} \subsetneq \mathcal{C}.$$

In spite of these negative results we will show in the remainder of this section that certain kinds of dichotomy do hold. Our main tool will be the Stone-Weierstrass theorem. Throughout this section we make the following assumption:

(GC_I) *Every G -continuous function $f : I \rightarrow \mathbf{R}$ is continuous, that is, $\mathcal{G}(I) \subset \mathcal{C}(I)$.*

As noted at the end of the previous section, we do not know of a regular method that does not satisfy this condition.

Theorem 7. *Let G be a regular method, and let $I = \mathbf{R}$ or $I = [a, b]$. Assuming (GC_I) , then*

$$\text{either } \mathcal{G}(I) = \mathcal{L}(I) \quad \text{or} \quad \mathcal{G}(I) \text{ is dense in } \mathcal{C}(I).$$

Proof. We first consider the case $I = \mathbf{R}$ and assume that $\mathcal{G} \neq \mathcal{L}$. We have to show that $\mathcal{H} := \overline{\mathcal{G}}$, the closure of \mathcal{G} in \mathcal{C} , coincides with \mathcal{C} . Recall that $\mathcal{C} = \mathcal{C}(\mathbf{R})$ carries the topology of uniform convergence on compact subsets of \mathbf{R} . We first show that the function $u \mapsto u^2$ belongs to \mathcal{H} .

Since $\mathcal{G} \neq \mathcal{L}$ there is a function $f \in \mathcal{G} \setminus \mathcal{L}$. By a shift of variable we may assume that $f|_{[0,1]} \notin \mathcal{L}[0,1]$. We now consider the functions f_n defined by

$$f_n(u) = \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}u\right), \quad n \in \mathbf{N}.$$

It is then clear that $f_n \in \mathcal{G}$. By $(GC_{\mathbf{R}})$, f is continuous on \mathbf{R} , so that we can define

$$g(u) = \begin{cases} \frac{1}{u} \int_0^u f(t) dt & \text{if } u \neq 0 \\ f(0), & \text{if } u = 0. \end{cases}$$

Then we have for $u \neq 0$,

$$\begin{aligned} |g(u) - f_n(u)| &= \left| \frac{1}{u} \sum_{k=1}^n \int_{(k-1)(u/n)}^{k(u/n)} \left(f(t) - f\left(\frac{k}{n}u\right) \right) dt \right| \\ &\leq \sup \left\{ |f(t) - f(\tau)| : |t - \tau| \leq \frac{|u|}{n}, \quad t, \tau \in [0, u] \text{ or } [u, 0] \right\}, \end{aligned}$$

the estimate being trivially true for $u = 0$. This shows that

$$f_n \longrightarrow g \quad \text{in } \mathcal{C},$$

so that $g \in \mathcal{H}$.

Since the functions $u \mapsto g(\alpha u)$ also belong to \mathcal{H} for any α , we can repeat the process to find that the function

$$h(u) = \begin{cases} \frac{1}{u} \int_0^u g(t) dt & \text{if } u \neq 0 \\ g(0) & \text{if } u = 0 \end{cases}$$

belongs to \mathcal{H} as well. It is clear that h is twice differentiable on $\mathbf{R} \setminus \{0\}$. In addition we have that $h''(u) \not\equiv 0$ on $(0, 1)$. Otherwise we would have

$$h(u) = au + b \quad \text{on } [0, 1],$$

hence

$$\int_0^u g(t) dt = au^2 + bu \quad \text{on } [0, 1].$$

This implies that

$$g(u) = 2au + b \quad \text{on } [0, 1],$$

hence

$$\int_0^u f(t) dt = 2au^2 + bu \quad \text{on } [0, 1],$$

which finally gives that

$$f(u) = 4au + b \quad \text{on } [0, 1],$$

contradicting the fact that $f|_{[0,1]} \notin \mathcal{L}[0, 1]$.

Writing $f = h$ we have thus found an element $f \in \mathcal{H}$ that is twice differentiable on $\mathbf{R} \setminus \{0\}$ with $f''(u_0) \neq 0$ for some $u_0 \in (0, 1)$. By a shift of variable we may assume that f is twice differentiable in a neighborhood of 0 with $f''(0) \neq 0$. Hence we can write

$$f(u) = f(0) + f'(0)u + \frac{1}{2}f''(0)u^2 + r(u) \quad \text{for } u \in \mathbf{R}$$

with

$$\frac{|r(u)|}{u^2} \longrightarrow 0 \quad \text{as } u \longrightarrow 0.$$

Substituting (u/n) for u and multiplying the equation by n^2 we obtain that

$$\frac{1}{2}f''(0)u^2 - \left(n^2 f\left(\frac{u}{n}\right) - n^2 f(0) - nf'(0)u \right) = -n^2 r\left(\frac{u}{n}\right)$$

for all $u \in \mathbf{R}$ and $n \in \mathbf{N}$. Now, since each function f_n given by

$$f_n(u) = n^2 f\left(\frac{u}{n}\right) - n^2 f(0) - nf'(0)u$$

lies in \mathcal{H} , and since for every $M > 0$

$$\sup_{|u| \leq M} \left| -n^2 r\left(\frac{u}{n}\right) \right| = \sup_{0 < |u| \leq M} \left| \frac{r(u/n)}{(u/n)^2} \right| |u|^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

we see that

$$f_n(u) \longrightarrow \frac{1}{2} f''(0) u^2 \quad \text{in } \mathcal{C},$$

hence that the function $u \mapsto u^2$ belongs to \mathcal{H} since $f''(0) \neq 0$.

Using this result we can show that $\mathcal{H} = \mathcal{C}$. It suffices to show that $\mathcal{H}|_{[a,b]} := \{f|_{[a,b]} : f \in \mathcal{H}\}$ is dense in $\mathcal{C}[a,b]$ for all $[a,b]$. Since $\mathcal{H} \supset \mathcal{L}$, $\mathcal{H}|_{[a,b]}$ separates the points of $[a,b]$ and contains the constant functions. In view of the Stone-Weierstrass theorem it remains to prove that $\mathcal{H}|_{[a,b]}$ is a subalgebra of $\mathcal{C}[a,b]$, for which we need only show that $fg \in \mathcal{H}$ if $f, g \in \mathcal{H}$. But since $fg = ((f+g)^2 - f^2 - g^2)/2$ this follows easily from the fact that the function $u \mapsto u^2$ belongs to \mathcal{H} .

The proof in the case $I = [a,b]$ is similar. Here we have to note that if the function $u \mapsto u^2$ belongs to $\mathcal{H}[a,b]$, the closure of $\mathcal{G}[a,b]$ in $\mathcal{C}[a,b]$, then it also belongs to $\mathcal{H}[c,d]$ for any interval $[c,d]$ so that again we can deduce that $f^2 \in \mathcal{H}[a,b]$ whenever $f \in \mathcal{H}[a,b]$. \square

In order to derive a ‘real’ dichotomy from Theorem 7 we have to find conditions under which $\mathcal{G}(I)$ is closed in $\mathcal{C}(I)$. We say that G -continuity is *boundedly determined*, if a function f is G -continuous on \mathbf{R} whenever $f|_{[a,b]} \in \mathcal{G}[a,b]$ for all compact intervals $[a,b]$. In addition, l_∞ denotes the space of bounded sequences endowed, as usual, with the supremum-norm $\|\cdot\|_\infty$.

Lemma 3. *Let G be a regular method. Assuming (GC_I) , suppose that $c_G \cap l_\infty$ is closed in l_∞ and that $G : (c_G \cap l_\infty, \|\cdot\|_\infty) \rightarrow \mathbf{R}$ is continuous. If*

(a) $I = [a,b]$ or

(b) $I = \mathbf{R}$ and G -continuity is boundedly determined, then $\mathcal{G}(I)$ is closed in $\mathcal{C}(I)$.

Proof. (a) Let $f_n \in \mathcal{G}[a,b]$ with $f_n(u) \rightarrow f(u)$ uniformly on $[a,b]$. Let \mathbf{x} be a G -convergent sequence in $[a,b]$ with $G(\mathbf{x}) = u \in [a,b]$.

Since each f_n is G -continuous and continuous on $[a, b]$ we have that $f_n(\mathbf{x}) \in c_G \cap l_\infty$ for all n and

$$\sup_k |f_n(x_k) - f(x_k)| \leq \|f_n - f\|_\infty \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By assumption this implies that $f(\mathbf{x}) \in c_G \cap l_\infty$ and $f_n(\mathbf{x}) \rightarrow f(\mathbf{x})$ in $(c_G \cap l_\infty, \|\cdot\|_\infty)$. Moreover, we have by the continuity of G and the G -continuity of each function f_n that

$$f_n(u) = f_n(G(\mathbf{x})) = G(f_n(\mathbf{x})) \rightarrow G(f(\mathbf{x})).$$

Since also $f_n(u) \rightarrow f(u)$, this shows that $G(f(\mathbf{x})) = f(u)$, so that f belongs to $\mathcal{G}[a, b]$.

(b) This follows directly from (a) since $f|_{[a, b]} \in \mathcal{G}[a, b]$ if $f \in \mathcal{G}(\mathbf{R})$. \square

As an immediate consequence of the last two results we obtain the following dichotomy.

Theorem 8. *Let G be a regular method. Assuming (GC_I) , suppose that $c_G \cap l_\infty$ is closed in l_∞ and that $G : (c_G \cap l_\infty, \|\cdot\|_\infty) \rightarrow \mathbf{R}$ is continuous. If*

(a) $I = [a, b]$ or

(b) $I = \mathbf{R}$ and G -continuity is boundedly determined,

then either $\mathcal{G}(I) = \mathcal{L}(I)$ or $\mathcal{G}(I) = \mathcal{C}(I)$.

If G is an ordinary or strong matrix method that is regular then $c_G \cap l_\infty$ is closed in l_∞ and $G : (c_G \cap l_\infty, \|\cdot\|_\infty) \rightarrow \mathbf{R}$ is continuous. For ordinary matrix methods, the first statement follows from the fact that by the Silverman-Toeplitz theorem, cf. [20], a regular matrix A defines a continuous linear mapping $A : l_\infty \rightarrow l_\infty$ with $c_G \cap l_\infty = A^{-1}(c)$, while the second then follows since $G(\mathbf{x}) = \lim A\mathbf{x}$ for $\mathbf{x} \in c_G \cap l_\infty$, and $\lim : c \rightarrow \mathbf{R}$ is continuous. For strong matrix methods the claim can be proved similarly, see also [10, Proposition 4 and Theorem 8].

We thus obtain the following dichotomy, using also the corollary to Theorem 5.

Corollary. *Let G be a regular ordinary or strong matrix method and let*

(a) $I = [a, b]$ or

(b) $I = \mathbf{R}$ and G -continuity is boundedly determined.

Then either $\mathcal{G}(I) = \mathcal{L}(I)$ or $\mathcal{G}(I) = \mathcal{C}(I)$.

It follows from Theorem 6 and its corollary that for ordinary or strong matrix methods G that are regular, G -continuity is in general not boundedly determined. For matrix summability, Example 1 gives a concrete such example if we note that the method defined there is a matrix method, cf. [31, Section 26].

6. Global consequences of G -continuity at a single point.

We return to the first result proved for G -continuity. Buck [7] had in fact shown that a function is already linear if it is only Cesàro continuous at a single point. Antoni [1, Theorems 1 and 2] and Spigel and Krupnik [28, Theorems 1 and 2] have obtained general results in this direction for matrix methods. We first generalize and strengthen Antoni's Theorem 2; note that Antoni's result also contains Theorem 1 of Spigel and Krupnik. The condition imposed on G is in the spirit of property (L_1) considered in Section 2. Calling two 0-1-sequences *disjoint* if their supports are disjoint we consider the following property for methods G :

(L_2) Disjoint G -convergent 0-1-sequences \mathbf{z} and \mathbf{z}' exist such that $G(\mathbf{z}) = \alpha$ and $G(\mathbf{z}') = \beta$ with $\alpha, \beta \neq 0$ and $\alpha + \beta \neq 1$.

Theorem 9. *Let G be a regular method with property (L_2) . Then every function $f : \mathbf{R} \rightarrow \mathbf{R}$ that is G -continuous at one point is linear.*

Proof. We can assume without loss of generality that f is G -continuous at 0 and $f(0) = 0$. Let \mathbf{z} and \mathbf{z}' be the sequences guaranteed by (L_2) , and define $z''_n = 1 - z_n - z'_n$. Then $\mathbf{z}'' = (z''_n)$ is a G -convergent 0-1-sequence with $\gamma := G(\mathbf{z}'') = 1 - (\alpha + \beta) \neq 0$, and \mathbf{z} , \mathbf{z}' and \mathbf{z}'' are pairwise disjoint. Let u and v be any real numbers and set $x_n = -z_n(\gamma/2\alpha)u - z'_n(\gamma/2\beta)v + z''_n((u+v)/2)$. Then \mathbf{x} is G -convergent with $G(\mathbf{x}) = -(\gamma/2)u - (\gamma/2)v + \gamma((u+v)/2) = 0$.

By G -continuity of f at 0 we have that $G(f(\mathbf{x})) = f(0) = 0$. Since $f(x_n) = z_n f(-(\gamma/2\alpha)u) + z'_n f(-(\gamma/2\beta)v) + z''_n f((u+v)/2)$, we see that

$$(3) \quad 0 = G(f(\mathbf{x})) = \alpha f\left(-\frac{\gamma}{2\alpha}u\right) + \beta f\left(-\frac{\gamma}{2\beta}v\right) + \gamma f\left(\frac{u+v}{2}\right).$$

If $v = 0$ or $u = 0$ in (3), then

$$\alpha f\left(-\frac{\gamma}{2\alpha}u\right) = -\gamma f\left(\frac{u}{2}\right)$$

and

$$\beta f\left(-\frac{\gamma}{2\beta}v\right) = -\gamma f\left(\frac{v}{2}\right).$$

Substituting this into (3) and dividing by γ we obtain that

$$f\left(\frac{u}{2}\right) + f\left(\frac{v}{2}\right) = f\left(\frac{u+v}{2}\right)$$

holds for all u and v . Setting $u = v$ we see that $f(u/2) = (f(u)/2)$, so that we get

$$\frac{1}{2}(f(u) + f(v)) = f\left(\frac{u+v}{2}\right)$$

for all u and v .

Lemma 1 shows that there is a dense subset D of \mathbf{R} and $a, b \in \mathbf{R}$ such that

$$f(u) = au + b \quad \text{for } u \in D.$$

Now let $u \in \mathbf{R}$ be arbitrary and choose $x_n \in D$ with $x_n \rightarrow 2u$. Then

$$\begin{aligned} f(u) &= \frac{1}{2}(f(x_n) + f(2u - x_n)) \\ &= \frac{1}{2}(ax_n + b + f(2u - x_n)). \end{aligned}$$

Since $(2u - x_n)_n$ is G -convergent to 0 the same holds for $(f(2u - x_n))_n$. Thus,

$$\begin{aligned} f(u) &= G\left(\left(\frac{1}{2}(ax_n + b + f(2u - x_n))\right)_n\right) = \frac{1}{2}(a2u + b + 0) \\ &= au + \frac{b}{2}. \end{aligned}$$

This shows that f is linear (note that in fact $b = 0$ because $f(0) = 0$). \square

Even in the case of matrix methods this result strengthens Theorem 2 of [1] since our result does not exclude that $\alpha = 1$, $\beta = 1$ or $\alpha + \beta = 0$. Theorem 9 also implies the main result of Borsík and Šalát [6] that every function that is G -continuous at one point for the method G of almost convergence must be linear. More generally, we obtain the following extension of a result of Spigel and Krupnik [28, Theorem 1]. Recall that a method G is called *strongly regular* if every almost convergent sequence is G -convergent to the same limit.

Corollary. *Let G be a strongly regular method. Then any function that is G -continuous at one point is linear.*

Proof. We can apply Theorem 9 to the sequences $\mathbf{z} = (1, 0, 0, 1, 0, 0, 1, \dots)$ and $\mathbf{z}' = (0, 1, 0, 0, 1, 0, 0, 1, 0, \dots)$ which are both almost convergent, hence G -convergent, to the value $1/3$. \square

In view of Theorem 1 one may ask if Theorem 9 remains true if the method G only has property (L_1) . This is not the case as was shown by Antoni and Šalát [2, Example]. They produce an example of a regular matrix method G with property (L_1) and a non-linear continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$ that is G -continuous at (exactly) one point. Now, by a result of Antoni [1, Lemma 1], that is the worst that can happen: For any such method G , if a function $f : \mathbf{R} \rightarrow \mathbf{R}$ is G -continuous at one point then it must be continuous on \mathbf{R} . Antoni's proof, however, contains a gap (the case when $y = \pm\infty$ is not treated). We will show that his result in fact holds for all regular methods. To this end we need the following lemma.

Lemma 4. *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a function such that at every point u_0 of discontinuity we have $\lim_{x \rightarrow u_0} |f(x)| = \infty$. Then for every such point u_0 and every $\lambda \neq 0$ there are sequences \mathbf{x} and \mathbf{x}' with $x_n \rightarrow u_0$, $x'_n \rightarrow u_0$ and $f(x_n) - f(x'_n) = \lambda$ for all n .*

Proof. Let u_0 be a point of discontinuity of f and set $U = \{u < u_0 : f \text{ is continuous at } u\}$. Every $u \in U$ has an open neighborhood in which f is bounded and hence continuous by our assumption. Thus U is an open set. Also, every neighborhood of u_0 intersects U because for every $\varepsilon > 0$ there is an $M > 0$ such that the set $\{u \in [u_0 - \varepsilon, u_0] : |f(u)| \leq M\}$ is infinite and hence has an accumulation point; this point has to belong to U by the assumption. We can therefore find maximal open intervals $I_n = (a_n, b_n) \subset U$ with $b_n \rightarrow u_0$ as $n \rightarrow \infty$ (possibly $b_n = u_0$ for all n). Then f is continuous on each interval (a_n, b_n) , and by maximality we have $\lim_{x \rightarrow b_n} |f(x)| = \infty$ for all n . Using the intermediate value theorem it is now easy to construct the desired sequences \mathbf{x} and \mathbf{x}' . \square

With this we can prove the extension of Antoni's lemma.

Theorem 10. *Let G be a regular method with property (L_1) . Then every function $f : \mathbf{R} \rightarrow \mathbf{R}$ that is G -continuous at one point is continuous on \mathbf{R} .*

Proof. Suppose that f is G -continuous at v_0 . Let $u_0 \in \mathbf{R}$. We first show that for all sequences $\mathbf{x} = (x_n)$ with $x_n \rightarrow u_0$ the sequences $f(\mathbf{x})$ are G -convergent to the same value. By (L_1) there is a G -convergent 0-1-sequence \mathbf{z} with $G(\mathbf{z}) = \alpha \neq 0, 1$. Then we consider the sequences \mathbf{y} and \mathbf{y}' with

$$y_n = z_n s_0 + (1 - z_n) x_n$$

and

$$y'_n = z_n x_n + (1 - z_n) t_0,$$

where $s_0 = ((v_0 - (1 - \alpha)u_0)/\alpha)$ and $t_0 = ((v_0 - \alpha u_0)/(1 - \alpha))$. Since $(z_n x_n - z_n u_0)_n$ is a null sequence the linearity and regularity of G implies that \mathbf{y} and \mathbf{y}' are G -convergent with

$$G(\mathbf{y}) = \alpha s_0 + (1 - \alpha) u_0 = v_0,$$

and similarly $G(\mathbf{y}') = v_0$. Since f is G -continuous at v_0 we see that the sequences \mathbf{w} and \mathbf{w}' given by

$$w_n = z_n f(s_0) + (1 - z_n) f(x_n)$$

and

$$w'_n = z_n f(x_n) + (1 - z_n) f(t_0)$$

are G -convergent to $f(v_0)$, cf. the proof of Theorem 1. Hence the sequence $\mathbf{w} + \mathbf{w}'$ with

$$w_n + w'_n = f(x_n) + z_n f(s_0) + (1 - z_n) f(t_0) =: f(x_n) + y''_n$$

is G -convergent to $2f(v_0)$. Since the sequence $\mathbf{y}'' = (y''_n)$ is G -convergent to $\alpha f(s_0) + (1 - \alpha) f(t_0)$ we deduce that $f(\mathbf{x})$ is G -convergent with

$$(4) \quad G(f(\mathbf{x})) = 2f(v_0) - \alpha f(s_0) - (1 - \alpha) f(t_0).$$

This value is the same for all sequences \mathbf{x} with $x_n \rightarrow u_0$.

We can now show that the assumptions of the lemma are satisfied, that is, that at every point u_0 of discontinuity of f we have $\lim_{x \rightarrow u_0} |f(x)| = \infty$. For otherwise there are sequences \mathbf{x} and \mathbf{x}' with $x_n \rightarrow u_0$, $x'_n \rightarrow u_0$ and $\lim_n f(x_n) \neq \lim_n f(x'_n)$ as $n \rightarrow \infty$; in fact, one may take $x'_n = u_0$ for all n . By the regularity of G we obtain that $G(f(\mathbf{x})) \neq G(f(\mathbf{x}'))$ contradicting (4). Hence Lemma 4 implies that if u_0 is a point of discontinuity of f then there are sequences \mathbf{x} and \mathbf{x}' with $x_n \rightarrow u_0$, $x'_n \rightarrow u_0$ and $f(x_n) - f(x'_n) = \lambda \neq 0$ for all n , which by regularity and linearity of G again leads to a contradiction with (4). Thus f cannot have any discontinuities. \square

Antoni has used his lemma to obtain another sufficient condition under which G -continuity at one point implies linearity [1, Theorem 1]. With Theorem 10 this can be extended to all regular methods. On the other hand, Spigel and Krupnik [28, Theorem 2] have also obtained this result (under a slightly stronger condition) for regular matrix methods, and a careful analysis of their proof shows that it also works under the weaker condition for general regular methods. The condition is the following:

(L_2) G -convergent 0-1-sequences \mathbf{z} and \mathbf{z}' exist such that $G(\mathbf{z}) = \alpha$ and $G(\mathbf{z}') = \beta$ with $\alpha \in (0, 1)$, $\beta \neq 0, 1$ and

$$\left(\frac{\alpha}{1 - \alpha} \right)^p \neq \left(\frac{\beta}{1 - \beta} \right)^q \quad \text{for all non-zero integers } p, q.$$

Theorem 11 (Spigel-Krupnik). *Let G be a regular method with property (L'_2) . Then every function $f : \mathbf{R} \rightarrow \mathbf{R}$ that is G -continuous at one point is linear.*

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