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SURFACES OF CONSTANT GAUSS CURVATURE IN LORENTZ-MINKOWSKI THREE-SPACE

RAFAEL LÓPEZ

ABSTRACT. Let M be a space-like or time-like surface in Lorentz-Minkowski three-space $\mathbf{L}^3$ generated by a one-parameter family of circular arcs. We show that if the Gauss curvature K is a nonzero constant, then M is a surface of revolution. We also describe the parametrizations for M when $K \equiv 0$.

1. Introduction and statement of results. Let $\mathbf{L}^3$ be the Lorentz-Minkowski three-dimensional space with scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$$

for vectors $\mathbf{x}$ and $\mathbf{y}$ given in terms of natural coordinates. A surface M is called *cyclic* if there is a one-parameter family of planes which meet M in pieces of circles. We will also say that M is *foliated by circles*. Surfaces of revolution are the best known examples of cyclic surfaces.

The author showed in [**8**] that the only cyclic surfaces with constant nonzero Gauss curvature K in Euclidean three-space $\mathbf{E}^3$ are surfaces of revolution. We also gave there a simple description of cyclic surfaces in $\mathbf{E}^3$ with zero Gauss curvature, in observing that such surfaces need not be surfaces of revolution. In this article we will consider nondegenerate cyclic surfaces in Lorentz-Minkowski ambient space with constant Gauss curvature. We prove the following statements.

Theorem 1. *Let M be a nondegenerate cyclic surface in $\mathbf{L}^3$ with constant Gauss curvature. Then M is a portion of a pseudosphere or a pseudohyperbolic surface unless the planes of the foliation are parallel.*

Theorem 2. *Let M be a nondegenerate surface M in $\mathbf{L}^3$ foliated by circles in parallel planes and with constant Gauss curvature K.*

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1. *If $K \neq 0$, then M is a surface of revolution.*

2. *If $K = 0$, then M can be described, up to a rigid motion of $\mathbf{L}^3$, by one of the following parametrizations*:

(a) $\mathbf{X}(u,v) = (a(u), b(u), u) + r(u)(\cos v, \sin v, 0)$ *with* $r(u) > 0$, $a(u)$ *and* $b(u)$ *linear functions, if the planes of the foliation are space-like.*

(b) $\mathbf{X}(u,v) = (u, a(u), b(u)) + r(u)(0, \cosh v, \sinh v)$ *or* $\mathbf{X}(u,v) = (u, a(u), b(u)) + r(u)(0, \sinh v, \cosh v)$, *with* $r(u) > 0$, $a(u)$ *and* $b(u)$ *linear functions, if the planes of the foliation are time-like.*

(c) $\mathbf{X}(u,v) = (a(u) + v, b(u) + u, b(u) - u) + r(u)(0, v^2/2, v^2/2)$ *with* $1/r(u)$, $a(u)$ *and* $b(u)$ *linear functions, if the planes of the foliating are light-like.*

As a consequence of Theorems 1 and 2, we have:

Corollary 1. *A nondegenerate cyclic surface in* $\mathbf{L}^3$ *with* $K = \text{constant} \neq 0$ *is a surface of revolution.*

In Eucliean ambient space, the study of cyclic surfaces with constant mean curvature was initiated by Riemann, showing a family of minimal (nonrotational) surfaces in $\mathbf{E}^3$ foliated by circles in parallel planes with the exception of a discrete set of straight lines [**13**]. Enneper proved that, if a minimal surface is foliated by circles, the planes containing the circles must be parallel, and then the surface is the catenoid or one of the examples done by Riemann [**1**]. These results have been studied to higher dimensions and constant mean curvature [**3, 4, 11**] as well as other ambient spaces [**4, 5, 6, 7**].

This paper is organized as follows. Section 2 gives the classical local formulas for K and a description of the surfaces of revolution in $\mathbf{L}^3$ with constant Gauss curvature. Sections 3 and 4 contain the proofs of Theorems 1 and 2, respectively.

2. Preliminaries. Let $\mathbf{L}^3$ be the three-dimensional Lorentz-Minkowski space. We need to recall some of the definitions in this ambient space and that can be viewed, for example, in [**12**] and [**14**]. A plane in $\mathbf{L}^3$ is said to be space-like, respectively time-like or light-like, if its Euclidean unit normal vector is time-like, respectively space-like or light-like. Let M be a smooth surface immersed in $\mathbf{L}^3$. We say that M is space-like, respectively, time-like, if the induced metric is a Riemannian, respectively Lorentzian, metric in each tangent plane. This is equivalent to each tangent plane is space-like, respectively time-like or that, locally, the unit normal vector is time-like, respectively space-like, everywhere. For simplicity, a nondegenerate surface is a space-like surface is a space-like or time-like surface in $\mathbf{L}^3$.

In Lorentz-Minkowski space $\mathbf{L}^3$, two surfaces play the same role as spheres in $\mathbf{E}^3$: the pseudohyperbolic surface and the pseudosphere. The pseudohyperbolic surface of radius $r > 0$ is the quadric

$$\mathbf{H}^{2,1}(r) = \{\mathbf{p} \in \mathbf{L}^3; \langle \mathbf{p}, \mathbf{p} \rangle = -r^2\}.$$

This surface is space-like. From the Euclidean viewpoint, $\mathbf{H}^{2,1}(r)$ is the hyperboloid of two sheets $x_1^2 + x_2^2 - x_3^2 = -r^2$ which is obtained by rotating the hyperbola $x_1^2 - x_3^2 = -r^2$ in the plane $x_2 = 0$ with respect to the x_3-axis. The second surface is the pseudosphere or Lorentz sphere $\mathbf{S}^{2,1}(r)$:

$$\mathbf{S}^{(2,1)}(r) = \{\mathbf{p} \in \mathbf{L}^3; \langle \mathbf{p}, \mathbf{p} \rangle = r^2\}.$$

This surface is time-like and obtained by rotating the hyperbola $x_1^2 - x_3^2 = 1$ in the plane $x_2 = 0$ with respect to the x_3-axis.

Let M be a nondegenerate connected surface in $\mathbf{L}^3$. Consider a unit normal vector field ν of M. If M is space-like, respectively time-like, the vector field ν defines the Gauss map $\nu : M \to \mathbf{H}^{2,1}(1)$, respectively $\nu : M \to \mathbf{S}^{2,1}(r)$. The Gauss curvature of M is defined as the determinant of the Weingarten endomorphism $-d\nu$:

$$K = \det(-d\nu).$$

In terms of a local parametrization $\mathbf{X}(u, v)$, K is given by

$$K = \frac{eg - f^2}{EG - F^2},$$

where $\{E, F, G\}$ and $\{e, f, g\}$ are the coefficients of the first and second fundamental forms, respectively, of the immersion according to the Gauss map

$$\nu = \frac{\mathbf{X}_u \wedge \mathbf{X}_v}{|\mathbf{X}_u \wedge \mathbf{X}_v|}.$$

Here $\wedge$ denotes the Lorentzian cross product and

$$\begin{aligned} E &= \langle \mathbf{X}_u, \mathbf{X}_u \rangle, \quad F = \langle \mathbf{X}_u, \mathbf{X}_v \rangle, \quad G = \langle \mathbf{X}_v, \mathbf{X}_v \rangle, \\ e &= \langle \nu, \mathbf{X}_{uv} \rangle, \quad\ \ f = \langle \nu, \mathbf{X}_{uv} \rangle, \quad\ \ g = \langle \nu, \mathbf{X}_{vv} \rangle, \end{aligned}$$

where the subscripts denote the corresponding derivatives. Notice that

$$W =: |\mathbf{X}_u \wedge \mathbf{X}_v|^2 = EG - F^2 \begin{cases} \text{is positive} & \text{if } M \text{ is space-like} \\ \text{is negative} & \text{if } M \text{ is time-like.} \end{cases}$$

A simple computation of K in local coordinates yields

$$[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}][\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}] - [\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}]^2 - \varepsilon K W^2 = 0, \tag{1}$$

where $\varepsilon = 1$, respectively, $\varepsilon = -1$, if M is space-like, respective if M is time-like, and $[,\,]$ denotes the determinant in $\mathbf{L}^3$:

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}] = \langle \mathbf{v}_1 \wedge \mathbf{v}_2, \mathbf{v} \rangle \quad \forall\, \mathbf{v} \in \mathbf{L}^3.$$

In particular the pseudohyperbolic surface $\mathbf{H}^{2,1}(r)$ and the pseudospheres $\mathbf{S}^{2,1}(r)$ have constant Gauss curvature $K = 1/r^2$ (note that $\mathbf{H}^{2,1}(r)$ and $\mathbf{S}^{2,1}(r)$ have $-1/r^2$ and $1/r^2$ as intrinsic curvature, respectively). Since in our reasonings we only use the constancy of the Gauss curvature, independently if the surface is space-like or time-like, we put $C = \varepsilon K$ and equation (1) writes as

$$[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}][\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}] - [\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}]^2 - C W^2 = 0. \tag{2}$$

Let us now introduce the concept of circle in $\mathbf{L}^3$. A *circle* in $\mathbf{L}^3$ is a planar curve having constant nonzero curvature. Though there is just one kind of circle in Euclidean space, the nature of a circle in $\mathbf{L}^3$ varies with the causal character (space-like, time-like or light-like) of the plane containing the curve.

To describe any circle in $\mathbf{L}^3$, we first choose an orthonormal basis $B = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in $\mathbf{L}^3(\langle \mathbf{e}_3, \mathbf{e}_3 \rangle = -1)$ adapted to the plane P containing the circle as follows (see [**5**] for a detailed explanation).

1. If P is space-like, take P parallel to span $(\mathbf{e}_1, \mathbf{e}_2)$. Then any circle in $\mathbf{L}^3$ included in P is given by

$$\alpha(s) = \mathbf{c} + r(\cos s\mathbf{e}_1 + \sin s\mathbf{e}_2) \tag{3}$$

with $r > 0$ and $\mathbf{c} \in P$.

2. If P is time-like, take P parallel to span $(\mathbf{e}_2, \mathbf{e}_3)$. Then any circle in $\mathbf{L}^3$ included in P is given by

$$\alpha(s) = \mathbf{c} + r(\sinh s\mathbf{e}_2 + \cosh s\mathbf{e}_3) \quad \text{type I} \tag{4}$$

or

$$\alpha(s) = \mathbf{c} + r(\cosh s\mathbf{e}_2 + \sinh s\mathbf{e}_3) \quad \text{type II} \tag{5}$$

with $r > 0$ and $\mathbf{c} \in P$.

3. If P is a light-like plane, take P parallel to span $(\mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3)$. Then any circle in $\mathbf{L}^3$ included in P is given by

$$\alpha(s) = \mathbf{c} + s\mathbf{e}_1 + \frac{rs^2}{2}(\mathbf{e}_2 + \mathbf{e}_3), \tag{6}$$

with $r > 0$ and $\mathbf{c} \in P$.

In the case that $B = ((1,0,0),(0,1,0),(0,0,1))$ is the usual basis in $\mathbf{L}^3$, then the circles are Euclidean circles, hyperbolas and parabolas if the plane containing the circle is space-like, time-like or light-like, respectively.

A *surface of revolution* is a surface whose image is stable under a one-parameter group of rotations of $\mathbf{L}^3$. In particular, the planes containing the circles are parallel. Recall that a surface M is *cyclic* if it is foliated by circles. Denote by u the parameter of this family. Then there are three types of cyclic surfaces in $\mathbf{L}^3$ depending on whether the planes containing the circles are space-like, time-like or light-like. The corresponding local parametrizations of the surfaces are given by (3), (4)–(5) and (6), where now $\mathbf{c}, r, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are smooth functions on the parameter u.

To close this section we describe the nondegenerate surfaces of revolution of constant Gauss curvature in $\mathbf{L}^3$. When the rotation axis is space-like, some partial descriptions of such surfaces were done in [**2**] for space-like surfaces. The classification depends on the causal character of the rotation axis. In each case the surface of revolution of constant Gauss curvature is governed by an ordinary differential equation of second order where a first integral will be obtained. We shall give explicit examples by solving some particular cases of these equations, in particular $\mathbf{H}^{2,1}(r)$ and $\mathbf{S}^{2,1}(r)$. We distinguish three cases.

1. *The rotation axis is time-like.* After an isometry of $\mathbf{L}^3$ and according to (3) we may suppose that the x_3-axis is the rotation axis of the surface. Then the surface is parametrized as

$$\mathbf{X}(u,v) = (r(u)\cos v, r(u)\sin v, u).$$

The metric ds^2 is given by

$$ds^2 = (r'^2 - 1)du^2 + r^2\, dv^2, \quad W = r^2(r'^2 - 1).$$

An easy computation of (2) yields

$$-Cr + 2Crr'^2 - Crr'^4 - r'' = 0,$$

where the prime denotes the derivative with respect to u. Letting $y = r'$ and $x = r$ as new dependent and independent variables, this equation transforms in $-Cx + 2Cxy^2 - Cxy^4 - yy' = 0$. This equation is integrable and leads to

$$r'^2 = \frac{-1 - Cr^2 + \lambda}{-Cr^2 + \lambda}, \quad \lambda \in \mathbf{R}. \tag{7}$$

We obtain explicit examples putting $\lambda = 0$. For $C \neq 0$, the solution of (7) is

$$r(u) = \sqrt{-\frac{1}{C} + u^2 + 2\sqrt{\pm C}\mu u + C\mu^2}.$$

With the change $u \to u + \sqrt{\pm C}\mu$, then $r(u) = \sqrt{-(1/C) + u^2}$ and the surface is

$$\mathbf{X}(u,v) = \Big(\sqrt{-\frac{1}{C} + u^2}\cos v, \sqrt{-\frac{1}{C} + u^2}\sin v, u - \sqrt{\pm C}\mu\Big).$$

This surface is the vertical translate of a pseudohyperbolic surface, $C > 0$, or a pseudosphere, $C < 0$. If $C = 0$, then $r'' = 0$ and r is a linear function. We remark that if $\lambda = 0$, the first fundamental form is $W = 1/C$. Therefore the case $K < 0$ is impossible, that is, $\varepsilon = 1$,

$K < 0$ and $\varepsilon = -1$, $K < 0$. Therefore, for $\lambda = 0$ in (7), there exist no (space-like or time-like) surfaces of revolution with constant negative Gauss curvature.

2. *The rotation axis is space-like.* Without loss of generality, we assume the x_1-axis is the rotation axis of the surface. The corresponding parametrization of the surface has the following possibilities:

$$\begin{aligned} \mathbf{X}(u,v) &= (u, r(u)\cosh v, r(u)\sinh v) \quad \text{type I}, \\ \mathbf{X}(u,v) &= (u, r(u)\sinh v, r(u)\cosh v) \quad \text{type II}. \end{aligned}$$

In this case the metric ds^2 is

$$\begin{aligned} ds^2 &= (1 + r'^2)\, du^2 - r^2\, dv^2, \quad W = -r^2(1 + r'^2) \quad \text{type I}, \\ ds^2 &= (1 - r'^2)\, du^2 - r^2\, dv^2, \quad W = r^2(1 - r'^2) \quad \text{type II}. \end{aligned}$$

(i) Surfaces of type I. The computation of C yields

$$-Cr - 2Crr'^2 - Crr'^4 + r'' = 0.$$

Again we put $y = r'$ and $x = r$. The above equation becomes an exact equation:

$$r'^2 = \frac{-1 - Cr^2 - \lambda}{Cr^2 + \lambda}.$$

In the searching of explicit examples, choose $\lambda = 0$. Then $C > 0$ is impossible. Assume $C < 0$. The corresponding solution is $r(u) = \sqrt{-(1/C) - u^2 - 2\sqrt{-C}\mu u + C\mu^2}$. Let us change u by $u - \sqrt{-C}\mu$ and we obtain

$$\mathbf{X}(u,v) = \Big(u - \sqrt{-C}\mu, \sqrt{-(1/C) - u^2}\cosh v, \sqrt{(-1/C) - u^2}\sinh v\Big).$$

This surface is a translate in the x_1-axis direction of a pseudosphere. The case $C = 0$ leads to $r'' = 0$ and r is a linear function.

(ii) Surfaces of type II. Now we have

$$-Cr + 2Crr'^2 - Crr'^4 + r'' = 0.$$

A first integral is obtained as above:

$$r'^2=\frac{-1+Cr^2+\lambda}{Cr^2+\lambda}.$$

Once again let $\lambda=0$ to obtain examples. Then $W=1/C$ and the case $K<0$ is again impossible. If $C\neq 0$, then $r(u)=\sqrt{(1/C)+u^2+2\sqrt{\pm C}u\mu u++C\mu^2}$ and it is possible to reparametrize the surface to get the pseudohyperbolic surface or the pseudosphere. The case $C=0$ yields that r is a linear function.

3. *The rotation axis is light-like.* After a rigid motion of the ambient space, we may assume that the axis of revolution is the line $\{x_1=0,x_2-x_3=0\}$. Following (6), take the parameter $v=s$ and the line of centers of the circles as a graph on the line $\{x_1=0,x_2+x_3=0\}$ in the plane $x_1=0$. Then the surface is parametrized as

$$\mathbf{X}(u,v)=(v,b(u)+u,b(u)-u)+r(u)\Big(0,\frac{v^2}{2},\frac{v^2}{2}\Big).$$

The metric of M is given by

$$ds^2=(4b'+2r'v^2)\,du^2+4rv\,du\,dv+dv^2,\quad W=2v^2(r'-2r^2)+4b'.$$

The computation of expression (2) leads to

$$\begin{aligned}(8)\quad 4(-4Cb'^2+rb'')+(32Cr^2b'-16Cb'r'-4r'^2+2rr'')v^2&\\ +4C(2r^2-r')v^4=0.&\end{aligned}$$

This expression is a polynomial on v and thus all coefficients vanish. In particular and for $C\neq 0$, the coefficient of v^4 yields $r'=2r^2$, whose solution is $r(u)=1/(-2u+\lambda)$, $\lambda\in\mathbf{R}$. We obtain explicit examples by choosing $\lambda=0$. In this case $r(u)=-1/(2u)$ and the independent coefficient in (7) is $8Cb'^2+b''/u=0$. Solving this differential equation we obtain (up a homothety on the parameter u):

$$\begin{aligned}C>0\quad b(u)&=\frac{\arctan\sqrt{C}u}{\sqrt{C\mu}},\quad \mu\in\mathbf{R},\\ C<0\quad b(u)&=\frac{\arctan h\sqrt{-C}u}{\sqrt{-C\mu}},\quad \mu\in\mathbf{R}.\end{aligned}$$

When $C = 0$, identity (8) writes as $-4rb'' + 2(-2r'^2 + rr'')v^2 = 0$. Then $b'' = 0$, that is, b is a linear function. Moreover, r satisfies $rr'' = 2r'^2$, whose solution is, up homothety in the parameter u, $r(u) = \mu e^u$, $\mu \in \mathbf{R}$.

3. Proof of Theorem 1. After a homothety in $\mathbf{L}^3$, we may assume that the constant C in equation (2) is $C = -1, 0$ or 1. The proof of Theorem 1 follows adopting the method in [**10**, pp. 85–89] in showing that the only minimal surfaces in Euclidean three-space generated by a family of circles are the catenoid and the Riemann's cyclic surfaces (see [**5**] for an example of this method in the Lorentzian context).

Consider a real interval I and $u \in I$ the parameter of each plane of the foliation that defines M. Let $\mathbf{N}(u)$ be a smooth unit vector field orthogonal in $\mathbf{E}^3$ to each u-plane. The reasoning is by contradiction. Assume that the u-planes are not parallel. Then $\mathbf{N}'(u) \neq 0$ in some real interval. In a first step we will assume that in this interval the planes that intersect M in circular arcs have the same causal character. Consider an integral curve Γ of the vector field $\mathbf{N}$. Then Γ is not a straight line and thus there exists a Frenet frame of Γ. Now we consider a suitable parametrization of our surface in terms of this frame. The computation of (2) yields a real trigonometric polynomial or a polynomial in one variable which vanishes in some real interval. The fact that the coefficients of these polynomials vanish will imply that our surface is a subset of the pseudohyperbolic surface or the pseudosphere. Explicit computation of these polynomials involves a hard task to do by hand. In this situation the aid of the computer and a symbolic program, such as Mathematica, makes the explicit computation of these polynomials easier (see [**9**] for a description of how Mathematica helps in these computations).

After this work, let S, respectively T, L, denote the subsets of the interval I where the corresponding u-plane is space-like, respectively time-like or light-like. The space-like and time-like causal character is an open condition and so S and U are open sets in I. The reasoning in the above paragraph proves that $\mathbf{N}$ is constant in open intervals of S, T and L. Since $\mathbf{N}$ is a continuous function, $\mathbf{N}$ is locally constant on the closure of these intervals. An argument of connectedness shows that $\mathbf{N}$ is constant on I and this means that the foliating planes are parallel.

In the next three subsections we distinguish the character causal of the planes of the foliation.

3.1 The planes are space-like. Let $\mathbf{e}_1(u), \mathbf{e}_2(u)$ be an orthonormal basis in each u-plane. Then M can be parametrized as (see (3))

$$\mathbf{X}(u,v) = \mathbf{c}(u) + r(u)(\cos v\mathbf{e}_1(u) + \sin v\mathbf{e}_2(u)).$$

Assume that the u-planes of the foliation are not parallel. The proof consists in showing that M is a subset of a pseudohyperbolic surface or a pseudosphere. Since the curve Γ is not a straight line, its curvature κ is not identically zero. Consider an open interval of the parameter u where $\kappa \neq 0$. In this interval the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is well defined, where $\mathbf{t} = \mathbf{N}$ is the unit tangent vector to Γ. The moving frame satisfies the following Frenet equations:

$$\begin{aligned} \mathbf{t}' &= \kappa\mathbf{n} \\ \mathbf{n}' &= \kappa\mathbf{t} + \sigma\mathbf{b} \\ \mathbf{b}' &= -\sigma\mathbf{n} \end{aligned}$$

Here $\mathbf{b} = \mathbf{t} \wedge \mathbf{n}$. After a change of coordinates, the surface M may be parametrized as

$$\mathbf{X}(u,v) = \mathbf{c}(u) + r(u)\cos v\mathbf{n}(u) + r(u)\sin v\mathbf{b}(u).$$

Put $\mathbf{c}' = \alpha\mathbf{t} + \beta\mathbf{n} + \gamma\mathbf{b}$, where α, β, γ are smooth functions on u. Equation (2) may be written as

$$\sum_{n=0}^{4} A_n(u)\cos nv + \sum_{n=1}^{4} B_n(u)\sin nv = 0,$$

for some functions A_n and B_n. The left side of this equation is a polynomial on the independent functions $\cos nv$, $\sin nv$. Therefore all functions A_n, B_n vanish. The computation of B_4 yields

$$B_4 = \frac{\beta\gamma r^4(-2C\beta^2 + 2C\gamma^2 + \kappa^2 + 2C\kappa^2 r^2)}{4}.$$

From $B_4 = 0$, we have three possibilities:

1. $\beta = 0$. Then

$$A_4 = \frac{-r^4(\gamma^2 + \kappa^2 r^2)(\kappa^2 + C\gamma^2 + C\kappa^2 r^2)}{8}.$$

Since $A_4 = 0$, it follows that $\gamma^2 = -\kappa^2(r^2 + C)$ (this occurs only for $C = -1$). If $\gamma = 0$ we are in the case 2 below. We then assume $r^2 \neq 1$ and the following holds

$$A_3 = \frac{\alpha\kappa^3 r^5}{4}, \quad B_3 = \pm\frac{\kappa^3 r^6 r'}{4\sqrt{1 - r^2}}.$$

Thus $\alpha = 0$ and $r' = 0$. Now the coefficient A_0 yields $A_0 = \kappa^4 r^6(r^2 - 1) = 0$, a contradiction.

2. $\gamma = 0$. Now

$$A_4 = \frac{r^4(-\beta^2 + \kappa^2 r^2)(C\beta^2 - \kappa^2 - C\kappa^2 r^2)}{8}.$$

We have two cases:

(a) $\beta^2 = \kappa^2 r^2$. Then

$$A_3 = -\frac{3\kappa^3 r^5(\alpha \mp r')}{4},$$

where $\mp$ corresponds to $\beta = \mp\kappa r$, respectively. This implies $\alpha = \pm r'$. But in both cases, we have $W = 0$ and this would imply that M is not an immersion, a contradiction.

(b) $\beta^2 = \kappa^2(r^2 + C)$, for $C \neq 0$. Now

$$A_3 = \frac{\kappa^3 r^5}{4}\Big(\alpha \mp \frac{rr'}{\sqrt{r^2 + C}}\Big) = 0,$$

depending on where $\beta = \pm\kappa\sqrt{r^2 + C}$, respectively. Then $\alpha = \pm(rr')/\sqrt{1 + r^2}$. This case is new and gives the possibility that the planes would not be parallel. In this case we shall show that M is included in $\mathbf{H}^{2,1}(1)$ or in $\mathbf{S}^{2,1}(1)$. For simplicity of presentation, choose the sign $+$ in β. We return with the expression of $\mathbf{c}'$. By using the Frenet equations, we obtain:

$$\mathbf{c}' = \frac{rr'}{\sqrt{r^2 + C}}\mathbf{t} + \kappa\sqrt{r^2 + C}\mathbf{n} = \frac{rr'}{\sqrt{r^2 + C}}\mathbf{t} + \sqrt{C + r^2}\mathbf{t}' = (\sqrt{r^2 + C}\mathbf{t})'.$$

Integrating this expression, there exists $\mathbf{c}_0 \in \mathbf{L}^3$ such that $\mathbf{c} = \mathbf{c}_0 + \sqrt{r^2 + C}\mathbf{t}$. Therefore the parametrization of M is

$$\mathbf{X}(u,v) = \mathbf{c}_0 + \sqrt{r^2 + C}\mathbf{t} + r\cos v\mathbf{n} + r\sin v\mathbf{b}.$$

This implies that $|\mathbf{X}(u,v) - \mathbf{c}_0|^2 = -C$m, that is, M is a subset of the pseudo-hyperbolic surface, $C = 1$, or the pseudosphere $C = -1$.

3. $-2C\beta^2 + 2C\gamma^2 + \kappa^2 + 2C\kappa^2 r^2 = 0$, for $C \neq 0$. Because $C^2 = 1$, we have

$$\beta^2 = \gamma^2 + \kappa^2 r^2 + C\kappa^2/2.$$

Substituting in $A_4 = 0$, we obtain

$$16\gamma^4 + (16\kappa^2 r^2 + 8C\kappa^2)\gamma^2 + \kappa^4 = 0,$$

or

$$(4\gamma^2 + C\kappa^2)^2 + 16\kappa^2 r^2 \gamma^2 = 0.$$

This implies $\gamma = C = 0$, a contradiction. $\square$

3.2 The planes are time-like. Consider $\mathbf{e}_1(u), \mathbf{e}_2(u)$ an orthonormal basis in each u-plane of the foliation, where $-\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 1$. Then a local parametrization of M is given by (see (4)–(5))

$$\mathbf{X}(u,v) = \mathbf{c}(u) + r(u)(\cosh v\mathbf{e}_1(u) + \sinh v\mathbf{e}_2(u)), \quad \text{type I},$$
$$\mathbf{X}(u,v) = \mathbf{c}(u) + r(u)(\sinh v\mathbf{e}_1(u) + \cosh v\mathbf{e}_2(u)), \quad \text{type II}.$$

For simplicity we only consider the first case. Assume that the foliating planes are not parallel and that in some u-interval the curvature κ of Γ satisfies $\kappa \neq 0$. Here the tangent vector field $\mathbf{t} = \mathbf{N}$ along Γ is space-like. We distinguish three cases depending on the causal character of the derivative $\mathbf{t}'$:

First case. $\mathbf{t}'$ is space-like. Consider $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the Frenet frame of $\Gamma(\mathbf{b} = \mathbf{t} \wedge \mathbf{n})$. Now $\mathbf{t}$ and $\mathbf{n}$ are space-like and β is time-like. In this situation the Frenet equations are

$$\mathbf{t}' = \kappa\mathbf{n},$$
$$\mathbf{n}' = -\kappa\mathbf{t} + \sigma\mathbf{b}$$
$$\mathbf{b}' = \sigma\mathbf{n}.$$

A change of coordinates leads to

$$\mathbf{X}(u,v)=\mathbf{c}(u)+r(u)\sinh v\mathbf{n}(u)+r(u)\cosh v\mathbf{b}(u).$$

Consider the identity (2). In this case we have

$$\sum_{n=0}^{4} A_n(u)\cosh nv+\sum_{n=1}^{4} B_n(u)\sinh nv=0.$$

Again all coefficients A_n, B_n vanish. Here

$$B_4=\frac{\beta\gamma r^4(2C\beta^2+2C\gamma^2+\kappa^2-2C\kappa^2r^2)}{4}.$$

From $B_4=0$, we have three cases:

1. $\beta=0$. Then

$$A_4=\frac{r^4(-\gamma^2+\kappa^2r^2)(C\gamma^2+\kappa^2-C\kappa^2r^2)}{8}.$$

Hence we have two possibilities:

(a) $\gamma^2=\kappa^2r^2$. Then $\gamma=\pm\kappa r$ and

$$A_3=\mp\frac{3\kappa^3r^5r'}{4},\quad B_3=-\frac{3\alpha\kappa^3r^5}{4}.$$

Thus $r'=\alpha=0$. We recalculate the coefficients and we see that $A_2=-\kappa^4r^6$, a contradiction.

(b) $\gamma^2=\kappa^2(r^2-C)$, for $C\neq 0$. When $\gamma=0$, this possibility is treated in the case 2 below. Thus we assume $r^2\neq C$. Now

$$A_3=-\frac{\kappa^3r^6r'}{4\sqrt{r^2-C}},\quad B_3=\frac{\alpha\kappa^3r^5}{4}.$$

Hence a computation of A_0 in (2) leads to $A_0=\kappa^4r^6(r^2-C)=0$, a contradiction.

2. $\gamma=0$. Then

$$A_4=\frac{r^4(-\beta^2+\kappa^2r^2)(C\beta^2+\kappa^2-C\kappa^2r^2)}{8}.$$

Hence $\beta^2 = \kappa^2 r^2$ or $\beta^2 = \kappa^2(r^2 - C)$.

(a) $\beta^2 = \kappa^2 r^2$. Now

$$B_3 = -\frac{3\kappa^3 r^5(a \mp r')}{4},$$

where $\mp$ corresponds with $\beta = \pm\kappa r$, respectively. Then $\alpha = \pm r'$ and, with this value for α, $W = 0$, a contradiction.

(b) $\beta^2 = \kappa^2(r^2 - C)$, for $C \neq 0$. Then $\beta = \pm\kappa\sqrt{r^2 - C}$ and

$$B_3 = \frac{\kappa^3 r^5}{4}\Big(\alpha \mp \frac{rr'}{\sqrt{r^2 - C}}\Big).$$

For simplicity, we assume that $\alpha = rr'/\sqrt{r^2 - C}$. By using the Frenet equations, we have

$$\mathbf{c}' = \frac{rr'}{\sqrt{r^2 - C}}\mathbf{t} + \kappa\sqrt{r^2 - C}\mathbf{n} = (\sqrt{r^2 - C}\mathbf{t})'.$$

Again there exists $\mathbf{c}_0 \in \mathbf{L}^3$ such that $\mathbf{c} = \mathbf{c}_0 + \sqrt{r^2 - C}\mathbf{t}$. The parametrization of M is given by

$$\mathbf{X}(u, v) = \mathbf{c}_0 + \sqrt{r^2 - C}\mathbf{t} + r\cosh v\mathbf{n} + r\sinh v\mathbf{b},$$

and this implies $|\mathbf{X}(u, v) - \mathbf{c}_0|^2 = -C$. Therefore, the surface is included in the pseudohyperbolic surface or the pseudosphere.

3. $2C\beta^2 + 2C\gamma^2 + \kappa^2 - 2C\kappa^2 r^2 = 0$, for $C \neq 0$. Since $C^2 = 1$,

$$\beta^2 = -\gamma^2 + \kappa^2 r^2 - C\kappa^2/2.$$

Then

$$A_4 = \frac{Cr^4((4\gamma^2 + C\kappa^2)^2 - 16\kappa^2 r^2\gamma^2)}{32}.$$

From $A_4 = 0$, it follows that $4\gamma^2 + C\kappa^2 = \pm 4\kappa r\gamma$. Thus

$$\gamma = \frac{\kappa}{2}(\pm r \pm \sqrt{r^2 - C}), \quad \beta^2 = \frac{\kappa^2}{4}(r \mp \sqrt{r^2 - C})^2.$$

Let us return to equation (2) and we recalculate the coefficients. In particular,

$$A_3 = \frac{\kappa^3 r^5}{8}\Big(4\alpha + r'\Big(-3 \pm \frac{r}{\sqrt{r^2 - C}}\Big)\Big).$$

Thus $A_3 = 0$ gives

$$\alpha = \frac{(3 \mp (r/\sqrt{r^2 - C}))r'}{4}.$$

With this value of α, we have

$$B_3 = \frac{3\kappa^3 r^5 r'}{16}\Big(1 \pm \frac{r}{\sqrt{r^2 - C}}\Big).$$

Then $B_3 = 0$ implies $r' = 0$. Hence

$$A_2 = -\frac{\kappa^4 r^5(r \pm \sqrt{r^2 - C}}{4},$$

and identity $A_3 = 0$ gets a contradiction.

Second case. $\mathbf{t}'$ is time-like. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame of Γ. Now

$$\begin{aligned} \mathbf{t}' &= \kappa \mathbf{n} \\ \mathbf{n}' &= \kappa \mathbf{t} + \sigma \mathbf{b} \\ \mathbf{b}' &= \sigma \mathbf{n}. \end{aligned}$$

Moreover, $\kappa \not\equiv 0$ because Γ is not a straight line and consider a u-interval where $\kappa \neq 0$. Equation (2) yields

$$\sum_{n=0}^{4} A_n(u) \cosh nv + \sum_{n=1}^{4} B_n(u) \sinh nv = 0,$$

for some functions A_n and B_n. Then

$$B_4 = \frac{\beta\gamma r^4(2C\beta^2 + 2C\gamma^2 + \kappa^2 - 2C\kappa^2 r^2)}{4}.$$

Once again, one has three possibilities:

1. $\beta = 0$. Then

$$A_4 = \frac{r^4(-\gamma^2 + \kappa^2 r^2)(C\gamma^2 + \kappa^2 - C\kappa^2 r^2)}{8}.$$

(a) $\gamma^2 = \kappa^2 r^2$. We recalculate (2) and we see that

$$A_3 = \frac{3\alpha\kappa^3 r^5}{4}, \quad B_3 = \pm\frac{3\kappa^3 r^5 r'}{4}.$$

From these expressions we get $r' = \alpha = 0$. Then $A_2 = \kappa^4 r^6$, a contradiction.

(b) $\gamma^2 = \kappa^2(r^2 - C)$, for $C \neq 0$. If $\gamma = 0$, we are in the case 2 below. Thus let us assume that $r^2 \neq C$. Hence,

$$A_3 = -\frac{\alpha\kappa^3 r^5}{4}, \quad B_3 = \mp\frac{\kappa^3 r^6 r'}{4\sqrt{r^2 - C}}.$$

This implies $r' = \alpha = 0$ and the coefficient A_0 in (2) is $A_0 = \pm C\kappa^4 r^6(r^2 - C)$, a contradiction.

2. $\gamma = 0$. Then

$$A_4 = \frac{r^4(-\beta^2 + \kappa^2 r^2)(C\beta^2 + \kappa^2 - C\kappa^2 r^2)}{8}.$$

From $A_4 = 0$, we have two cases:

(a) $\beta^2 = \kappa^2 r^2$. Then $\beta = \pm\kappa r$. Hence,

$$A_3 = \frac{3\kappa^3 r^5(\alpha \mp r')}{4}.$$

A calculation of W yields $W = 0$, a contradiction.

(b) $\beta^2 = \kappa^2(r^2 - C)$, for $C \neq 0$. Now $\beta = \pm\kappa\sqrt{r^2 - C}$. Then

$$A_3 = \frac{\kappa^3 r^5}{4}\Big(- \alpha \pm \frac{rr'}{\sqrt{r^2 - C}}\Big).$$

Without loss of generality, we assume $\alpha = rr'/\sqrt{r^2 - C}$. By using the Frenet equations, it follows that

$$\mathbf{c}' = \frac{rr'}{\sqrt{r^2 - C}}\mathbf{t} + \kappa\sqrt{r^2 - C}\mathbf{n} = (\sqrt{r^2 - C}\mathbf{t})'.$$

Then $\mathbf{c} = \mathbf{c}_0 + \sqrt{r^2 - C}\mathbf{t}$ for some $\mathbf{c}_0 \in \mathbf{L}^3$ and

$$\mathbf{X}(u, v) = \mathbf{c}_0 + \sqrt{r^2 - C}\mathbf{t} + r \cosh v \mathbf{n} + r \sinh v \mathbf{b}.$$

Thus the surface M satisfies $|\mathbf{X}(u, v) - \mathbf{c}_0|^2 = -C$ and this means that M is a subset of the pseudohyperbolic surface or the pseudosphere.

3. $2C\beta^2 + 2C\gamma^2 + \kappa^2 - 2C\kappa^2 r^2 = 0$, for $C \neq 0$. Since $C^2 = 1$, we obtain

$$\beta^2 = -\gamma^2 + \kappa^2 r^2 - C\kappa^2/2.$$

A calculation of the coefficient A_4 leads to

$$A_4 = \frac{Cr^4((4\gamma^2 + C\kappa^2)^2 - 16\kappa^2 r^2 \gamma^2)}{32}.$$

Then $A_4 = 0$ implies $4\gamma^2 + C\kappa^2 = \pm 4\kappa r \gamma$. Thus

$$\gamma = \frac{\kappa}{2}(\pm r \pm \sqrt{r^2 - C}),$$

and then

$$\beta^2 = \frac{\kappa^2}{4}(r \mp \sqrt{r^2 - C})^2.$$

It follows that

$$A_3 = \frac{\kappa^3 r^5}{8}\Big(2\alpha - r'\Big(\pm 3 + \frac{r}{\sqrt{r - C}}\Big)\Big).$$

Thus $A_3 = 0$ leads to

$$\alpha = \frac{[\pm 3 + (r/\sqrt{r^2 - C})]r'}{2}.$$

Now

$$B_3 = \frac{3\kappa^3 r^5 r'}{16}\Big(\mp 1 - \frac{r}{\sqrt{r^2 - C}}\Big).$$

Hence $r' = 0$. The coefficient A_2 yields

$$A_2 = \frac{\kappa^4 r^5 (r \pm \sqrt{r^2 - C})}{4},$$

and $A_2 = 0$ gets a contradiction.

Third case. $\mathbf{t}'$ is lightlike. Then $\langle \mathbf{t}', \mathbf{t}' \rangle = 0$. Since the plans are not parallel, $\mathbf{t}' \neq 0$. Put $\mathbf{n} = \mathbf{t}'$ and consider $\mathbf{b}$ the unique light-like vector orthogonal to $\mathbf{t}$ such that $\langle \mathbf{b}, \mathbf{n} \rangle = 1$ and $[\mathbf{t}, \mathbf{n}, \mathbf{b}] = 1$. The Frenet equations with respect to $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ are

$$\begin{aligned} \mathbf{t}' &= \mathbf{n} \\ \mathbf{n}' &= \sigma \mathbf{n} \\ \mathbf{b}' &= -\mathbf{t} - \sigma \mathbf{b}. \end{aligned}$$

Again a change of parameters yields that M may be written in the form

$$\mathbf{X}(u,v) = \mathbf{c}(u) + r(u) v \mathbf{n}(u) - \frac{r(u)}{2v} \mathbf{b}(u).$$

Then equation (2) is written as

$$\sum_{n=0}^{8} A_n(u) \frac{1}{v^n} = 0.$$

This is a real polynomial on the variable $1/v$. Therefore $A_n = 0$ for each n. The computation of the coefficient A_8 gives

$$A_8 = \frac{r^4(-\beta^2 + r^2)(1 + C\beta^2 - Cr^2)}{16}.$$

From $A_8 = 0$ we distinguish two cases:

1. $\beta^2 = r^2$. Hence $\beta = \pm r$. Then $A_7 = 3r^5(a \mp r')/8$ and so $\alpha = \pm r'$. Hence $A_6 = \pm \gamma r^5/2$. This implies $\gamma = 0$ and, from here, $W = 0$, a contradiction.

2. $\beta^2 = r^2 - C$, for $C \neq 0$. Hence $\beta = \pm\sqrt{r^2 - C}$. Then $A_0 = -\gamma^4 r^4$ and

$$A_7 = \frac{r^5}{8}\Big(- \alpha \pm \frac{rr'}{\sqrt{r^2 - C}} \Big).$$

Hence $\alpha = \pm rr'/\sqrt{r^2 - C}$. Assuming without loss of generality the choice $+$ for α, the Frenet equations of Γ lead $\mathbf{c}' = (\sqrt{r^2 - C}\mathbf{t})'$. Thus there is a $\mathbf{c}_0 \in \mathbf{L}^3$ such that

$$\mathbf{X}(u,v) = \mathbf{c}_0 + \sqrt{r^2 - C}\mathbf{t} + rv\mathbf{n} - \frac{r}{2v}\mathbf{b}.$$

In this case $\mathbf{X}$ satisfies the identity $|\mathbf{X}(u,v)-\mathbf{c}_0|^2=-C$. Therefore, the surface is a portion of the pseudohyperbolic surface or the pseudosphere.

3.3 The planes are light-like. In each u-plane of the foliation, let $\mathbf{e}_1(u), \mathbf{e}_2(u)$ be vectors such that $\langle \mathbf{e}_1, \mathbf{e}_1\rangle = 1$ and $\langle \mathbf{e}_2, \mathbf{e}_2\rangle = 0$. Then M may be written in the form (see (6))

$$\mathbf{X}(u,v)=\mathbf{c}(u)+v\mathbf{e}_1(u)+r(u)v^2\mathbf{e}_2(u).$$

Write $\mathbf{n}=\mathbf{e}_1$ and $\mathbf{t}=\mathbf{e}_2$. For each u, let $\mathbf{b}(u)$ be the unique light-like vector orthogonal to $\mathbf{n}(u)$ such that

$$\langle \mathbf{t}, \mathbf{b}\rangle = 1, \quad [\mathbf{t}, \mathbf{n}, \mathbf{b}] = 1.$$

Each u-plane of the foliation contains the vector $\mathbf{t}(u)$. Therefore, the planes containing the pieces of circles are parallel if and only if $\mathbf{t}$ is a constant vector. We can assume that $\mathbf{t}'=\kappa\mathbf{n}$ (in the contrary case, change $\mathbf{t}$ for $\bar{\mathbf{t}}=\phi\mathbf{t}$ for some function $\phi=\phi(u)$ and the function r changes into $\bar{r}=r/\phi$).

Assume that the u-planes are not parallel. Then $\kappa\neq 0$ in some interval of the parameter u and we work in this interval. Moreover, the Frenet equations are

$$\begin{aligned}\mathbf{t}' &= \kappa\mathbf{n}\\ \mathbf{n}' &= \sigma\mathbf{t}-\kappa\mathbf{b}\\ \mathbf{b}' &= -\sigma\mathbf{n}.\end{aligned}$$

In the above notation the surface is parametrized as

$$\mathbf{X}(u,v)=\mathbf{c}(u)+v\mathbf{n}(u)+r(u)v^2\mathbf{t}.$$

In this case equation (2) writes as

$$\sum_{n=0}^{6} A_n(u)v^n=0.$$

Then all coefficients A_n vanish. The computations of A_6 and A_5 lead to

$$\begin{aligned}A_6 &= -4C\kappa^2(r'-2\gamma r^2)^2\\ A_5 &= 2\kappa(r'-2\gamma r^2)(8\beta C\kappa r-8C\gamma^2r^2-3\kappa^2r^2+4C\gamma r'-4C\kappa\sigma).\end{aligned}$$

Thus $r' = 2\gamma r^2$ (if $C = 0$ we use the value of A_5). Now

$$A_4 = 4\kappa^2(2C\beta r - 2\kappa r^2 - C\sigma)(\sigma - 2\beta r).$$

From $A_4 = 0$, we have the following possibilities:

1. $\sigma = 2\beta r$. Then $A_3 = -10\alpha\kappa^3 r^2 = 0$, that is, $\alpha = 0$. But now $W = 0$, a contradiction.

2. $\sigma = 2\beta r - 2C\kappa r^2$, for $C \neq 0$. Then $A_3 = 6\kappa^3 r^2(\alpha - 2C\gamma r^2)$ and $A_3 = 0$ yields $\alpha = 2C\gamma r^2$. Then Frenet equations lead to

$$\mathbf{c}' = 2C\gamma r^2\mathbf{t} + \beta\mathbf{n} + \gamma\mathbf{b} = \left(Cr\mathbf{t} - \frac{\mathbf{b}}{\mathbf{2r}}\right)'.$$

Hence there exists $\mathbf{c}_0 \in \mathbf{L}^3$ such that $\mathbf{c} = \mathbf{c}_0 + Cr\mathbf{t} - (\mathbf{b}/2r)$. Thus

$$\mathbf{X}(u,v) = \mathbf{c}_0 + v\mathbf{n} + (rv^2 + Cr)\mathbf{t} - \frac{\mathbf{b}}{\mathbf{2r}}.$$

This implies $|\mathbf{X}(u,v) - \mathbf{c}_0|^2 = -C$ and M is indeed a subset of the pseudohyperbolic surface or the pseudosphere.

4. Proof of Theorem 2. Theorem 2 treats the case that the planes containing the pieces of circles are parallel. The proof is divided into three parts according to the causal character of the foliating planes. Recall that $C = -1, 0$ or 1 in equation (2).

1. *The planes are space-like.* After a rigid motion in $\mathbf{L}^3$, we may assume the planes are parallel to the plane $x_3 = 0$. In this situation the circles are Euclidean circles and M can be parametrized by

$$\mathbf{X}(u,v) = (a(u), b(u), u) + r(u)(\cos v, \sin v, 0),$$

where a, b and $r > 0$ are smooth functions on u. Equation (2) is written as

$$\sum_{n=0}^{r} A_n(u)\cos nv + \sum_{n=1}^{4} B_n(u)\sin nv = 0,$$

for some functions A_n and B_n. This expression is a polynomial on $\cos nv$, $\sin nv$ and thus $A_n = B_n = 0$ for all n. A simple computation of the coefficients A_4 and B_4 gives

$$A_4 = C\frac{r^4(-a'^4 + 6a'^2b'^2 - b'^4)}{8}$$
$$B_4 = C\frac{r^4a'b'(-a'^2 + b'^2)}{2}.$$

If $C \neq 0$ it follows from $A_4 = B_4 = 0$ that $a' = b' = 0$, and so the functions a and b are constant. This means that the curve of centers of the circles is a straight line orthogonal to each plane of the foliation. Thus all circles that define M are coaxial and M is a surface of revolution. In the case $C = 0$, we have that equation (2) yields

$$r^3(a''\cos v + b''\sin v + r'') = 0.$$

This immediately concludes $a'' = b'' = r'' = 0$, and the functions a, b and r are linear.

2. *The planes are time-like.* A rigid motion in $\mathbf{L}^3$ allows us to assume that the planes are parallel to the plane $x_1 = 0$. Then M may be written in the next two forms:

$$\mathbf{X}(u,v) = (u, a(u), b(u)) + r(u)(0, \cosh v, \sinh v) \quad \text{type I}$$
$$\mathbf{X}(u,v) = (u, a(u), b(u)) + r(u)(0, \sinh v, \cosh v) \quad \text{type II.}$$

Equation (2) yields in both cases:

$$\sum_{n=0}^{4} A_n(u)\cosh nv + \sum_{n=1}^{4} B_n(u)\sinh nv = 0.$$

Again all coefficients A_n, B_n vanish. Here

$$A_4 = C\frac{-r^4(a'^4 + 6a'^2b'^2 - b'^4)}{8}.$$

If $C \neq 0$, identity $A_4 = 0$ implies $a' = b' = 0$. Then M is again a surface of revolution. If $C = 0$, we obtain

$$r^3(a''\cosh v - b''\sinh v + r'') = 0 \quad \text{type I.}$$
$$r^3(-a''\sinh v + b''\cosh v + r'') = 0 \quad \text{type II.}$$

In both cases one concludes that $a'' = b'' = r'' = 0$.

3. *The planes are light-like.* After a rigid motion in $\mathbf{L}^3$, we may assume the foliating planes are parallel to the plane $x_2 - x_3 = 0$. Then M can be parametrized as

$$\mathbf{X}(u,v) = (a(u)+v, b(u)+u, b(u)-u) + r(u)\left(0, \frac{v^2}{2}, \frac{v^2}{2}\right),$$

where a, b and $r > 0$ are smooth functions on u. Then (2) leads to

$$\sum_{n=0}^{4} A_n(u)v^n = 0,$$

for some functions A_n. As a consequence, all coefficients A_n vanish. Assume $C \neq 0$. Then

$$A_4 = -4C(r' - 2r^2)^2.$$

Thus $r' = 2r^2$ and r is given by

$$r(u) = \frac{1}{-2u+\lambda}, \quad \lambda \in \mathbf{R}.$$

Now $A_2 = -16Cr^2a'^2 = 0$. From $A_2 = 0$ we have $a' = 0$ and thus M is a surface of revolution. If $C = 0$, then equation (2) is written as

$$(2rr'' - 4r'^2)v^2 - 4r^2a''v + 4rb'' = 0.$$

In particular, $a'' = b'' = 0$. Moreover, $rr'' - 2r'^2 = 0$. Therefore the radius function $r(u)$ is given by $r(u) = 1/(-\lambda u + \mu)$, $\lambda, \mu \in \mathbf{R}$.

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Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain
E-mail address: `rcamino@ugr.es`