STRONG ASYMPTOTICS FOR
RELATIVISTIC HERMITE POLYNOMIALS

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ABSTRACT. Strong asymptotic results for relativistic Hermite polynomials $H_n^N(z)$ are established as $n, N \to \infty$, for the cases where $N = an + \alpha + 1/2$, $a \geq 0, \alpha > -1$, or $N/n \to \infty$, thereby supplementing recent results on weak asymptotics for these polynomials. Depending on growth properties of the ratio N/n for the rescaled polynomials $H_n^N(c_n z)$ (c_n being suitable positive numbers, $n, N \to \infty$), formulae of Plancherel-Rotach type are derived on the oscillatory interval, in the complex plane away from the oscillatory region, and near the endpoints of the oscillatory interval.

1. Introduction and summary. In this paper we continue the study of asymptotic properties of relativistic Hermite polynomials H_n^N. This set of polynomials has been introduced for the investigation of the harmonic oscillator in the frame of relativistic quantum theory [1]. Here n denotes the principal quantum number, being a nonnegative integer, and N is a positive parameter describing the underlying relativistic effect such that the system approaches the classical (nonrelativistic) model as $N \to \infty$. This transition is made precise by the limit relation

$$(1.1) \qquad \lim_{N \to \infty} H_n^N(z) = H_n(z), \quad z \in \mathbf{C}, \quad n \in \mathbf{N}_0,$$

where H_n denotes the well-known Hermite polynomials [17, Chapter V]. Similar to H_n its relativistic counterpart can be characterized by several properties, for instance as polynomial solution of the second order linear differential equation [1, 11, 15, 19, 20]

$$(1.2) \qquad \left(1 + \frac{z^2}{N}\right)y'' - \frac{2}{N}(N + n - 1)zy' + \frac{n}{N}(2N + n - 1)y = 0,$$

by the Rodrigues formula [**20**]

$$(1.3) \qquad H_n^N(z) = (-1)^n \left(1 + \frac{z^2}{N} \right)^{N+n} \left(\frac{d}{dz} \right)^n \frac{1}{(1 + z^2/N)^N},$$

by the three term recurrence relation [**1, 15, 19**]

$$(1.4) \qquad \begin{aligned} H_{n+1}^N(z) = &\, 2 \left(1 + \frac{n}{N} \right) z H_n^N(z) \\ &- \frac{n}{N} (2N + n - 1) \left(1 + \frac{z^2}{N} \right) H_{n-1}^N(z), \quad n \geq 1, \end{aligned}$$

with $H_0^N(z) \equiv 1$, $H_1^N(z) = 2z$, and through the orthogonality formulae [**1, 11, 19**]

$$(1.5) \quad \int_{-\infty}^{\infty} x^k H_n^N(x) \frac{dx}{(1 + x^2/N)^{N+n}}$$

$$= \begin{cases} 0 & k = 0, \dots, n-1, \\ n! \sqrt{\pi N} \, \dfrac{\Gamma(N - 1/2)}{\Gamma(N)} & k = n, \end{cases} \qquad N > \frac{1}{2},$$

$$(1.6) \quad \int_{-\infty}^{\infty} H_n^N(x) H_m^N(x) \frac{dx}{(1 + x^2/N)^{N+1+(m+n)/2}}$$

$$= \delta_{n,m} \frac{\sqrt{\pi} n! \sqrt{N} \Gamma(2N + n) \Gamma(N + (1/2))}{(n + N) N^n \Gamma(2N) \Gamma(N)}, \quad n, m \in \mathbf{N}_0, \quad N > \frac{1}{2}.$$

The two latter identities express the fact that the relativistic Hermite polynomials form a system of orthogonal polynomials with respect to a varying weight, that is, a weight function depending on the degree n and the varying parameter N.

Recent publications which are concerned with the asymptotic theory of the relativistic Hermite polynomials deal with weak asymptotics, that is, the computation of the limit distribution of the zeros of H_n^N. More precisely, denoting by $x_{n\nu}^N, \nu = 1, \dots, n$, the zeros of H_n^N, which are all real and simple due to the orthogonality (1.5), and by

$$(1.7) \qquad \xi_n^N(x) := \frac{1}{n} \sum_{\nu=1}^n \delta_{x_{n\nu}^N}(x), \quad x \in \mathbf{R},$$

the corresponding normalized counting measure ($\delta_{x_{n\nu}^N}$ is the Dirac measure at the point $x_{n\nu}^N$), then among other theorems in [**7, 11, 20**] the following results were established. Suppose that N depends on n such that

$$(1.8) \qquad \lim_{n\to\infty} \frac{N}{n} = a \in [0, \infty],$$

then in the sense of weak convergence, $n \to \infty$,

$$(1.9) \qquad \xi_n^N\big(\sqrt{N}x\big) \to \xi(x), \quad \text{if} \quad 0 < a < \infty,$$

where the measure ξ is absolutely continuous with

$$(1.10) \qquad \frac{d}{dx}\,\xi(x) = \frac{\sqrt{1 + 2a - a^2 x^2}}{\pi(1 + x^2)} \quad \text{for } |x| < \sqrt{1 + 2a}\,/\,a$$

and

$$(1.11) \qquad \xi_n^N\big(\sqrt{2n}x\big) \to \xi(x), \quad \text{if } a = \infty,$$

where now the measure ξ is Wigner's semicircle law with density

$$(1.12) \qquad \frac{d}{dx}\,\xi(x) = \frac{2}{\pi}\sqrt{1 - x^2}, \quad |x| < 1.$$

The papers [**11, 20**] make use of logarithmic potential theory with external fields, whereas the approach in [**7**, Theorem 3.2] is based on the method of canonical moments essentially. It is worthwhile to mention that both results (1.9) and (1.11) can be obtained from [**13**] using the varying recurrence relation (1.4) and also from [**8**] on the basis of the differential equation (1.2). A slight modification of the method in [**8**] confirms the validity of (1.9) too in the case $a = 0$. Then (1.10) is Cauchy's density on the real line. If $N = 1$ (see problem 57 in [**16**, Chapter V] for a very closely related case), then the zeros $x_{n\nu}^1$ of H_n^1 can be computed explicitly using (1.3) and then (1.9) follows immediately. On the other hand the limit relation (1.1) suggests that the relativistic polynomials H_n^N should behave like its classical companion H_n if the parameter N runs "far ahead" of the degree n. This speculation is made precise by the relation (1.11) above saying that the limit distribution

of the zeros for H_n^N in this case is the same as for the standard Hermite polynomials H_n [**9**].

In view of the weak asymptotics it is natural to ask for strong asymptotics; that is, for asymptotic forms of the rescaled polynomials $H_n^N(c_n z), (c_n)$ being a suitable sequence of positive numbers.

A direct approach to answer this question is to start from (1.3) which gives the integral representation

$$(1.13) \quad H_n^N(\sqrt{N} z) = \frac{(-1)^n}{2\pi i N^{n/2}} (1 + z^2)^{N+n} \int_C \frac{dt}{(1 + t^2)^N (t - z)^{n+1}},$$

$z \in \mathbf{C} \setminus \{i, -i\}$, where C is a simple closed contour with positive orientation encircling $t = z$ but not $t = i, -i$. A saddle point approximation is very cumbersome, in particular concerning a control of the uniform dependence on the variable z. Therefore in the sequel we make use of the connection formula

$$(1.14) \quad H_n^N(\sqrt{N} z) = c_{nN}(1 + z^2)^{n/2} P_n^{(N-(1/2), N-(1/2))} \left(\frac{z}{\sqrt{1 + z^2}} \right)$$

derived by Nagel [**14**] using the differential equation (1.2) and several transformation formulae for the hypergeometric function. Here $P_n^{(N-(1/2), N-(1/2))}$ denotes the Jacobi polynomial the definition of which we take from Szegö's book [**17**, Chapter IV] together with the normalization and some standard formulae as well. The numbers c_{nN} are given by

$$(1.15) \qquad c_{nN} = \frac{n!}{N^{n/2}} \frac{\Gamma(N + (1/2))\Gamma(n + 2N)}{\Gamma(2N)\Gamma(n + N + (1/2))}$$

[**14, 15**], and the power $(1 + z^2)^{n/2}$ is defined through $(1 + z^2)^{n/2} = \exp\big((n/2)\log(1 + z^2)\big)$ where the logarithm is the principal branch in

$$(1.16) \qquad \mathbf{C}_i := \mathbf{C} \setminus \Big(i[1, \infty) \cup i(-\infty, -1] \Big),$$

that is $\log(1 + z^2)$ reduces to 0 for $z = 0$ and is continuous throughout $\mathbf{C}_i$. Further

$$(1.17) \qquad \zeta := \frac{z}{\sqrt{1 + z^2}}$$

is that branch mapping the domain $\mathbf{C}_i$ of the z-plane conformally onto the cut ζ-plane

$$(1.18) \qquad \mathbf{C}_1 := \mathbf{C} \setminus \Big((-\infty, -1] \cup [1, \infty) \Big)$$

such that the origins of both regions correspond to each other. Thus the question of finding strong asymptotics for H_n^N is transferred to the same problem for the symmetric Jacobi polynomials $P_n^{(N-(1/2),N-(1/2))}$ with varying weights. The connection of general relativistic orthogonal polynomials and Jacobi polynomials with varying weights has been pointed out systematically in a recent note by Ismail [**12**].

It is the main purpose of this paper to derive and to work out the details of strong asymptotics for the relativistic Hermite polynomials. We will consider the cases where $N = an + \alpha + 1/2$, $a \geq 0$, $\alpha > -1$, or $N/n \to \infty$. For these special parameters N satisfying (1.8) the strong asymptotics will be performed on the basis of known Darboux type formulae for Jacobi polynomials with varying weights (Sections 3 and 4) which we take from [**2, 3, 10**] and by the use of a Riemann-Hilbert approach (Section 5). The central results are formulae of Plancherel-Rotach type for H_n^N including Airy asymptotics.

2. Jacobi polynomials with varying weights. In this section we collect some known Darboux type formulae which after some modifications we take from [**2, 10**]. In view of the general assumption (1.8) and the connection formula (1.14) we consider the Jacobi polynomials in ultraspherical form $P_n^{(\alpha_n, \alpha_n)}$ where (α_n) is a sequence of real numbers satisfying

$$(2.1) \qquad \alpha_n = an + \alpha,$$

where $a \geq 0$, $\alpha > -1$, or

$$(2.2) \qquad \lim_{n \to \infty} \frac{\alpha_n}{n} = \infty.$$

The assumption (2.1) is made in the light of the simplicity of some formulae given by [**10**] although Darboux type asymptotic forms for $P_n^{(\alpha_n, \alpha_n)}$ are available from the recent literature [**2**] under the general condition $\lim_{n \to \infty} \alpha_n/n = a$ too.

First we impose (2.1). After some easy computations from Theorem 1 in [**10**] we readily obtain

Lemma 2.1. *Suppose that $\theta \in (0, \pi)$ is fixed and the sequence (α_n) is given by (2.1). Further assume that the function τ is defined by*

(2.3)
$$\tau(\theta) := \theta - \arctan\left(\frac{1 + a + \sqrt{1 + 2a}}{a} \ \tan\frac{\theta}{2}\right)$$
$$- \arctan\left(\frac{a}{1 + a + \sqrt{1 + 2a}} \ \tan\frac{\theta}{2}\right).$$

If

(2.4)
$$\xi = \frac{\sqrt{1 + 2a}}{1 + a} \ \cos\theta,$$

then

(2.5)
$$P_n^{(\alpha_n, \alpha_n)}(\xi) = \left(\pi n \frac{1 + 2a}{1 + a} \ \sin\frac{\theta}{2} \ \cos\frac{\theta}{2}\right)^{-1/2}$$
$$\cdot \left(\frac{(1 + a)^2}{1 + 2a}\right)^{n/2} \left(\frac{1 + 2a}{(1 + a)^2}\right)^{-an - \alpha}$$
$$\cdot \left(\frac{1}{4}\left(a^2 + (1 + 2a)\sin^2\theta\right)\right)^{-(an + \alpha)/2}$$
$$\cdot \left\{\sin\left(n(\theta + a\tau(\theta)) + \frac{\theta}{2} + \frac{\pi}{4} + \alpha\tau(\theta)\right) + \mathcal{O}\left(\frac{1}{n}\right)\right\},$$

as $n \to \infty$. Moreover the remainder holds uniformly on $[\varepsilon, \pi - \varepsilon]$ for any $\varepsilon \in (0, \pi)$.

In (2.3), "arctan" denotes the principal branch, that is, $-\pi/2 < \arctan x < \pi/2$ for real x and, if $a = 0$, then we use the relation $\lim_{a \to 0+} \tau(\theta) = \theta - \pi/2$.

Next, we provide the companion of Lemma 2.1, that is, the asymptotics on the complement of the interval of zeros. Here and throughout, for real r, s with $r < s$, we use the notation

(2.6)
$$\mathbf{C}_{[r,s]} := \mathbf{C} \setminus [r, s],$$

i.e., the complex plane with a cut along the interval $[r, s]$, and the generalized Jukowski function

$$(2.7) \qquad w = \phi_s(\zeta) := \frac{\sqrt{\zeta + s} + \sqrt{\zeta - s}}{\sqrt{\zeta + s} - \sqrt{\zeta - s}} = \frac{1}{s}\left(\zeta + \sqrt{\zeta^2 - s^2}\right)$$

with

$$(2.8) \qquad s := \frac{\sqrt{1 + 2a}}{1 + a}$$

mapping $\mathbf{C}_{[-s,s]}$ conformally onto the exterior of the unit circle $|w| > 1$ such that $\zeta = \infty$ corresponds to $w = \infty$. For later reference here we already mention the inverse of the function ϕ_s which is given by

$$(2.9) \qquad \zeta = \phi_s^{-1}(w) = s\frac{w + w^{-1}}{2}, \quad |w| > 1.$$

Now Theorem 2 in [**10**] immediately implies

Lemma 2.2. *Suppose that the real sequence (α_n) is given by (2.1) and that the function w is defined in (2.7). Then for $\zeta \in \mathbf{C}_{[-s,s]}$, the Jacobi polynomial satisfies*

$$(2.10) \qquad \begin{aligned} P_n^{(\alpha_n, \alpha_n)}(\zeta) = &\left(\pi n \frac{1 + 2a}{1 + a}\right)^{-1/2} \left(\frac{(1 + a)^2}{1 + 2a}\right)^{n/2} \\ &\cdot \left(\frac{4(1 + a)^2 \ w^2}{(1 + 2a)^2(w^2 - \frac{1}{1+2a})}\right)^{an+\alpha} \\ &\cdot \frac{w^{n+1}}{(w^2 - 1)^{1/2}}\left\{1 + \mathcal{O}\left(\frac{1}{n}\right)\right\}, \end{aligned}$$

as $n \to \infty$. Here the branches of the fractional powers are positive when w is real and greater than 1. Moreover the remainder holds uniformly on compact subsets of $\mathbf{C}_{[-s,s]}$.

The reader who is interested in the general case $\lim_{n\to\infty} \alpha_n/n = a \in [0, \infty)$ is referred to [**2**, Section 4]. The next auxiliary results on Jacobi

polynomials with varying weights deal with the assumption (2.2). To this end we put

$$(2.11) \qquad s_n := \sqrt{\frac{(n+1)(2\alpha_n + n - 1)}{n(2\alpha_n + n)}},$$

$$(2.12) \qquad D_n := \frac{n^2 + 2n\alpha_n}{(n+\alpha_n)^2}$$

and observe that $s_n \to 1$, $D_n \to 0$, by (2.2). Now Theorem 3.5 in [**2**] reduces to

Lemma 2.3. *Suppose that the real sequence (α_n) satisfies (2.2) and that s_n, D_n are defined by (2.11), (2.12), respectively. Further, assume that $\theta \in (0, \pi)$ is fixed and the functions ρ_n and a_n are defined through*

$$\rho_n(\theta) := n\theta + \frac{\theta}{2} + \frac{\pi}{4} + \alpha_n\theta$$

$$(2.13) \qquad - \alpha_n \Bigg\{ \arctan\left(\frac{\sqrt{n+2\alpha_n - 1} + \sqrt{n+1}}{\sqrt{n+2\alpha_n - 1} - \sqrt{n+1}} \tan\frac{\theta}{2} \right)$$

$$+ \arctan\left(\frac{\sqrt{n+2\alpha_n - 1} - \sqrt{n+1}}{\sqrt{n+2\alpha_n - 1} + \sqrt{n+1}} \tan\frac{\theta}{2} \right) \Bigg\},$$

$$(2.14) \qquad a_n(\theta) := \frac{2^{\alpha_n + (1/2)}(n+\alpha_n)^{2\alpha_n + n + (1/2)}}{(\pi \sin\theta)^{1/2}(n+1)^{(n+1)/2}(n+2\alpha_n - 1)^{\alpha_n + (n+1)/2}}$$

$$\cdot \left((\alpha_n - 1)^2 + (n+1)(n+2\alpha_n - 1)\sin^2\theta \right)^{-\alpha_n/2}.$$

If $\xi = s_n \cos\theta$, then the rescaled Jacobi polynomial $P_n^{(\alpha_n,\alpha_n)}(\sqrt{D_n}\zeta)$ satisfies

$$(2.15) \qquad P_n^{(\alpha_n,\alpha_n)}\left(\sqrt{D_n}\xi\right) = a_n(\theta)\Big\{ \sin\rho_n(\theta) + o(1) \Big\},$$

as $n \to \infty$. Moreover the remainder holds uniformly with respect to $\theta \in [\varepsilon, \pi - \varepsilon]$ for any $\varepsilon \in (0, \pi)$.

In (2.13) "arctan" is the principal branch as defined above. At this stage we point out that only under stronger growth conditions than (2.2) the expressions in Lemma 2.3 may be simplified.

Finally in this section we derive asymptotics for $P_n^{(\alpha_n,\alpha_n)}\left(\sqrt{D_n}\zeta\right)$ from [**2**] for ζ belonging to the cut plane $\mathbf{C}_{[-s_n,s_n]}$ (see (2.6), (2.11)). Looking at (2.7) and (2.8), we consider the variables $\zeta \in \mathbf{C}_{[-s_n,s_n]}$ and $w_n, |w_n| > 1$, being connected by

$$(2.16) \qquad\qquad \zeta = s_n \frac{w_n + w_n^{-1}}{2}$$

and

$$(2.17) \qquad w_n = \phi_{s_n}(\zeta) = \frac{\sqrt{\zeta + s_n} + \sqrt{\zeta - s_n}}{\sqrt{\zeta + s_n} - \sqrt{\zeta - s_n}} = \frac{1}{s_n}\left(\zeta + \sqrt{\zeta^2 - s_n^2}\right).$$

This means that the domain $\mathbf{C}_{[-s_n,s_n]}$ is mapped conformally onto the exterior of the unit circle in the w_n-plane such that the points $\zeta = \infty$ and $w_n = \infty$ correspond to each other. Now from Theorem 3.3 in [**2**] we readily obtain

Lemma 2.4. *Suppose that the real sequence (α_n) satisfies (2.2) and that D_n is defined by (2.12). If the function w_n is defined in (2.17), then for $\zeta \in \mathbf{C}_{[-1,1]}$ the rescaled Jacobi polynomial satisfies*

$$P_n^{(\alpha_n,\alpha_n)}\left(\sqrt{D_n}\zeta\right) = \frac{4^{\alpha_n}(n+\alpha_n)^{2\alpha_n+n+(1/2)}}{\pi^{1/2}(n+1)^{(n+1)/2}(n+2\alpha_n-1)^{2\alpha_n+(n+1)/2}}$$

$$\cdot \left(\frac{w_n^2}{w_n^2 - \frac{n+1}{n+2\alpha_n-1}}\right)^{\alpha_n} \frac{w_n^{n+1}}{(w_n^2-1)^{1/2}}\left(1 + o(1)\right),$$

as $n \to \infty$. Here the branches of the fractional powers are positive when w_n is real and greater than s_n (see (2.11)). Moreover the o-term holds uniformly on compact subsets of $\mathbf{C}_{[-1,1]}$.

3. Asymptotics on the interval of zeros. In this section we use the preparations given by Lemmata 2.1 and 2.3 to derive oscillating asymptotics for the relativistic Hermite polynomials H_n^N on the interval of zeros. For reasons explained above we restrict our considerations to the cases

$$(3.1) \qquad\qquad N = an + \alpha + \frac{1}{2},$$

where $a \geq 0, \alpha > -1$ (see (1.14)) and

$$
\text{(3.2)} \qquad \lim_{n \to \infty} \frac{N}{n} = \infty.
$$

In view of the basic connection formula (1.14) and besides the conformal mapping ζ introduced in (1.17) we also consider its inverse given by

$$
\text{(3.3)} \qquad z = \frac{\zeta}{\sqrt{1 - \zeta^2}}.
$$

According to the properties stated in the introduction we have to take that branch which maps the cut ζ-plane $\mathbf{C}_1$ (see (1.18)), onto the domain $\mathbf{C}_i$ (see (1.16)) in the z-plane such that the origins of both regions correspond to each other.

First we consider the case (3.1). By (1.14), (1.17), (3.3) and (2.4), we observe that the asymptotic interval of zeros for $P_n^{(N-(1/2),N-(1/2))}(\zeta)$ in the ζ-plane is given by $(-\sqrt{1+2a}/(1+a), \sqrt{1+2a}/(1+a))$ which by (3.3) is mapped onto the interval $\left(-\sqrt{1+2a}/a, \sqrt{1+2a}/a\right)$ in the z-plane. If $a = 0$, then we have the obvious correspondence of the interval $(-1, 1)$ to the whole real line. Now on the basis of (1.14) Lemma 2.1 gives the following asymptotics by straightforward computations.

Theorem 3.1. *Suppose that $\theta \in (0, \pi)$ is fixed, $N > 0$, c_{nN} are given by (3.1), (1.15), respectively and that the function τ is defined in (2.3). If*

$$
\text{(3.4)} \qquad x = \frac{\sqrt{1+2a}\cos\theta}{\sqrt{a^2 + (1+2a)\sin^2\theta}},
$$

then

$$
H_n^N\left(\sqrt{N}x\right) = 2^{N-(1/2)} c_{nN} \frac{(1+a)^{2n+2N-1}}{(1+2a)^{(n/2)+N-(1/2)}}
$$

$$
\text{(3.5)} \qquad \cdot \left(\pi n \frac{1+2a}{1+a} \sin\frac{\theta}{2} \cos\frac{\theta}{2}\right)^{-1/2}
$$

$$
\cdot \left(a^2 + (1+2a)\sin^2\theta\right)^{-(n+N)/2+(1/4)}
$$

$$
\cdot \left\{ \sin\left(n(\theta + a\tau(\theta)) + \frac{\theta}{2} + \frac{\pi}{4} + \alpha\tau(\theta)\right) + \mathcal{O}\left(\frac{1}{n}\right)\right\},
$$

as $n \to \infty$. *Moreover the remainder holds uniformly on $[\varepsilon, \pi - \varepsilon]$ for any $\varepsilon \in (0, \pi)$.*

Second we deal with the case (3.2) which is handled by Lemma 2.3 via (1.14). After some easy computations we obtain

Theorem 3.2. *Suppose that $N > 0$ satisfies (3.2) and that the numbers c_{nN} are given by (1.15). Further assume that $\theta \in (0, \pi)$ is fixed and the functions ρ_n^* and a_n^* are defined through*
(3.6)
$$\rho_n^*(\theta) := n\theta + \frac{\theta}{2} + \frac{\pi}{4} + \left(N - \frac{1}{2}\right)\theta$$
$$- \left(N - \frac{1}{2}\right)\left\{ \arctan\left(\frac{\sqrt{n+2N-2} + \sqrt{n+1}}{\sqrt{n+2N-2} - \sqrt{n+1}} \tan\frac{\theta}{2}\right) \right.$$
$$\left. + \arctan\left(\frac{\sqrt{n+2N-2} - \sqrt{n+1}}{\sqrt{n+2N-2} + \sqrt{n+1}} \tan\frac{\theta}{2}\right)\right\},$$

(3.7)
$$a_n^*(\theta) := \frac{c_n N}{(\pi \sin\theta)^{1/2}}$$
$$\cdot \frac{2^{N/2}\left(n+N-\frac{1}{2}\right)^{2N+n-(1/2)}\left\{\left(N-\frac{3}{2}\right)^2 + (n+1)(n+2N-2)\right\}^{n/2}}{(n+1)^{(n+1)/2}(n+2N-2)^{(N+n)/2}}$$
$$\cdot \left\{\left(N-\frac{3}{2}\right)^2 + (n+1)(n+2N-2)\sin^2\theta\right\}^{-(n+N-(1/2))/2}.$$

If
$$x = \frac{\sqrt{(n+1)(n+2N-2)}\cos\theta}{\sqrt{(N-(3/2))^2 + (n+1)(n+2N-2)\sin^2\theta}},$$

then

(3.8)
$$H_n^N\left(\sqrt{N}x\right) = a_n^*(\theta)\left\{\sin\rho_n^*(\theta) + o(1)\right\},$$

as $n \to \infty$. Moreover the remainder holds uniformly with respect to $\theta \in [\varepsilon, \pi - \varepsilon]$ for any $\varepsilon \in (0, \pi)$.

In (3.6) "arctan" is the principal branch as above. The weak asymptotics in (1.11) supports the asymptotic equivalence of H_n^N and its classical companion H_n under the condition (3.2). This fact also is made precise by the strong asymptotics (3.8), since $\rho_n^*(\theta) = (n/2)(2\theta - \sin 2\theta)(1 + o(1))$, $n \to \infty$, which indicates the close similarity to the corresponding Plancherel-Rotach formula in [**17**, p. 201] for the Hermite polynomials H_n.

4. Asymptotics off the interval of zeros. In the sequel we derive asymptotics for $H_n^N(d_n z)$ where z belongs to the cut planes

$$\mathbf{C}_{[-c_a,c_a]} \qquad \text{if (3.1) holds with } a > 0,$$
$$\mathbf{C} \setminus \mathbf{R} \qquad \text{if (3.1) holds with } a = 0,$$
$$\mathbf{C}_{[-1,1]} \qquad \text{if (3.2) holds}$$

and (d_n) is a properly chosen sequence of positive numbers suggested by the previous results. Here and throughout the paper for positive a we use the abbreviation

$$c_a := \frac{\sqrt{1 + 2a}}{a}.$$

Theorem 4.1. *Suppose that $N > 0$ satisfies* (3.1) *with $a > 0$ and c_{nN} is given by* (1.15). *If the function w is defined in* (2.7) *and* (1.17), *then for $z \in \mathbf{C}_{[-c_a,c_a]}$ we have*

$$
\begin{aligned}
(4.1) \quad H_n^N(\sqrt{N}z) = {}& 4^{N+(n-1)/2} c_{nN} \left(\frac{(1+a)^2}{1+2a} \right)^{n/2} \left(\frac{1+a}{1+2a} \right)^{2N-1} \\
& \cdot \left(\pi n \frac{1+2a}{1+a} \right)^{-1/2} \frac{w^{n+1}}{(w^2-1)^{1/2}} \\
& \cdot \left(\frac{w^2}{(1+2a-w^2)\left(w^2 - \frac{1}{1+2a}\right)} \right)^{n/2} \\
& \cdot \left(\frac{w^2}{w^2 - \frac{1}{1+2a}} \right)^{N-(1/2)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right) \right),
\end{aligned}
$$

as $n \to \infty$. Here the branches of the fractional powers are positive when w is real and in the interval $(1, \sqrt{1+2a}\,)$. The remainder holds uniformly on compact subsets of $\mathbf{C}_{[-c_a,c_a]}$.

Proof. We start from (1.14) and verify the identity

$$(4.2) \qquad 1 + z^2 = \frac{4(1+a)^2 w^2}{(1+2a)(1+2a-w^2)\left(w^2 - \frac{1}{1+2a}\right)}$$

by (3.3), (2.9), (2.8). Now an application of Lemma 2.2 completes the proof.   □

The next result deals with the case $a = 0$ in (3.1).

Theorem 4.2. *Suppose that* $\alpha > -(1/2)$, $N = \alpha + (1/2)$ *and* c_{nN} *is given by* (1.15). *Then we have*

$$(4.3) \quad H_n^N(\sqrt{N}z)$$
$$= \begin{cases} c_{nN} \dfrac{(z+i)^n}{\sqrt{\pi n}} \left(\dfrac{z+i}{2i}\right)^{1/2} \left(\dfrac{2}{i}(z+i)\right)^\alpha \left(1+\mathcal{O}\!\left(\dfrac{1}{n}\right)\right) & \text{if } \operatorname{Im} z > 0 \\[3mm] c_{nN} \dfrac{(z-i)^n}{\sqrt{\pi n}} \left(\dfrac{z-i}{-2i}\right)^{1/2} \left(\dfrac{2}{-i}(z-i)\right)^\alpha \left(1+\mathcal{O}\!\left(\dfrac{1}{n}\right)\right) & \text{if } \operatorname{Im} z < 0, \end{cases}$$

as $n \to \infty$. *Here the branches of the fractional powers are positive for purely imaginary* z. *Moreover the remainders hold uniformly on compact subsets of the upper and the lower half plane respectively.*

Proof. Again we start from (1.14) and by Lemma 2.2 we obtain

$$H_n^N(\sqrt{N}z) = \frac{c_{nN}}{(\pi n)^{1/2}}(1+z^2)^{n/2}\frac{w^{n+1}}{(w^2-1)^{1/2}}\left(\frac{4w^2}{w^2-1}\right)^\alpha\left(1+\mathcal{O}\!\left(\frac{1}{n}\right)\right),$$

the variables z and w being connected by (1.17) and (2.9). According to the determination of the fractional powers given above we have

$$w = \begin{cases} \left(\dfrac{z+i}{z-i}\right)^{1/2} & \text{if } \operatorname{Im} z > 0 \text{ and } z \in \mathbf{C}_i \\[3mm] \left(\dfrac{z-i}{z+i}\right)^{1/2} & \text{if } \operatorname{Im} z < 0 \text{ and } z \in \mathbf{C}_i \end{cases}$$

($\mathbf{C}_i$ is defined in (1.16)) where the square roots are such that, for $z = it$,

$$
w = \begin{cases}
i\sqrt{\dfrac{1+t}{1-t}} & \text{if } 0 < t < 1 \\[2ex]
-i\sqrt{\dfrac{1-t}{1+t}} & \text{if } -1 < t < 0
\end{cases}
$$

and continuous throughout $\mathbf{C}_i\left(\sqrt{\dfrac{1+t}{1-t}} > 0\right)$. Now straightforward computations yield (4.3) and the uniformity statements are provided by Lemma 2.2 again. $\quad\square$

Finally we establish the companion of Theorem 3.2.

Theorem 4.3. *Suppose that $N > 3/2$ satisfies (3.2) and that the numbers c_{nN} are given by (1.15). Further assume that the quantities r_n, a_n^{**} are defined through*

$$
(4.4) \qquad\qquad r_n := \frac{\sqrt{(n+1)(n+2N-2)}}{N-(3/2)},
$$

$$
(4.5) \qquad a_n^{**} := \frac{4^{N-(1/2)} c_{nN} \left(n+N-(1/2)\right)^{2N+n-(1/2)}}{\pi^{1/2}(n+1)^{(n+1)/2}(n+2N-2)^{2N+(n-1)/2}}.
$$

If the function w_n is defined as in (2.17) with $\alpha_n = N - (1/2)$, then for $z \in \mathbf{C}_{[-1,1]}$ we have

$$
\begin{aligned}
H_n^N\left(\sqrt{N}\, r_n z\right) = a_n^{**}(1 + r_n^2 z^2)^{n/2}\, \frac{w_n^{n+1}}{(w_n^2 - 1)^{1/2}} \\
\cdot \left(\frac{w_n^2}{w_n^2 - \frac{n+1}{n+2N-2}}\right)^{N-(1/2)} \left(1 + o(1)\right),
\end{aligned}
$$
(4.6)

as $n \to \infty$. The determination of the fractional powers involving w_n is given in Lemma 2.3 and the branch of $(1 + r_n^2 z^2)^{n/2}$ is specified in the introduction. Moreover the o-term holds uniformly on compact subset of $\mathbf{C}_{[-1,1]}$.

Proof. With $\alpha_n = N - 1/2$ in (2.1) and (2.12) we have

$$s_n = \sqrt{\frac{(n+1)(n+2N-2)}{n(n+2N-1)}}, \quad D_n = \frac{n(n+2N-1)}{(n+N-1/2)^2}$$

and replacing (3.3) by

(4.7)
$$r_n z = \frac{\sqrt{D_n}\,\zeta}{\sqrt{1-D_n\zeta^2}}$$

the cut z-plane $\mathbf{C}_{[-1,1]}$ corresponds to $\mathbf{C}_{[-s_n,s_n]}$ in the ζ-plane and (1.14) can be rewritten as

$$H_n^N\big(\sqrt{N}\,r_n z\big) = c_{nN}(1 + r_n^2 z^2)^{n/2}\,P_n^{(N-(1/2),N-(1/2))}\big(\sqrt{D_n}\,\zeta\big).$$

Now Lemma 2.4 proves the theorem. □

At this stage we mention the possibility of deriving the weak asymptotics described in the introduction from the strong asymptotics in Theorems 4.1, 4.2, 4.3, provided that (3.1) and (3.2) hold. Due to the uniform validity of (4.1), (4.3), (4.6) logarithmic differentiation of these asymptotic forms immediately leads to the Stieltjes transforms of the limit distributions of the zeros and a subsequent Stieltjes inversion gives the densities of these distributions stated in the introduction. For a worked out example following this approach, see [**2**, Section 3]. The "asymptotic similarity" of H_n^N and H_n under the condition (3.2) is expressed by the weak asymptotics (1.11). The final result in this section establishes a relationship between these polynomials from the viewpoint of strong asymptotics.

To this end we recall the Plancherel-Rotach asymptotics for H_n, $w = z + \sqrt{z^2 - 1} = \phi_1(z)$ (see (2.7)),

(4.8)
$$H_n\big(\sqrt{2n+1}\,z\big) = \frac{2^{n/2}\sqrt{n!}}{(2\pi n)^{1/4}} \frac{w^{n+1}}{(w^2-1)^{1/2}} \exp\left(\left(n+\frac{1}{2}\right)\frac{w^2+1}{2w^2}\right)(1+o(1)),$$

as $n \to \infty$, holding uniformly on compact subsets of $\mathbf{C}_{[-1,1]}$, e.g., [**9, 17, 18**]. We restrict the comparison to two cases which are formulated by the following relative asymptotics.

Theorem 4.4. *Suppose that $w = z + \sqrt{z^2 - 1}$ is Jukowski's function as defined above. If the parameter N satisfies*

$$\lim_{n \to \infty} \frac{n^2}{N} = 0,$$

and r_n is defined in (4.4), then

$$(4.9) \qquad \lim_{n \to \infty} \frac{H_n^N(\sqrt{N}\, r_n z)}{H_n(\sqrt{2n+1}\, z)} = \exp\left(\frac{w^2 + 1}{4w^2}\right),$$

whereas

$$(4.10) \qquad \lim_{n \to \infty} \frac{H_n^{n^2}(n r_n z)}{H_n(\sqrt{2n+1}\, z)} = \exp\left(\frac{w^4 + 3w^2 - 2}{8w^4}\right)$$

and both formulae (4.9), (4.10) hold uniformly on compact subsets of $\mathbf{C}_{[-1,1]}$.

Both limit relations of this theorem can be derived from Theorem 4.3 and (4.8) utilizing the substitutions (4.7) and (2.17). The computations make frequent use of Taylor approximations for several elementary functions and Stirling's formula for Euler's Γ-function. The verifications are tedious but straightforward; therefore, we omit the detailed calculations.

5. Riemann-Hilbert approach. In this section we will use the Riemann-Hilbert approach for orthogonal polynomials, as described in [**4**], to obtain the strong asymptotics (3.5) on the interval of the zeros, and (4.1) off the interval once more, but this time we also get strong asymptotics in the neighborhood of the largest zeros, which will turn out to be in terms of the Airy function. Uniform asymptotics for orthogonal polynomials with a weight $w_n(x) = e^{-nV(x)}$ have been obtained earlier, using the Riemann-Hilbert approach, by Deift, Kriecherbauer, McLaughlin, Venakides and Zhou, for the case when $V(x)$ is a polynomial of even degree [**6**] and the case when $V(x)$ is real analytic, with growth condition $V(x)/\log(1 + x^2) \to \infty$ as $|x| \to \infty$ [**5**]. Relativistic Hermite polynomials correspond to the case where $V(x) = (1 + a)\log(1 + x^2)$, with $a > 0$. Even though this particular

case is not covered by these authors, it turns out that the analysis in [**5**] still works. We will repeat the analysis but in addition we get explicit formulas in our asymptotic results. The analysis is facilitated by the fact that the asymptotic density of the zeros is supported on one interval.

Recall from (1.5) that $H_n^N(z)$ is a polynomial of degree n which is orthogonal to all polynomials of degree $< n$ with respect to the weight function

$$\frac{1}{(1 + (x^2/N))^{N+n}}$$

on the real line $(-\infty, \infty)$. Note that this weight function depends on the degree n. The scaled polynomial $H_n^N(\sqrt{N}\, z)$ is therefore a polynomial of degree n which is orthogonal to all polynomials of degree $< n$ with respect to the weight function

$$w_{n,N}(x) = \frac{1}{(1 + x^2)^{N+n}}.$$

In a similar way we see that $H_{n-1}^{N+1}(\sqrt{N+1}\, z)$ is a polynomial of degree $n - 1$ which is orthogonal to all polynomials of degree $< n - 1$ with respect to the same weight function $w_{n,N}(x)$. Furthermore, it is not so difficult to show that the leading coefficient $k_{n,N}$ of $H_n^N(z)$ is given by

$$k_{n,N} = \frac{2\Gamma(2N + n)}{N^{n-1}\Gamma(2N + 1)},$$

hence the monic orthogonal polynomial of degree n for the weight function $w_n(x)$ is

$$P_{n,N}(z) = \frac{N^{n/2}\Gamma(2N)}{\Gamma(2N + n)}\, H_n^N(\sqrt{N}\, z).$$

Consider the following Riemann-Hilbert problem. Find $Y_n(z)$ such that

1. $Y_n(z)$ is a 2×2 matrix valued function which is analytic in $\mathbf{C} \setminus \mathbf{R}$,

2. on the real line the following *jump condition* holds

$$Y_n^+(x) = Y_n^-(x)\begin{pmatrix} 1 & w_{n,N}(x) \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbf{R},$$

3. near infinity Y_n has the *growth condition*

$$Y_n(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty.$$

Then the solution of this Riemann-Hilbert problem is given by

$$Y_n(z) = \begin{pmatrix} P_{n,N}(z) & \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{P_{n,N}(x)}{x-z} w_{n,N}(x)\, dx \\ d_n P_{n-1,N+1}(z) & \dfrac{d_n}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{P_{n-1,N+1}(x)}{x-z} w_{n,N}(x)\, dx \end{pmatrix},$$

where d_n is such that

$$-\frac{d_n}{2\pi i} \int_{-\infty}^{\infty} x^{n-1} P_{n-1,N+1}(x) w_{n,N}(x)\, dx = 1,$$

which is achieved for

$$d_n = -2\pi i \frac{(2N+1)\Gamma(2N+n+1)}{(n-1)!4^{N+1}\Gamma^2(N+(3/2))}.$$

We will perform some transformations on this Riemann-Hilbert problem in order to be able to obtain strong asymptotics for the matrix Y_n uniformly over the complex plane. For the first transformation we need some information about the complex logarithmic potential of the density (1.10) describing the asymptotic distribution of the zeros.

Lemma 5.1. *Let*

$$g(z) = \frac{1}{\pi} \int_{-c_a}^{c_a} \frac{\sqrt{1+2a-a^2 x^2}}{1+x^2} \log(z-x)\, dx, \quad z \in \mathbf{C}_{[-c_a,c_a]},$$

where $\log z$ is chosen to be analytic in $\mathbf{C} \setminus (-\infty, 0]$ with $\log z > 0$ for $z > 1$, then

$$(5.1) \quad \begin{aligned} g(z) = {}& \log \frac{z + \frac{1}{a}\sqrt{a^2 z^2 - 1 - 2a}}{2} \\ & + (a+1)\log\left(1 + \frac{a^2}{(1+2a)^2}\left(z - \frac{1}{a}\sqrt{a^2 z^2 - 1 - 2a}\right)^2\right), \end{aligned}$$

whenever $z \notin [-c_a, c_a]$, and for $t \in [-c_a, c_a]$

(5.2)
$$
\begin{aligned}
g_\pm(t) &= \lim_{\varepsilon \to 0+} g(t \pm i\varepsilon) \\
&= a \log 2a - \frac{1+2a}{2} \log(1+2a) + \frac{1+a}{2} \log(1+t^2) \\
&\quad \pm i \left(a \arcsin \frac{at}{\sqrt{1+2a}} - (a+1) \arctan \frac{(1+a)t}{\sqrt{1+2a-a^2t^2}} + \frac{\pi}{2} \right).
\end{aligned}
$$

Moreover, for $z > c_a$, we have

(5.3)
$$
\begin{aligned}
g(z) &= a \log 2a - \frac{1+2a}{2} \log(1+2a) \\
&\quad + \frac{1+a}{2} \log(1+z^2) - \int_{c_a}^z \frac{\sqrt{a^2 y^2 - 1 - 2a}}{1+y^2} \, dy.
\end{aligned}
$$

Proof. First we change variables and put

$$
z = c_a \xi, \quad x = c_a y,
$$

so that $\xi \in \mathbf{C} \setminus [-1, 1]$ and $y \in [-1, 1]$. This gives

$$
g(z) = \frac{a}{\pi} \int_{-1}^1 \frac{\sqrt{1-y^2}}{\frac{a^2}{1+2a} + y^2} \left(\log \frac{\sqrt{1+2a}}{a} + \log(\xi - y) \right) dy.
$$

If we use partial fractions and the integral

$$
\frac{1}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1-y^2}} \frac{dy}{\xi - y} = \frac{1}{\sqrt{\xi^2 - 1}}, \quad \xi \notin [-1, 1],
$$

taking into account that our definition of the square root implies that

$$
\sqrt{\left(\frac{\pm ia}{\sqrt{1+2a}} \right)^2 - 1} = \frac{\pm i(a+1)}{\sqrt{1+2a}},
$$

one easily finds

$$
\frac{a}{\pi} \int_{-1}^1 \frac{\sqrt{1-y^2}}{\frac{a^2}{1+2a} + y^2} \, dy = 1.
$$

Hence

$$g(z) = \log \frac{\sqrt{1+2a}}{a} + \frac{a}{\pi} \int_{-1}^{1} \frac{\sqrt{1-y^2}}{\frac{a^2}{1+2a} + y^2} \log(\xi - y)\, dy.$$

Now use the Fourier series

$$\log(\xi - y) = \log \frac{\xi + \sqrt{\xi^2 - 1}}{2} - 2 \sum_{k=1}^{\infty} \frac{(\xi - \sqrt{\xi^2 - 1})^k}{k} T_k(y),$$

where $T_k(y) = \cos k\theta$, for $y = \cos\theta$, is the Chebyshev polynomial of the first kind of degree k, then

(5.4)

$$\frac{a}{\pi} \int_{-1}^{1} \frac{\sqrt{1-y^2}}{\frac{a^2}{1+2a} + y^2} \log(\xi - y)\, dy$$

$$= \log \frac{\xi + \sqrt{\xi^2 - 1}}{2} - 2 \sum_{k=1}^{\infty} \frac{(\xi - \sqrt{\xi^2 - 1})^k}{k} \frac{a}{\pi} \int_{-1}^{1} T_k(y) \frac{\sqrt{1-y^2}}{\frac{a^2}{1+2a} + y^2}\, dy.$$

This series is uniformly convergent, with respect to y, since $|\xi - \sqrt{\xi^2 - 1}| < 1$. A decomposition into partial fractions and the integral

$$\frac{1}{\pi} \int_{-1}^{1} \frac{T_k(y)}{\sqrt{1-y^2}} \frac{dy}{\xi - y} = \frac{(\xi - \sqrt{\xi^2 - 1})^k}{\sqrt{\xi^2 - 1}}, \quad \xi \notin [-1,1],$$

now give

$$\frac{a}{\pi} \int_{-1}^{1} T_k(y) \frac{\sqrt{1-y^2}}{\frac{a^2}{1+2a} + y^2}\, dy = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ (-1)^l \dfrac{a+1}{(1+2a)^l}, & \text{if } k = 2l \text{ is even,} \end{cases}$$

so that the infinite sum in (5.4) is

$$-2(a{+}1) \sum_{l=1}^{\infty} \frac{(\xi - \sqrt{\xi^2 - 1})^{2l}}{2l} \frac{(-1)^l}{(1{+}2a)^l} = (a{+}1) \log\left(1 + \frac{(\xi - \sqrt{\xi^2 - 1})^2}{1 + 2a}\right).$$

Returning to the original variable z then gives the required expression (5.1).

For the boundary values $g_\pm$ we observe that

$$\lim_{\varepsilon \to 0+} \sqrt{a^2(t \pm i\varepsilon)^2 - 1 - 2a} = \pm i \sqrt{1 + 2a - a^2 t^2},$$

so that

$$g_\pm(t) = \log \frac{t \pm \frac{i}{a} \sqrt{1 + 2a - a^2 t^2}}{2}$$
$$+ (a+1) \log \left(1 + \frac{a^2}{(1+2a)^2} \left(t - \frac{i}{a} \sqrt{1 + 2a - a^2 t^2} \right)^2 \right).$$

The real part of this expression is

$$\log \frac{\left| t \pm \frac{i}{a} \sqrt{1 + 2a - a^2 t^2} \right|}{2}$$
$$+ (a+1) \log \left| 1 + \frac{a^2}{(1+2a)^2} \left(t - \frac{i}{a} \sqrt{1 + 2a - a^2 t^2} \right)^2 \right|$$

and the first term is easily evaluated as

$$\log \frac{\left| t \pm (i/a)\sqrt{1 + 2a - a^2 t^2} \right|}{2} = \log \frac{\sqrt{1 + 2a}}{2a}.$$

The second term is

$$(a+1) \log \left| 1 + \frac{a^2}{(1+2a)^2} \left(t - \frac{i}{a} \sqrt{1 + 2a - a^2 t^2} \right)^2 \right|$$
$$= (a+1) \log \left(\frac{2a}{1+2a} \sqrt{1 + t^2} \right).$$

Combining both expressions gives the real part in (5.2). For the imaginary part we observe that our choice of the logarithm gives

$$\lim_{\varepsilon \to 0+} \log(t \pm i\varepsilon - x) = \begin{cases} \log |t - x| & \text{if } t > x, \\ \log |t - x| \pm i\pi & \text{if } t < x, \end{cases}$$

and hence the imaginary part of $g_\pm(t)$ is given by

$$(5.5) \qquad \Im\, g_\pm(t) = \pm \int_t^{\sqrt{1+2a}/a} \frac{\sqrt{1 + 2a - a^2 x^2}}{1 + x^2}\, dx.$$

Some straightforward calculus gives

$$\int_t^{c_a} \frac{\sqrt{1 + 2a - a^2 x^2}}{1 + x^2}\, dx = a \arcsin \frac{at}{\sqrt{1 + 2a}} - (a + 1)$$
$$\cdot \arctan \frac{(1 + a)t}{\sqrt{1 + 2a - a^2 t^2}} + \frac{\pi}{2},$$

which can be verified by taking derivatives. This proves the expression (5.2).

Finally, in order to prove (5.3) we use that for $z > c_a > x$

$$\log(z - x) = \int_{c_a}^z \frac{dy}{y - x} + \log(c_a - x).$$

Use this in the definition of $g(z)$ and interchange the order of integration to find

$$g(z) = g(c_a) + \int_{c_a}^z \frac{1}{\pi} \int_{-c_a}^{c_a} \frac{\sqrt{1 + 2a - a^2 x^2}}{1 + x^2} \frac{dx}{y - x}\, dy.$$

The inner integral can be worked out using a partial fraction decomposition and the integral

$$\frac{1}{\pi} \int_{-1}^1 \sqrt{1 - x^2} \frac{dx}{z - x} = z - \sqrt{z^2 - 1}.$$

Furthermore, from (5.1) we can work out $g(c_a)$. Combining these results then gives (5.3).   □

We transform the matrix Y_n to a new matrix M_n by

(5.6)
$$M_n(z) = \begin{pmatrix} e^{nl_a/2} & 0 \\ 0 & e^{-nl_a/2} \end{pmatrix} Y_n(z) \begin{pmatrix} e^{-ng(z)} & 0 \\ 0 & e^{ng(z)} \end{pmatrix}$$
$$\cdot \begin{pmatrix} e^{-nl_a/2} & 0 \\ 0 & e^{nl_a/2} \end{pmatrix},$$

where g is given by Lemma 5.1 and

$$l_a = -2a \log 2a + (1 + 2a) \log(1 + 2a).$$

Clearly $M_n(z)$ is a 2×2 matrix valued function which is analytic in $\mathbf{C} \setminus \mathbf{R}$. Furthermore, since $e^{ng(z)} = z^n (1 + \mathcal{O}(1/z))$, we see that the matrix $M_n(z)$ is *normalized* as $z \to \infty$

$$(5.7) \qquad M_n(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty.$$

Some matrix algebra shows that the jump condition for M_n on the real line is

$$M_n^+(x) = M_n^-(x) \begin{pmatrix} e^{n(g_-(x)-g_+(x))} & w_{n,N}(x)e^{n(g_-(x)+g_+(x)+l_a)} \\ 0 & e^{n(g_+(x)-g_-(x))} \end{pmatrix},$$
$$x \in \mathbf{R}.$$

On the interval $[-c_a, c_a]$ we have, according to Lemma 5.1

$$g_-(x) + g_+(x) = 2a \log 2a - (1 + 2a) \log(1 + 2a) + (1 + a) \log(1 + x^2),$$

and since $w_{n,N}(x) = \exp(-n(1 + a) \log(1 + x^2))$ for $N = na$, we see that
$$w_{n,N}(x)\, e^{n(g_-(x)+g_+(x)+l_a)} = 1, \quad -c_a \leq x \leq c_a.$$

Lemma 5.1 also gives
(5.8)
$$\phi_+(x) := \frac{g_+(x) - g_-(x)}{2}$$
$$= i\left(a \arcsin \frac{ax}{\sqrt{1+2a}} - (a+1) \arctan \frac{(1+a)x}{\sqrt{1+2a-a^2x^2}} + \frac{\pi}{2} \right),$$
$$-c_a \leq x \leq c_a,$$

which is purely imaginary. If we also define $\phi_-(x) = -\phi_+(x) = \overline{\phi_+(x)}$, for $x \in [-c_a, c_a]$, then the jump matrix for M_n on $[-c_a, c_a]$ has the form

$$V_2(x) := \begin{pmatrix} e^{n(g_-(x)-g_+(x))} & w_{n,N}(x)\, e^{n(g_-(x)+g_+(x)+l_a)} \\ 0 & e^{n(g_+(x)-g_-(x))} \end{pmatrix}$$
$$= \begin{pmatrix} e^{2n\phi_-(x)} & 1 \\ 0 & e^{2n\phi_+(x)} \end{pmatrix}, \quad -c_a \leq x \leq c_a.$$

For $x > c_a$ the function $g(x)$ is analytic, hence $g_-(x) = g_+(x) = -U(x;\xi)$, where

$$U(x;\xi) = \int \log \frac{1}{|x-t|} \, d\xi(t)$$

is the logarithmic potential of the measure ξ in (1.10). For $x < -c_a$ we have $g_\pm(x) = -U(x;\xi) \pm i\pi$. Hence the jump matrix for M_n on $\mathbf{R} \setminus [-c_a, c_a]$ is

$$V_{1,3}(x) := \begin{pmatrix} e^{n(g_-(x)-g_+(x))} & w_{n,N}(x)\, e^{n(g_-(x)+g_+(x)+l_a)} \\ 0 & e^{n(g_+(x)-g_-(x))} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & w_{n,N}\, e^{-n(2U(x;\xi)-l_a)} \\ 0 & 1 \end{pmatrix}, \quad x < -c_a \quad \text{or } x > c_a.$$

If we use (5.3), then we find for $x > c_a$

$$-2U(x;\xi) + l_a - (1+a)\log(1+x^2)$$
$$= 2g(x) + l_a - (1+a)\log(1+x^2),$$
$$= -2 \int_{c_a}^x \frac{\sqrt{a^2 y^2 - 1 - 2a}}{1+y^2} \, dy < 0,$$

and a similar expression for $x < -c_a$. Hence

$$V_{1,3}(x) = \begin{pmatrix} 1 & e^{2n\phi(x)} \\ 0 & 1 \end{pmatrix}, \quad x < -c_a \quad \text{or } x > c_a,$$

with

$$\phi(x) = -\int_{c_a}^x \frac{\sqrt{a^2 y^2 - 1 - 2a}}{1+y^2} \, dy < 0, \quad x \in \mathbf{R} \setminus [-c_a, c_a].$$

The matrix function M_n thus solves the *normalized* Riemann-Hilbert problem (because (5.7) holds) on the three curves $\Sigma_1 = (-\infty, -c_a)$, $\Sigma_2 = [-c_a, c_a]$, $\Sigma_3 = (c_a, \infty)$ with jumps respectively V_1, V_2, V_3. The jump V_2 on $[-c_a, c_a]$ factors into three simpler matrices:

$$V_2(x) = \begin{pmatrix} 1 & 0 \\ e^{-2n\phi_-(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-2n\phi_+(x)} & 1 \end{pmatrix},$$
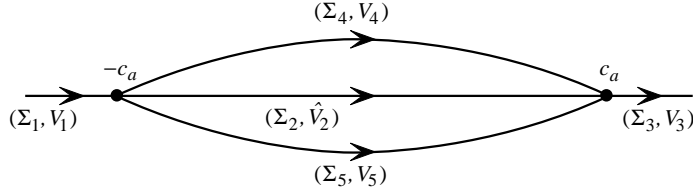
FIGURE 1. Riemann-Hilbert problem with 5 curves.

so that we can jump over this interval in three steps. Rather than jumping over this interval in three steps, we prefer to open a lens (see Figure 1) with upper curve Σ_4 and lower curve Σ_5 and jump over $\Sigma_4, \Sigma_2, \Sigma_5$ using the matrices in the product (5.9). For this, we need to find an analytic continuation of $\phi_\pm(x)$ to the upper and lower complex plane. This is achieved by taking

$$(5.10) \qquad \phi(z) = \int_z^{c_a} \frac{\sqrt{a^2 y^2 - 1 - 2a}}{1 + y^2} \, dy,$$

where the path connecting z and c_a does not intersect the interval $[-c_a, c_a]$. This defines ϕ in a neighborhood of $[-c_a, c_a]$ (modulo $2\pi i$) and ϕ is independent of the path connecting z and c_a, as long as the paths stay in the neighborhood of $[-c_a, c_a]$ and do not cross $[-c_a, c_a]$ (the poles of $1/(1 + x^2)$ will cause problems if paths are allowed to go around $\pm i$). Observe that

$$\lim_{\varepsilon \to 0+} \phi(t \pm i\varepsilon) = \pm i \int_t^{c_a} \frac{\sqrt{1 + 2a - a^2 x^2}}{1 + x^2} \, dx = \phi_\pm(t), \quad -c_a \le t \le c_a,$$

so that the function defined in (5.10) has $\phi_\pm$ as boundary values on $[-c_a, c_a]$. We define a new matrix function $\widehat{M_n}$ by

$$(5.11) \qquad \widehat{M_n}(z) = \begin{cases} M_n(z) & z \in \mathbf{C} \setminus (\mathbf{R} \cup \text{Int}\,(\Sigma_4 \cup \Sigma_5)), \\ M_n(z)\Phi_n(z)^{-1} & z \in \text{Int}\,(\Sigma_4 \cup \Sigma_2), \\ M_n(z)\Phi_n(z) & z \in \text{Int}\,(\Sigma_2 \cup \Sigma_5), \end{cases}$$

where $\text{Int}\,(\Sigma)$ is the (open) interior of a closed curve Σ and

$$\Phi_n(z) = \begin{pmatrix} 1 & 0 \\ e^{-2n\phi(z)} & 1 \end{pmatrix},$$

then $\widehat{M}_n(z)$ solves the normalized Riemann-Hilbert problem on the 5 curves $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5)$ with jumps given by

$$V_1 = V_3 = \begin{pmatrix} 1 & e^{2n\phi(x)} \\ 0 & 1 \end{pmatrix}, \ \widehat{V}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ V_4 = V_5 = \begin{pmatrix} 1 & 0 \\ e^{-2n\phi(z)} & 1 \end{pmatrix}.$$

The function $\phi(z) = u(x, y) + iv(x, y)$, with $z = x + iy$, has a positive real part u on Σ_4 because $u(x + 0, 0) = 0$ and $\partial v/\partial x = -\sqrt{1 + 2a - a^2 x^2}/(1 + x^2) < 0$ on $(-c_a, c_a)$, so that the Cauchy-Riemann equation gives $\partial u/\partial y = -\partial v/\partial x > 0$ and $u(x, y) > 0$ for y small. This means that $e^{-2n\phi(z)}$ decreases exponentially fast to zero on Σ_4. Similarly $\phi(z)$ has also a positive real part on Σ_5 so that $e^{-2n\phi(z)}$ also decreases exponentially fast to zero on Σ_5. This means that

$$\lim_{n \to \infty} V_1(x) = I, \quad x \in \Sigma_1,$$
$$\lim_{n \to \infty} V_3(x) = I, \quad x \in \Sigma_3,$$
$$\lim_{n \to \infty} V_4(x) = I, \quad x \in \Sigma_4,$$
$$\lim_{n \to \infty} V_5(x) = I, \quad x \in \Sigma_5,$$

so that we expect that $\widehat{M}_n$ for $n \to \infty$ behaves like the solution $\widehat{M}_\infty$ of the normalized Riemann-Hilbert problem on $\Sigma_2 = [-c_a, c_a]$ with jump $\widehat{V}_2$. This particular matrix function is known (see, e.g., [**4**, pp. 200–201]) and is given by

$$(5.12) \qquad \widehat{M}_\infty(z) = \begin{pmatrix} \dfrac{\beta + 1/\beta}{2} & \dfrac{\beta - 1/\beta}{2i} \\ -\dfrac{\beta - 1/\beta}{2i} & \dfrac{\beta + 1/\beta}{2} \end{pmatrix},$$

where

$$\beta(z) = \left(\frac{az - \sqrt{1 + 2a}}{az + \sqrt{1 + 2a}} \right)^{1/4}.$$

However, in order to be able to compare $\widehat{M}_n$ with $\widehat{M}_\infty$, we need to have *uniform convergence* of V_1, V_3, V_4, V_5 to the identity matrix on $\Sigma_1, \Sigma_3, \Sigma_4, \Sigma_5$ respectively. This fails to be true in the neighborhood of $\pm c_a$. To overcome this difficulty, we introduce a parametrix around $\pm c_a$. This consists of two small closed curves Σ_6 and Σ_7 around $-c_a$
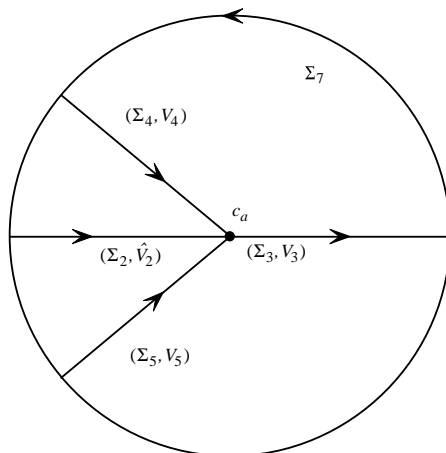
FIGURE 2. The Riemann-Hilbert problem near c_a.

and c_a. Locally, the Riemann-Hilbert problem for $\widehat{M}_n$ inside Σ_7 looks like Figure 2. The jumps V_3, V_4 and V_5 contain the function ϕ which near c_a behaves like

$$
\phi(z) = a \int_z^{c_a} \sqrt{x - c_a} \, \frac{\sqrt{x + c_a}}{1 + x^2} \, dx
$$
$$
= \frac{-2a}{3} \frac{\sqrt{2c_a}}{1 + c_a^2} \, (z - c_a)^{3/2} + \mathcal{O}((z - c_a)^{5/2})
$$
$$
=: -(z - c_a)^{3/2} \, G(z),
$$
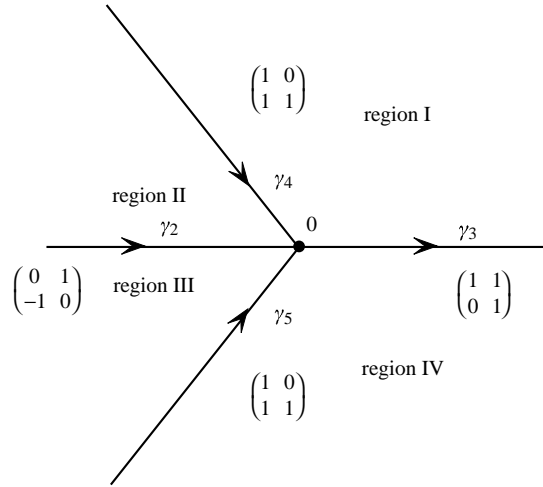
where $G(c_a) > 0$, then clearly $\phi(z)$ tends to zero as $z \to c_a$; in particular we have

$$
V_3(c_a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad V_4(c_a) = V_5(c_a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},
$$

and

$$
V_j(z) = \begin{pmatrix} e^{n\phi(z)} & 0 \\ 0 & e^{-n\phi(z)} \end{pmatrix}_- V_j(c_a) \begin{pmatrix} e^{-n\phi(z)} & 0 \\ 0 & e^{n\phi(z)} \end{pmatrix}_+, \quad j = 3, 4, 5.
$$

We will solve the problem in $\mathrm{Int}\,(\Sigma_7)$ exactly using Airy functions, as is explained in detail in [**4**, Section 7.6]. Let $\omega = e^{2\pi i/3}$ be a primitive

FIGURE 3. The Riemann-Hilbert problem for Ψ.

third root of unity and put

$$\Psi(s) = \begin{pmatrix} \mathrm{Ai}(s) & \mathrm{Ai}(\omega^2 s) \\ \mathrm{Ai}'(s) & \omega^2 \mathrm{Ai}'(\omega^2 s) \end{pmatrix} \begin{pmatrix} e^{-i\pi/6} & 0 \\ 0 & e^{i\pi/6} \end{pmatrix}, \qquad s \in \text{ region I,}$$

$$\Psi(s) = \begin{pmatrix} \mathrm{Ai}(s) & \mathrm{Ai}(\omega^2 s) \\ \mathrm{Ai}'(s) & \omega^2 \mathrm{Ai}'(\omega^2 s) \end{pmatrix} \begin{pmatrix} e^{-i\pi/6} & 0 \\ -e^{i\pi/6} & e^{i\pi/6} \end{pmatrix}, \qquad s \in \text{ region II,}$$

$$\Psi(s) = \begin{pmatrix} \mathrm{Ai}(s) & -\omega^2 \mathrm{Ai}(\omega s) \\ \mathrm{Ai}'(s) & -\mathrm{Ai}'(\omega s) \end{pmatrix} \begin{pmatrix} e^{-i\pi/6} & 0 \\ e^{i\pi/6} & e^{i\pi/6} \end{pmatrix}, \qquad s \in \text{ region III,}$$

$$\Psi(s) = \begin{pmatrix} \mathrm{Ai}(s) & -\omega^2 \mathrm{Ai}(\omega s) \\ \mathrm{Ai}'(s) & -\mathrm{Ai}'(\omega s) \end{pmatrix} \begin{pmatrix} e^{-i\pi/6} & 0 \\ 0 & e^{i\pi/6} \end{pmatrix}, \qquad s \in \text{ region IV,}$$

then $\Psi(s)$ solves the Riemann-Hilbert problem on the four curves and the constant jump matrices given in Figure 3, with asymptotic behavior that can be described using the asymptotic behavior of the Airy function $\mathrm{Ai}(s)$ and its derivative $\mathrm{Ai}'(s)$.

Define the matrix function M_p by

$$(5.14) \quad M_p(z) = E(z)\Psi(n^{2/3}\lambda(z)) \begin{pmatrix} e^{-n\phi(z)} & 0 \\ 0 & e^{n\phi(z)} \end{pmatrix}, \quad z \in \text{Int}\,(\Sigma_7),$$

where $E(z)$ is an analytic matrix function and $\lambda(z) = (z - c_a)G^{2/3}(z)$ with G as defined in (5.13). The function λ maps a neighborhood of c_a to a neighborhood of 0. Choose $\varepsilon > 0$ such that $D_\varepsilon = \{|\lambda| < \varepsilon\}$ is in this neighborhood around 0. Then we choose $\Sigma_7 = \lambda^{-1}(\partial D_\varepsilon)$ and some matrix calculus shows that M_p has the jump conditions

$$
\begin{aligned}
M_{p+}(z) &= M_{p-}(z)V_2 && z \in \lambda^{-1}(\gamma_2 \cap D_\varepsilon) := \Sigma_2 \cap \text{Int}\,(\Sigma_7), \\
M_{p+}(z) &= M_{p-}(z)V_3 && z \in \lambda^{-1}(\gamma_3 \cap D_\varepsilon) := \Sigma_3 \cap \text{Int}\,(\Sigma_7), \\
M_{p+}(z) &= M_{p-}(z)V_4 && z \in \lambda^{-1}(\gamma_4 \cap D_\varepsilon) := \Sigma_4 \cap \text{Int}\,(\Sigma_7), \\
M_{p+}(z) &= M_{p-}(z)V_5 && z \in \lambda^{-1}(\gamma_5 \cap D_\varepsilon) := \Sigma_5 \cap \text{Int}\,(\Sigma_7).
\end{aligned}
$$

The function $E(z)$ can be chosen in such a way that $M_p(z)$ is nearly equal to $\widehat{M}_\infty$ on Σ_7, by using the asymptotic behavior of the Airy function (this is in fact the reason why we need Airy functions), as is explained in detail in [**4**, Section 7.6]. The appropriate choice is
(5.15)
$$
E(z) = \sqrt{\pi}\, e^{i\pi/6} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}
$$
$$
\cdot \begin{pmatrix} n^{1/6}[(z + c_a)G^{2/3}(z)]^{1/4} & 0 \\ 0 & n^{-1/6}[(z + c_a)G^{2/3}(z)]^{-1/4} \end{pmatrix}.
$$

With this choice we have

$$
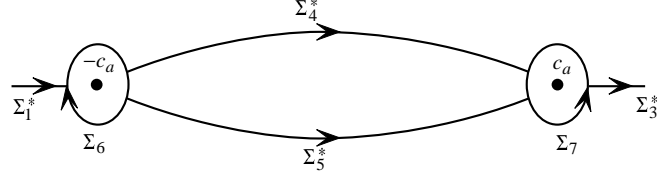M_p(z) = \widehat{M}_\infty(z)V_p(z), \quad z \in \Sigma_7,
$$

with

$$
V_p(z) = I + \mathcal{O}\left(\frac{1}{n}\right),
$$

uniformly on Σ_7.

A similar analysis can be done in the neighborhood of $-c_a$ with contour Σ_6, or one can just use the symmetry of the problem to obtain the parametrix and the solution

$$
M_p(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M_p(-z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

around $-c_a$ in $\text{Int}\,(\Sigma_6)$. Outside $\text{Int}\,(\Sigma_6) \cup \text{Int}\,(\Sigma_7)$ we define $M_p(z) = \widehat{M}_\infty$.

FIGURE 4. The Riemann-Hilbert problem for $L = \widehat{M}_n M_p^{-1}$.

Now we can consider the matrix $L(z) = \widehat{M}_n(z) M_p^{-1}(z)$. We see that L solves the Riemann-Hilbert problem on the curves in Figure 4. Inside Σ_6 and Σ_7 the functions M_p and $\widehat{M}_n$ have the same jumps, hence L has no jumps. Similarly, on Σ_2 the functions M_p and $\widehat{M}_n$ have the same jump, hence L has no jumps on the interval $[-c_a, c_a]$. The only remaining jumps are on the curves Σ_6 and Σ_7 and the curves Σ_1^*, Σ_3^*, Σ_4^* and Σ_5^*, which are the parts of the curves Σ_1, Σ_3, Σ_4 and Σ_5 outside Σ_6 and Σ_7. Here the jumps are

$$
\begin{aligned}
V_L(z) &= \widehat{M}_\infty V_{1,3} \widehat{M}_\infty^{-1} &&= \mathcal{O}(e^{-cn}), && z \in \Sigma_{1,3}^*, \\
V_L(z) &= \widehat{M}_\infty V_{4,5} \widehat{M}_\infty^{-1} &&= \mathcal{O}(e^{-cn}), && z \in \Sigma_{4,5}^*, \\
V_L(z) &= \widehat{M}_\infty V_p^{-1} \widehat{M}_\infty^{-1} &&= \mathcal{O}(1/n), && z \in \Sigma_{6,7}.
\end{aligned}
$$

All the jumps now tend uniformly to the identity matrix I as $n \to \infty$. We can then use a perturbation theorem, such as Theorem 7.103 or Corollary 7.108 in [4], to conclude that $L(z) = I + \mathcal{O}(1/n)$ in the L_2-norm on the curves in Figure 4, and uniformly on compact subsets of the complex plane away from these curves.

All of this now finally gives the following asymptotic results

Theorem 5.1. *If $N = an$ and $n \to \infty$, then*

$$
\frac{N^{n/2}\Gamma(2N)}{\Gamma(2N+n)} H_n^N\left(\sqrt{N}z\right) = \left(\frac{z + \sqrt{z^2 - c_a^2}}{2}\right)^{n+(1/2)}
$$

$$
\cdot \left(1 + \left(\frac{a}{1+2a}\right)^2 \left(z - \sqrt{z^2 - c_a^2}\right)^2\right)^{n+N}
$$

$$
\cdot (z^2 - c_a^2)^{-1/4}\left(1 + \mathcal{O}(1/n)\right),
$$

uniformly on compact subsets of $\mathbf{C}_{[-c_a, c_a]}$.

This result coincides with the case $\alpha = -1/2$ of Theorem 4.1. We omit the somewhat tedious but straightforward verification.

Proof. We take the curves $\Sigma_4, \Sigma_5, \Sigma_6, \Sigma_7$ small enough so that the compact set in $\mathbf{C} \setminus [-c_a, c_a]$ is outside these curves. Using the transformations (5.6) and (5.11) we then see that

$$P_{n,N}(z) = (Y_n)_{1,1} = (\widehat{M}_n)_{1,1} e^{ng(z)}.$$

But

$$(\widehat{M}_n)_{1,1} = L_{1,1}(\widehat{M}_\infty)_{1,1} + L_{1,2}(\widehat{M}_\infty)_{2,1},$$

and since $L = I + \mathcal{O}(1/n)$, we see that

$$(\widehat{M}_n)_{1,1} = (\widehat{M}_\infty)_{1,1} + \mathcal{O}(1/n).$$

The result now follows by using (5.1) and (5.12). □

Theorem 5.2. *On the interval* $(-c_a, c_a)$ *we put* $x = c_a \cos\theta$ *and have for* $N = an$ *and* $n \to \infty$

$$\frac{N^{n/2}\Gamma(2N)}{\Gamma(2N+n)} H_n^N(\sqrt{N}\,x) = \sqrt{2}\,(2a)^N(1+2a)^{-N-n/2}$$
$$\cdot (1+x^2)^{(n+N)/2}(\sin\theta)^{-1/2}$$
$$\cdot \left[\sin\left(n\Im g_+(x) + \frac{\theta}{2} + \frac{\pi}{4}\right) + \mathcal{O}(1/n)\right],$$

uniformly on closed intervals of $(-c_a, c_a)$, *where* $g_+(x)$ *is given in* (5.2).

Again this result corresponds to the case $\alpha = -1/2$ of Theorem 3.1 but in somewhat different notation.

Proof. We are interested in $P_{n,N}(x) = (Y_n)_{1,1}$ on the interval $(-c_a, c_a)$. Since this is a polynomial, this is also equal to $(Y_n^+)_{1,1}$.

Hence we will investigate Y_n^+ on the interval $(-c_a, c_a)$. Using the transformations (5.6) and (5.11) we get on $(-c_a, c_a)$

$$(Y_n^+) = \left[(\widehat{M}_n^+)_{1,1} e^{n\phi_+(x)} + (\widehat{M}_n^+)_{1,2} e^{-n\phi_+(x)} \right] e^{n(g_+(x)+g_-(x))/2}.$$

For a closed interval of $(-c_a, c_a)$ we can always make the Σ_6 and Σ_7 small enough so that they don't intersect this closed interval. Then on this closed interval

$$(\widehat{M}_n^+)_{1,1} = L_{1,1}(\widehat{M}_\infty)_{1,1} + L_{1,2}(\widehat{M}_\infty)_{2,1} = (\widehat{M}_\infty)_{1,1} + \mathcal{O}(1/n),$$
$$(\widehat{M}_n^+)_{1,2} = L_{1,1}(\widehat{M}_\infty)_{1,2} + L_{1,2}(\widehat{M}_\infty)_{2,2} = (\widehat{M}_\infty)_{1,2} + \mathcal{O}(1/n).$$

If we put $x = c_a \cos\theta$, then

$$
\begin{aligned}
(\widehat{M}_\infty^+)_{1,1} &= \frac{\beta_+ + 1/\beta_+}{2} \\
&= \frac{1}{2}\left[e^{i\pi/4}\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/4} + e^{-i\pi/4}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{1/4} \right], \\
(\widehat{M}_\infty^+)_{1,2} &= \frac{\beta_+ - 1/\beta_+}{2i} \\
&= \frac{1}{2}\left[e^{-i\pi/4}\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/4} + e^{i\pi/4}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{1/4} \right] \\
&= \overline{(\widehat{M}_\infty^+)_{1,1}}.
\end{aligned}
$$

We thus get

$$(Y_n^+)_{1,1} = 2 e^{n(g_+(x)+g_-(x))/2} \Re\left((\widehat{M}_\infty^+)_{1,1} e^{n\phi_+(x)} \right).$$

Recall that on one hand we have

$$\frac{g_+(x) + g_-(x)}{2} = a\log(2a) - \frac{1+2a}{2}\log(1+2a) + \frac{1+a}{2}\log(1+x^2),$$

which gives the amplitude in our asymptotic expression, and on the other hand, simple trigonometry gives

$$\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/4} = \left(\frac{\sin\theta/2}{\cos\theta/2}\right)^{1/2},$$

so that the oscillatory part simplifies to

$$2\Re\left((\widehat{M}_{\infty}^{+})_{1,1}e^{n\phi_{+}(x)}\right)$$

$$= \frac{\sin\theta/2\cos(\pi/4 + n\Im g_{+}(x)) + \cos\theta/2\sin(\pi/4 + n\Im g_{+}(x))}{(\cos\theta/2\sin\theta/2)^{1/2}}$$

$$= \frac{\sqrt{2}\sin(n\Im g_{+}(x) + \theta/2 + \pi/4)}{(\sin\theta)^{1/2}}.$$

A combination of these results gives the required asymptotic expression.
□

Finally, in the neighborhood of $\pm c_a$ we get asymptotics in terms of Airy functions.

Theorem 5.3. *In the neighborhood of c_a we have*

$$\frac{N^{n/2}\Gamma(2N)}{\Gamma(2N+n)}H_{n}^{N}(\sqrt{N}z) = \sqrt{\pi}\left(n^{1/6}\left[(z+c_a)G^{2/3}(z)\right]^{1/4}\mathrm{Ai}(n^{2/3}\lambda(z))\right.$$

$$\left. - n^{-1/6}\left[(z+c_a)G^{2/3}(z)\right]^{-1/4}\mathrm{Ai}'(n^{2/3}\lambda(z))\right)$$

$$\cdot\left[1 + \mathcal{O}(1/n)\right],$$

where the functions G and λ are given in (5.13) *and* (5.14) *respectively.*

Proof. Again we will investigate $P_{n,N}(z) = (Y_{n}^{+})_{1,1}$ but now for $z \in \mathrm{Int}\,(\Sigma_7)$. The analysis around $-c_a$ is similar. We will consider the case $z < c_a$; the case $z > c_a$ can be handled in the same way. For $z < c_a$ we are in region II of Figure 3. This means that

$$(Y_{n}^{+})_{1,1} = \left[(\widehat{M}_{n}^{+})_{1,1}\,e^{n\phi_{+}(z)} + (\widehat{M}_{n}^{+})_{1,2}\,e^{-n\phi_{+}(z)}\right]e^{n(g_{+}(z)+g_{-}(z))/2},$$

and $\widehat{M}_{n}^{+} = LM_{p}^{+}$. Since $L = I + \mathcal{O}(1/n)$ we therefore get

$$(\widehat{M}_{n}^{+})_{1,1} = L_{1,1}(M_{p}^{+})_{1,1} + L_{1,2}(M_{p}^{+})_{2,1} = (M_{p}^{\infty})_{1,1} + \mathcal{O}(1/n),$$

$$(\widehat{M}_{n}^{+})_{1,2} = L_{1,1}(M_{p}^{+})_{1,2} + L_{1,2}(M_{p}^{+})_{2,2} = (M_{p}^{\infty})_{1,2} + \mathcal{O}(1/n).$$

This gives

$$(Y_n^+)_{1,1} = \left[(M_p^+)_{1,1}\, e^{n\phi_+(z)} + (M_p^+)_{1,2}\, e^{-n\phi_+(z)} + \mathcal{O}(1/n)\right]$$
$$\cdot\, e^{n(g_+(z)+g_-(z))/2}.$$

In region II we have

$$(M_p^+)_{1,1} = (E_{1,1}\Psi_{1,1} + E_{1,2}\Psi_{2,1})\, e^{-n\phi_+(z)}$$
$$(M_p^+)_{1,2} = (E_{1,1}\Psi_{1,2} + E_{1,2}\Psi_{2,2})\, e^{n\phi_+(z)},$$

so that

$$(Y_n^+)_{1,1} = \left[E_{1,1}(\Psi_{1,1} + \Psi_{1,2}) + E_{1,2}(\Psi_{2,1} + \Psi_{2,2}) + \mathcal{O}(1/n)\right]$$
$$\cdot\, e^{n(g_+(z)+g_-(z))/2}$$
$$= \sqrt{\pi}\, e^{i\pi/6} n^{1/6}[(z+c_a)G^{2/3}(z)]^{1/4}(\Psi_{1,1} + \Psi_{1,2})$$
$$- \sqrt{\pi}\, e^{i\pi/6} n^{-1/6}[(z+c_a)G^{2/3}(z)]^{-1/4}(\Psi_{2,1} + \Psi_{2,2}).$$

Simple calculus now gives

$$\Psi_{1,1} + \Psi_{1,2} = e^{-i\pi/6}\mathrm{Ai}(s),$$
$$\Psi_{2,1} + \Psi_{2,2} = e^{-i\pi/6}\mathrm{Ai}'(s),$$

where $s = n^{2/3}\lambda(z)$ and $\lambda(z) = (z-c_a)G^{2/3}(z)$. The result now follows by combining these expressions. $\square$

6. Conclusion. Relativistic Hermite polynomials are basically Jacobi polynomials, as was pointed out by Ismail [**12**]. Strong asymptotic formulae for Jacobi polynomials (with parameters depending on the degree) have been worked out earlier and we have shown that these results can be transferred to relativistic Hermite polynomials, which gives strong asymptotics on the oscillatory region, Theorem 3.1 for $N = aN + \alpha + 1/2$, $a \geq 0, \alpha \geq -1$ and Theorem 3.2 when $N/n \to \infty$. Away from the oscillatory region, the strong asymptotics are given by Theorem 4.1 for $N = an + \alpha + 1/2$ with $a > 0$; Theorem 4.2 for $N = \alpha + 1/2$ with $\alpha > -1/2$; and Theorem 4.3 when $N/n \to \infty$.

These formulas cover the strong asymptotics on $[-c_a + \varepsilon, c_a - \varepsilon]$ and $\varepsilon > 0$, with $c_a = \sqrt{1 + 2a}/a$ (the oscillatory region) and compact

subsets of $\mathbf{C} \setminus [-c_a, c_a]$. The results are not valid in the neighborhood of the endpoints $\pm c_a$. Classically, the strong asymptotic formulae of Plancherel and Rotach for Hermite polynomials deal with three regions [**17**, Theorem 8.22.9]: the oscillatory interval where the Hermite polynomials behave like a trigonometric function; the region away from this interval, where the Hermite polynomials behave like hyperbolic functions; and the regions in the neighborhood of the endpoints of the interval, where the Hermite polynomials behave like Airy functions. Recently, a very interesting technique, based on a Riemann-Hilbert problem for matrix valued functions, has been used to sharpen these results for Hermite polynomials. The Riemann-Hilbert method gives uniform strong asymptotics which hold everywhere in the complex plane. Furthermore the method can be used for a much wider class of orthogonal polynomials [**4, 5, 6**]. We use the same Riemann-Hilbert approach in Section 5 and show that the analysis in [**5**] still works for relativistic Hermite polynomials. All the details are worked out in Section 5 for the case $N = an$, $a > 0$, and as a result we obtain the strong asymptotics on closed intervals in $(-c_a, c_a)$ (Theorem 5.2); the strong asymptotics on $\mathbf{C} \setminus [-c_a, c_a]$ (Theorem 5.1), and the Airy type asymptotics near the endpoints $\pm c_a$ (Theorem 5.3). The three regions are covered simultaneously with this Riemann-Hilbert approach.

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