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ASYMPTOTIC BEHAVIOR AND OSCILLATION OF DELAY PARTIAL DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT. We obtain sufficient conditions for the oscillation of all solutions of the linear partial difference equations with positive and negative coefficients of the form

$$A_{m-1,n} + A_{m,n-1} - A_{mn} + pA_{\substack{m+k\\n+l}} - qA_{\substack{m+k'\\n+l'}} = 0,$$

and

$$A_{m-1,n} + A_{m,n-1} - A_{mh} + p_{mn}A_{\substack{m+k\\n+l}} - q_{mn}A_{\substack{m+k'\\n+l'}} = 0,$$

where $m, n = 0, 1, \dots,$ and k, k', l', l are nonnegative integers $p, q \in (0, \infty)$, and coefficients $\{q_{mn}\}$ and $\{p_{mn}\}$ are sequences of nonnegative real numbers. In this paper $A_{\substack{m\\n}} = A_{m,n}$.

1. Introduction. Partial difference equations arise from various practical problems and numerical analysis of partial difference equations [**1-2**]. In this area, the oscillatory and nonoscillatory behaviors of delay partial difference equations have been investigated in, for example, [**3, 4, 6–11**].

In this paper we consider the linear partial difference equations with positive and negative coefficients in the form

$$A_{m-1,n} + A_{m,n-1} - A_{mn} + pA_{\substack{m+k\\n+l}} - qA_{\substack{m+k'\\n+l'}} = 0, \tag{1.1}$$

and

$$A_{m-1,n} + A_{m,n-1} - A_{mn} + p_{mn}A_{\substack{m+k\\n+l}} - q_{mn}A_{\substack{m+k'\\n+l'}} = 0. \tag{1.2}$$

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Let $N_i = \{i, i+1, i+2, \dots, |i = 0, 1, 2, \dots\}$, where

$$k, k', l', l \in N_0, \quad p, q \in (0, \infty), \quad p_{mn}, q_{mn} \in [N_0^2, (0, \infty)], \\ k > k', \quad l > l'. \tag{1.3}$$

Note that in the case of $q = q_{mn} = 0$, some results for the oscillation of (1.1) and (1.2) have been obtained in [**3, 6**–**8**]. Regarding the definition of the oscillation as well as initial conditions, the reader is referred to [**3, 8**]. As can be easily seen, a detailed and specific study of (1.1) and (1.2) in such a general form is very difficult. Nevertheless, in this paper, we are able to obtain some sufficient conditions for the oscillatory behavior of all solutions of (1.1) and (1.2).

2. Preliminary lemmas. Consider the delay partial difference equation

$$A_{m-1,n} + A_{m,n-1} - A_{mn} + p_{mn} A_{\substack{m+k \\ n+l}} = 0. \tag{2.1}$$

The following results are obtained based on [**3**, pp. 237–240]:

Lemma 1 [**3**]. *Assume that one of the following two conditions is satisfied*:

(i)

$$\liminf_{m,n\to\infty} \frac{1}{kl} \left(\sum_{i=m+1}^{m+k} \sum_{j=n+1}^{n+l} p_{ij} \right) > \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}, \tag{2.2}$$

where $\alpha = \max(k, l)$.

(ii) *For all large enough* m *and* n,

$$p_{mn} \geq \xi > \frac{(k+l)^{(k+l)}}{(k+l+1)^{(k+l+1)}}. \tag{2.3}$$

Then every solution of (2.1) *oscillates.*

Lemma 2 [**3**].

$$\sum_{i=m_1}^{m}\sum_{j=n_1}^{n}(A_{i-1,j}+A_{i,j-1}-A_{ij}) = \sum_{i=m_1}^{m}\sum_{j=n_1}^{n-1}+\sum_{i=m_1}^{m}A_{i,n_1-1}+A_{m_1-1,n}-A_{m,n}. \tag{2.4}$$

Next we consider the following equation:

$$A_{m-1,n}+A_{m,n-1}-pA_{mn}+p_{mn}A_{\substack{m+k\\n+l}}=0. \tag{2.5}$$

The following results can be obtained based on [**4**]:

Lemma 3 [**4**]. *Assume that* (2.2) *or* (2.3) *holds. Then the partial difference inequality*

$$A_{m-1,n}+A_{m,n-1}-A_{mn}+p_{mn}A_{\substack{m+k\\n+l}}\le 0 \tag{2.6}$$

cannot have eventually positive solutions, and

$$A_{m-1,n}+A_{m,n-1}-A_{mn}+p_{mn}A_{\substack{m+k\\n+l}}\ge 0 \tag{2.7}$$

cannot have eventually negative solutions.

Lemma 4 [**4**]. *Assume that for all large enough m and n,*

$$p_{mn}>\xi>\frac{p^{2(k+l)+1}(k+l)^{(k+l)}}{(k+l+1)^{(k+l+1)}}. \tag{2.8}$$

Then every solution of equation (2.7) *oscillates.*

Now let s,t be positive integers and c be a positive real number, such that

$$s\le m,\quad t\le n,\quad 1<c\le 2 \tag{2.9}$$

and

(2.10)

$$\begin{aligned}C_{st} &= 2^{s+t-m-n} c A_{\substack{m+k\\n+l}} + A_{st} - \left(\frac{1}{2}\right)^{2(s+t-m-n)+1}\\ &\quad\times\left(\sum_{i=m-k}^{s} q_{in} A_{\substack{i+k'\\n+l'}} + \sum_{j=n-l}^{t} q_{mj} A_{\substack{m+k'\\j+l'}}\right)\\ &\quad-\frac{1}{2}\left(\sum_{i=m-k}^{s} q_{\substack{i-k'+k\\n-l'+l}} A_{\substack{i+k\\n+l}} + \sum_{j=n-l}^{t} q_{\substack{m-k'\\j-l'+l}} A_{\substack{m+k\\j+l}} + 2A_{mn}\right).\end{aligned}$$

Let also

$$\alpha_{mn} \overset{\text{def}}{=} p_{mn} - q_{\substack{m-k'+k\\n-l'+l}} > 0, \quad \text{for } m \geq k-k', \quad n \geq l-l'. \tag{2.11}$$

Then

$$p_{mn} - \frac{1}{2} q_{\substack{m-k'+k\\n-l'+l}} \geq p_{mn} - q_{\substack{m-k'+k\\n-l'+l}} > 0. \tag{2.12}$$

The following results can be established based on (2.10):

Lemma 5. *Assume that* (1.3) *holds and* $\{A_{mn}\}$ *is an eventually positive solution of* (1.2)*, that is, there exist positive integers* M, N *such that* $A_{mn} > 0$ *as* $m \geq M$, $n \geq N$. *Then*

(i) C_{mn} *is increasing in* m, n, *that is,*

$$C_{m-1,n} < C_{mn}, C_{m,n-1} < C_{mn}. \tag{2.13}$$

(ii) *For sufficiently large* M, N, *when* $m \geq M$, $n \geq N$, *we have*

$$C_{mn} \leq cA_{mn}. \tag{2.14}$$

(iii)

$$C_{m-1,n} + C_{m,n-1} - C_{mn} = -\alpha_{mn} A_{\substack{m+k\\ n+l}} - \beta_{mn}(A) < 0, \tag{2.15}$$

where

$$\begin{aligned}
\beta_{mn}(A) &= \frac{7}{2}\Delta_1 + \frac{1}{2}\Delta_2 + 2q_{mn} A_{\substack{m+k'\\ n+l'}} > 0,\\
\Delta_1 &= \sum_{i=m-k}^{m-1} q_{in} A_{\substack{i+k'\\ n+l'}} + \sum_{j=n-1}^{n-1} q_{mj} A_{\substack{m+k'\\ j+l'}}\\
\Delta_2 &= \sum_{i=m-k}^{m} q_{\substack{i-k'+k\\ n-l'+l}} A_{\substack{i+k\\ n+l}} + \sum_{j=n-l}^{n} q_{\substack{m-k'+k\\ j-l'+l}} A_{\substack{m+k\\ j+l}} + 2A_{mn}.
\end{aligned}$$

Proof. (i) From (2.10), we obtain

(2.16)

$$\begin{aligned}
C_{mn} = cA_{\substack{m+k\\ n+l}} + A_{mn} &- \frac{1}{2}\left(\sum_{i=m-k}^{m} q_{in} A_{\substack{i+k'\\ n+l'}} + \sum_{j=n-l}^{n} q_{mj} A_{\substack{m+k'\\ j+l'}} \right)\\
&- \frac{1}{2}\left(\sum_{i=m-k}^{m} q_{\substack{i-k'+k\\ n-l'+l}} A_{\substack{i+k\\ n+l}} + \sum_{j=n-l}^{n} q_{\substack{m-k'+k\\ j-l'+l}} A_{\substack{m+k\\ j+l}} + 2A_{mn} \right)\\
&= cA_{\substack{m+k\\ n+l}} + A_{mn} - \frac{1}{2}\left(\sum_{i=m-k}^{m-1} q_{in} A_{\substack{i+k'\\ n+l'}} + \sum_{j=n-l}^{n-1} q_{mj} A_{\substack{m'_k\\ j+l'}} \right)\\
&- \frac{1}{2}\left(\sum_{i=m-k}^{m} q_{\substack{i-k'+k\\ n-l'+l}} A_{\substack{i+k\\ n+l}} + \sum_{j=n-l}^{n} q_{\substack{m-k'+k\\ j-l'+l}} A_{\substack{m+k\\ j+l}} + 2A_{mn} \right)\\
&- q_{mn} A_{\substack{m+k'\\ n+l'}}\\
&= cA_{\substack{m+k\\ n+l}} + A_{mn} - \frac{1}{2}\Delta_1 - \frac{1}{2}\Delta_2 - q_{mn} A_{\substack{m+k'\\ n+l'}},
\end{aligned}$$

$$\begin{aligned}
C_{m-1,n} &= \frac{1}{2}\, cA_{\substack{m+k\\n+l}} + A_{m-1,n} - 2\Bigg(\sum_{i=m-k}^{m-1} q_{in} A_{\substack{i+k'\\n+l'}} + \sum_{j=n-l}^{n} q_{mj} A_{\substack{m+k'\\j+l'}}\Bigg) \\
&\quad - \frac{1}{2}\Bigg[\sum_{i=m-k}^{m-1} q_{\substack{i-k'+k\\n-l'+l}} A_{\substack{i+k\\n+l}} + \sum_{j=n-l}^{n} q_{\substack{m-k'+k\\j-l'+l}} A_{\substack{m+k\\j+l}} + 2A_{mn}\Bigg] \\
&= \frac{1}{2}\, cA_{\substack{m+k\\n+l}} + A_{m-1,n} - 2\Bigg(\sum_{i=m-k}^{m-1} q_{in} A_{\substack{i+k'\\n+l'}} + \sum_{j=n-l}^{n-1} A_{mj} A_{\substack{m+k'\\j+l'}}\Bigg) \\
&\quad - \frac{1}{2}\Bigg[\sum_{i=m-k}^{m} q_{\substack{i-k'+k\\n-l'+l}} A_{\substack{i+k\\n+l}} + \sum_{j=n-l}^{n} q_{\substack{m-k'+k\\j-l'+l}} A_{\substack{m+k\\j+l}} + 2A_{mn}\Bigg] \\
&\quad - 2q_{mn} A_{\substack{m+k'\\n+l'}} + \frac{1}{2} q_{\substack{m-k'+k\\n-l'+l}} A_{\substack{m+k\\n+l}} \\
&= \frac{1}{2}\, cA_{\substack{m+k\\n+l}} + A_{m-1,n} - 2\Delta_1 - \frac{1}{2}\Delta_2 - 2q_{mn} A_{\substack{m+k'\\n+l'}} \\
&\quad + \frac{1}{2}\, q_{\substack{m-k'+k\\n-l'+l}} A_{\substack{m+k\\n+l}}.
\end{aligned}$$

Since $A_{mn} > 0$, we have

$$\begin{aligned}
C_{m-1,n} - C_{mn} &= \frac{1}{2}\, cA_{\substack{m+k\\n+l}} + A_{m-1,n} - 2\Delta_1 - \frac{1}{2}\Delta_2 - 2q_{mn} A_{\substack{m+k'\\n+l'}} \\
&\quad + \frac{1}{2}\, q_{\substack{m-k'+k\\n-l'+l}} A_{\substack{m+k\\n+l}} \\
&\quad - \Big[cA_{\substack{m+k\\n+l}} + A_{mn} - \frac{1}{2}\Delta_1 - \frac{1}{2}\Delta_2 - q_{mn} A_{m+k' n+l'}\Big] \\
&= -\frac{1}{2}\, cA_{\substack{m+k\\n+l}} + A_{m-1,n} - A_{mn} - \frac{3}{2}\Delta_1 - q_{mn} A_{\substack{m+k'\\n+l'}} \\
&\quad + \frac{1}{2}\, q_{\substack{m-k'+k\\n-l'+l}} A_{\substack{m+k\\n+l}} \\
&\le -\frac{1}{2}\, cA_{\substack{m+k\\n+l}} + A_{m-1,n} + A_{m,n-1} - A_{mn} \\
&\quad - \frac{3}{2}\Delta_1 - q_{mn} A_{\substack{m+k'\\n+l'}} + \frac{1}{2} q_{\substack{m-k'+k\\n-l'+l}} A_{\substack{m+k\\n+l}} \\
&= -\frac{1}{2}\, cA_{\substack{m+k\\n+l}} - p_{mn} A_{\substack{m+k\\n+l}} + q_{mn} A_{\substack{m+k'\\n+l'}}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{3}{2}\Delta_1 - q_{mn} A_{\substack{m+k'\\ n+l'}} + \frac{1}{2} q_{\substack{m-k'+k\\ n-l'+l}} A_{\substack{m+k\\ n+l}} \\
&= -\Big(p_{mn} - \frac{1}{2}\, q_{\substack{m-k'+k\\ n-l'+l}}\Big) A_{\substack{m+k\\ n+l}} - \frac{1}{2}\, c A_{\substack{m+k\\ n+l}} - \frac{3}{2}\Delta_1 \\
&\le -\Big(p_{mn} - \frac{1}{2}\, q_{\substack{m-k'+k\\ n-l'+l}}\Big) A_{\substack{m+k\\ n+l}} < 0.
\end{aligned}
$$

That is, $C_{m-n-1} \le C_{mn}$. Similarly we have $C_{m,n-1} - C_{mn} < 0$.

(ii) From (2.16), we have

$$
\begin{aligned}
C_{mn} &= c A_{\substack{m+k\\ n+l}} + A_{mn} - \frac{1}{2}\Delta_1 - \frac{1}{2}\Delta_2 - q_{mn} A_{\substack{m+k'\\ n+l'}} \\
&= c A_{\substack{m+k\\ n+l}} - \frac{1}{2}\Delta_1 \\
&\quad - \frac{1}{2}\Big(\sum_{i=m-k}^{m} q_{\substack{i-k'+k\\ n-l'+l}} A_{\substack{i+k\\ n+l}} + \sum_{j=n-l}^{n} q_{\substack{m-k'+k\\ j-l'+l}} A_{\substack{m+k\\ j+l}} \Big) \\
&\quad - q_{mn} A_{\substack{m+k'\\ n+l'}} \\
&\le c A_{\substack{m+k\\ n+l}}.
\end{aligned}
$$

Thus, for sufficiently large M, N, when $m \ge M$, $n \ge N$, we have

$$C_{mn} \le c A_{mn}.$$

Furthermore, note that

$$
\begin{aligned}
C_{m,n-1} &= \frac{1}{2}\, c A_{\substack{m+k\\ n+l}} + A_{m,n-1} \\
&\quad - 2\Big(\sum_{i=m-k}^{m} q_{in} A_{\substack{i+k'\\ n+l'}} + \sum_{j=n-l}^{n-1} q_{mj} A_{\substack{m+k'\\ j+l'}} \Big) \\
&\quad - \frac{1}{2}\Big(\sum_{i=m-k}^{m} q_{\substack{i-k'+k\\ n-l'+l}} A_{\substack{i+k\\ n+l}} + \sum_{j=n-l}^{n} q_{\substack{m-k'+k\\ j-l'+l}} A_{\substack{m+k\\ j+l}} + 2A_{mn} \Big) \\
&= \frac{1}{2}\, c A_{\substack{m+k\\ n+l}} + A_{m,n-1} - 2\Delta - 1
\end{aligned}
$$

$$- \frac{1}{2}\Delta_2 - 2q_{mn}A_{\substack{m+k' \\ n+l'}} + \frac{1}{2}q_{\substack{m-k'+k \\ n-l'+l}}A_{\substack{m+k \\ n+l}}.$$

Thus we have

$$\begin{aligned}
&C_{m-1,n} + C_{m,n-1} - C_{mn} \\
&\qquad = A_{m-1,n} + A_{m,n-1} + 2\bigg(\frac{1}{2}\, cA_{\substack{m+k \\ n+l}} - 2\Delta 1 - \frac{1}{2}\Delta_2 \\
&\qquad\quad - 2q_{mn}A_{\substack{m+k' \\ n+l'}} + \frac{1}{2}\, q_{\substack{m-k'+k \\ n-l'+l}}A_{\substack{m+k \\ n+l}}\bigg) \\
&\qquad\quad - \bigg(cA_{\substack{m+k \\ n+l}} + A_{mn} - \frac{1}{2}\Delta_1 - \frac{1}{2}\Delta_2 - q_{mn}A_{\substack{m+k' \\ n+l'}}\bigg) \\
&\qquad = A_{m-1,n} + A_{m,n-1} - A_{mn} - \frac{7}{2}\Delta_1 - \frac{1}{2}\Delta_2 \\
&\qquad\quad - 3q_{mn}A_{\substack{m+k' \\ n+l'}} + q_{\substack{m-k'+k \\ n-l'+l}}A_{\substack{m+k \\ n+l}} \\
&\qquad = -p_{mn}A_{\substack{m+k \\ n+l}} + q_{mn}A_{\substack{m+k' \\ n+l'}} - \frac{7}{2}\Delta_1 - \frac{1}{2}\Delta_2 \\
&\qquad\quad - 3q_{mn}A_{\substack{m+k' \\ n+l'}} + q_{\substack{m-k'+k \\ n-l'+l}}A_{\substack{m+k \\ n+l}} \\
&\qquad = -\alpha_{mn}A_{\substack{m+k \\ n+l}} - \frac{7}{2}\Delta_1 - \frac{1}{2}\Delta_2 - 2q_{mn}A_{\substack{m+k' \\ n+l'}} \\
&\qquad = -\alpha_{mn}A_{\substack{m+k \\ n+l}} - \beta_{mn}(A) \\
&\qquad \leq -\alpha_{mn}A_{\substack{m+k \\ n+l}} < 0.
\end{aligned}$$

In particular, for the case of constant coefficients, we obtain the following result:

Corollary 1. *Assume that the conditions of Lemma* 5 *hold. Then*

(i) *The C_{mn} is increasing in m, n, that is,*

$$C_{m-1,n} < C_{mn}, C_{m,n-1} < C_{mn}.$$

(ii) *For sufficiently large M, N as $m \geq M$, $n \geq N$, we have*

$$C_{mn} \leq cA_{mn} \quad \textit{and} \quad C_{m-1,n} + C_{m,n-1} - C_{mn} = -\alpha A_{\substack{m+k \\ j+l}} - \beta(A),$$

where $\alpha = p - q > 0$ *and*

$$\beta(A) = q\Bigg[2A_{\substack{m+k' \\ n+l'}} + \frac{7}{2}\Bigg(\sum_{i=m-k}^{m-1} A_{\substack{i+k' \\ n+l'}} + \sum_{j=n-l}^{n-1} qA_{\substack{m+k' \\ j+l'}}\Bigg) + \frac{1}{2}\Bigg(\sum_{i=m-k}^{m} A_{\substack{i+k \\ n+l}} + \sum_{j=n-l}^{n} A_{\substack{m+k \\ j+l}} + 2A_{mn}\Bigg)\Bigg].$$

Lemma 6. *Assume that* (1.3), (2.9)–(2.11) *hold. Further, assume that for* $m \geq k - k'$, $n \geq l - l'$,

$$\frac{1}{2}\Bigg(\sum_{i=m-k}^{m} q_{in} + \sum_{j=n-l}^{n} q_{mj}\Bigg) + \frac{1}{2}\Bigg(\sum_{i=m-k}^{m} q_{\substack{i-k'+k \\ n-l'+l}} + \sum_{j=n-l}^{n} q_{\substack{m-k'+k \\ j-l'+l}} + 2\Bigg) < c. \tag{2.17}$$

Let $\{A_{mn}\}$ *be an eventually positive solution of equation* (1.2). *Then* $\{C_{mn}\}$ *is increasing and eventually positive.*

Proof. By Lemma 5, $\{C_{mn}\}$ is increasing in m, n. Next we shall show that the $\{C_{mn}\}$ is an eventually positive. Because $\{A_{mn}\}$ is an eventually positive solution of equation (1.2) and the $\{C_{mn}\}$ is increasing, thus the limit of $\{C_{mn}\}$ exists. If $\lim_{m,n\to\infty} C_{mn} = -\infty$, as $m, n \to \infty$, then $\{A_{mn}\}$ must be unbounded. There exists $\{(m_k, n_k)\}$ such that $\lim_{m\to\infty} m_k = \infty$, $\lim_{m\to\infty} n_k = \infty$ and $\lim_{m\to\infty} A_{m_k+k,n_k+l} = +\infty$, $A_{m_k+k,n_k+l} = \max_{M\leq m\leq m_k, N\leq n\leq n_k} A_{m+k,n+l}$. On the other hand,

$$\begin{aligned} &C_{m_k n_k} \\ &= cA_{\substack{m_k+k \\ n_l+l}} + A_{m_k n_k} \\ &\quad - \frac{1}{2}\Bigg(\sum_{i=m_k-k}^{m_k} qin_k A_{\substack{i+k' \\ n_k+l'}} + \sum_{j=n_k-l}^{n_k} q_{m_k j} A_{\substack{m_k+k' \\ j+l'}}\Bigg) \\ &\quad - \frac{1}{2}\Bigg(\sum_{i=m_k-k}^{m_k} q_{\substack{i-k'+k \\ n_k-l'+l}} A_{\substack{i+k \\ n_k+l}} + \sum_{j=n_k-l}^{n_k} q_{\substack{m_k-k'+k \\ j-l'+l}} A_{\substack{m_k+k \\ j+l}} + 2A_{m_k n_l}\Bigg) \end{aligned}$$

$$\begin{aligned}
&\geq A_{\substack{m_k+k\\ n_l+l}}\bigg[c-\frac{1}{2}\bigg(\sum_{i=m_k-k}^{m_k} q_{in_k}+\sum_{j=n_k-l}^{n_k} q_{m_kj}\bigg)\\
&\quad-\frac{1}{2}\bigg(\sum_{i=m_k-k}^{m_k} q_{\substack{i-k'+k\\ n_k-l'+l}}+\sum_{j=n_k-l}^{n_k} q_{\substack{m_k-k'+k\\ j-l'+l}}+2\bigg)\bigg]\\
&\geq 0,
\end{aligned}$$

a contradiction. Hence, $\lim_{m,n\to\infty} C_{mn} = \beta$ exists. If $\{A_{mn}\}$ is unbounded, then $\beta \geq 0$. Now we consider the case that $\{A_{mn}\}$ is bounded. Let $\bar{\beta} = \limsup_{m,n\to\infty} A_{mn} = \lim_{m'\to\infty,n'\to\infty} A_{m',n'}$. Then

$$\begin{aligned}
&cA_{\substack{m'+k\\ n'+l}}-C_{m',n'}\\
&\leq cA_{\substack{m'+k\\ n'+l}}+A_{m',n'}-C_{m',n'}\\
&=\frac{1}{2}\bigg(\sum_{i=m'-k}^{m'} q_{in'}A_{\substack{i+k'\\ n'+l'}}+\sum_{j=n-l}^{n} q_{m'j}A_{\substack{m'+k'\\ j+l'}}\bigg)\\
&\quad+\frac{1}{2}\bigg(\sum_{i=m'-k}^{m'} q_{\substack{i-k'+k\\ n'-l'+l}}A_{\substack{i+k\\ n'+l}}+\sum_{j=n'-l}^{n'} q_{\substack{m'-k'+k\\ j-l'+l}}A_{\substack{m'+k\\ j+l}}+2\bigg)\\
&\leq A(\xi_m,\eta_n)\bigg[\frac{1}{2}\bigg(\sum_{i=m'-k}^{m'} q_{in'}+\sum_{j=n'-l}^{n'} q_{m'j}\bigg)\\
&\quad+\frac{1}{2}\bigg(\sum_{i=m'-k}^{m'} q_{\substack{i-k'+k\\ n'-l'+l}}+\sum_{j=n'-l}^{n'} q_{\substack{m'-k'+k\\ j-l'+l}}+2\bigg)\bigg],
\end{aligned}$$

where $A(\xi_m,\eta_n) = \max\{A_{i+k,j+l}|_{j=n'-l,\dots,n'}^{i=m'-k,\dots,m'}\}$. Taking superior limit on both sides of the above inequality, we have $c\bar{\beta}-\beta \leq c\bar{\beta}$, therefore $\beta \geq 0$. Hence $C_{mn} > 0$ for $m \geq M$, $n \geq N$.

In particular, in the constant coefficient case, we obtain the following conclusion:

Corollary 2. *Assume that* (1.3)–(1.5) *and* (2.9)–(2.11) *hold and that, for* $m \geq k-k'$, $n \geq l-l'$, *we have*

$$q(k+l+3) < c.$$

Let $\{A_{mn}\}$ *be an eventually positive solution of* (1.1). *Then* $\{C_{mn}\}$ *is increasing and eventually positive in* m, n.

3. Oscillation of equation (1.1).

Theorem 3.1. *Assume that*

(i) $k \geq k' \geq 1$, $l \geq l' \geq 1$, $p > q > 0$,

(ii) $\alpha \geq \xi > (k+l)^{(k+l)}/(k+l+1)^{(k+l+1)}$,

(iii) $c - q(k+l+3) \geq 0$.

Then every solution of (1.1) *oscillates.*

Proof. (a) If $k = k'$, $l = l'$, then (1.1) becomes

$$A_{m-1,n} + A_{m,n-1} - A_{mn} + (p-q)A_{\substack{m+k\\ n+l}} = 0, \tag{3.1}$$

and the oscillatory behavior of (3.1) has been studied in [**3, 6, 7**–**8**].

(b) If $k > k'$, $l > l'$, then we let $\{A_{mn}\}$ be an eventually positive solution of (1.1). This means that m_0, n_0 exists such that when $m \geq m_0$, $n \geq n_0$, we have

$$A_{mn} > 0. \tag{3.2}$$

By Corollary 2 $\{C_{mn}\}$ is eventually positive and increasing. However, by (ii) of Lemma 5, we have $cA_{mn} > C_{mn}$. Also by Corollary 1, we have

$$C_{m-1,n} + C_{m,n-1} - C_{mn} + (p-q)A_{\substack{m+k\\ n+l}} \leq 0. \tag{3.3}$$

Hence

$$\begin{aligned} C_{m-1,n} + C_{m,n-1} - C_{mn} &+ (p-q)\frac{1}{c}\,C_{\substack{m+k\\ n+l}} \\ &\leq C_{m-1,n} + C_{m,n-1} - C_{mn} + (p-q)A_{\substack{m+k\\ n+l}} \leq 0. \end{aligned}$$

It follows from condition (ii) of Theorem 3.1 and Lemma 1 that every solution of

$$C_{m-1,n} + C_{m,n-1} - C_{mn} + (p-q)\frac{1}{c}\,C_{\substack{m+k\\ n+l}} = 0$$

oscillates. However, Lemma 3 leads to a contradiction. The proof is thus completed. □

Example 1. Consider the partial difference equation

$$A_{m-1,n} + A_{m,n-1} - A_{mn} + \frac{53}{18} A_{\substack{m+2\\n+2}} - \frac{1}{18} A_{\substack{m+1\\n+2}} = 0, \tag{3.4}$$

where $k = 2$, $k' = 1$, $l = l' = 2$, $q = (1/18)$, $p = (53/18)$, $k-k' = 1 > 0$, $l-l' = 0$. For $1 < c \le 2$, we have $c-q(k+l+3) = c-(7/18) > 0$. Taking $\xi = 1$ yields $[(k+l)^{(k+l)}/(k+l+1)^{(k+l+1)}] = (4^4/5^5)$ and $\alpha = p-q = (52/18) > 1 > (256/3125) = 4^4/5^5 = [(k+l)^{(k+l)}/(k+l+1)^{(k+l+1)}]$. It can be verified that all the hypotheses of Theorem 3.1 are satisfied. Therefore, all solutions of (3.4) are oscillatory. In fact (3.4) has a unique oscillatory solution given by $\{A_{mn}\} = \{(-1)^{m+n}\}$.

4. Stability of equation (1.2).

Theorem 4.1. *Assume that* (1.3), (2.11) *and* (2.17) *hold and that one of the following two conditions is satisfied*:

(i) *There exists a positive integer* α_0 *such that*

$$p_{mn} - q_{\substack{m+k-k'\\n+l-l'}} \ge \alpha_0 \quad \text{for } m \ge k-k', \quad n \ge l-l', \tag{4.1}$$

or

(ii) *There exists a positive constant* $\beta_0 \in (0,1)$ *such that*

(4.2)
$$\frac{1}{2}\bigg(\sum_{i=m-k}^{m} q_{in} + \sum_{j=n-1}^{n} q_{mj}\bigg) + \frac{1}{2}\bigg(\sum_{i=m-k}^{m} q_{\substack{i-k'+k\\n-l'+l}} + \sum_{j=n-l}^{n} q_{\substack{m-k'+k\\j-l'+l}} + 2\bigg) \le c - \beta_0 \quad \text{for } m \ge k_0, \quad n \ge l_0,$$

where $k_0 = k+k'$, $l_0 = l+l'$ *and*

$$\sum_{i=k+k'}^{\infty} \sum_{j=l+l'+1}^{\infty} (p_{ij} - q_{\substack{i+k-k'\\j+l-l'}}) = \infty. \tag{4.3}$$

Then every nonoscillatory solution of (1.2) *tends to zero as* $m, n \to \infty$.

Proof. It suffices to show that every eventually positive solution $\{A_{mn}\}$ of (1.2) tends to zero as $m, n \to \infty$.

It follows from Lemma 5 that $\{C_{mn}\}$ is increasing and positive. Hence,

$$\lim_{m,n\to\infty} C_{mn} = \xi \in R^+, \tag{4.4}$$

where $R^+ = (0, \infty)$. By Lemma 5 it is easy to see that

$$C_{m-1,n} + C_{m,n-1} - C_{mn} = -\alpha_{mn} A_{\substack{m+k\\ j+l}}. \tag{4.5}$$

Taking m_1, n_1 sufficiently large, and summing both sides of (4.5) form $m_1 + 1$, $n_1 + 1 \to \infty$, we get

$$\begin{aligned} &\sum_{i=m_1+1}^{\infty} \sum_{j=n_1+1}^{\infty} (C_{i-1,j} + C_{i,j-1} - C_{ij}) \\ &\qquad \le -\sum_{i=m_1+1} \sum_{j=n_1+1}^{\infty} (p_{ij} - q_{\substack{i+k-k'\\ j+l-l'}}) A_{\substack{i+k\\ j+l}}. \end{aligned} \tag{4.6}$$

Since $C_{ij} > 0$, we have

$$\begin{aligned} -2C_{m_1,n_1} &\le \sum_{i=m_1}^{\infty} C_{in_1} + \sum_{j=n_1}^{\infty} C_{m_1 j} - 2C_{m_1,n_1} \\ &= \sum_{i=m_1+1}^{\infty} \sum_{j=n_1+1}^{\infty} [(C_{i-1,j} - C_{ij}) + (C_{i,j-1} - C_{ij})] \\ &= \sum_{i=m_1+1}^{\infty} \sum_{j=n_1+1}^{\infty} (C_{i-1,j} + C_{i,j-1} + 2C_{ij}) \\ &\le \sum_{i=m_1+1}^{\infty} \sum_{j=n_1+1}^{\infty} (C_{i-1,j} + C_{i,j-1} - C_{ij}) \\ &\le -\sum_{i=m_1+1}^{\infty} \sum_{j=n_1+1}^{\infty} (p_{ij} - q_{\substack{i+k-k'\\ j+l-l'}}) A_{\substack{i+k\\ j+l}}, \end{aligned}$$

or

$$-2C_{m_1,n_1} \le -\sum_{i=m_1+1}^{\infty}\sum_{j=n_1+1}^{\infty}(p_{ij} - q_{\substack{i+k-k' \\ j+l-l'}})A_{\substack{i+k \\ j+l}}. \tag{4.8}$$

First assume that (4.1) holds. Then (4.8) implies that

$$\sum_{i=m_1+1}^{\infty}\sum_{j=n_1+1}^{\infty}(p_{ij} - q_{\substack{i+k-k' \\ j+l-l'}}) < \infty.$$

Since $A_{i-k,j-l}$ is a positive solution of (1.2), and from (4.1) and (4.8), we have

$$\lim_{m,n\to\infty} A_{m,n} = 0.$$

Thus the proof is completed when (4.1) holds.

Next assume that (4.2) and (4.3) hold. From (4.8), it follows that

$$\liminf_{m,n\to\infty} A_{m,n} = 0.$$

Also (2.16) implies $C_{mn} \le cA_{mn}$ and, in view of (4.4), $\xi = 0$.

Now we claim that $\{A_{mn}\}$ is bounded. Otherwise, there would exist a subsequence, $\{A_{m_r+k,n_r+l}\}$ of $\{A_{mn}\}$ such that

$$A_{\substack{m_r+k \\ n_r+l}} = \max\{A_{\substack{m+k \\ n+l}} \mid {}_{\substack{m\le m_r \\ n\le n_r}} \text{ for } r = 1, 2, \dots\}$$

and

$$\lim_{r\to\infty} A_{\substack{m_r+k \\ n_r+l}} = \infty.$$

Then by (2.10) and (4.2), we have

$$\begin{aligned} C_{m_r n_r} = {} & cA_{\substack{m_r+k \\ n_r+l}} + A_{m_r n_r} \\ & -\frac{1}{2}\bigg(\sum_{i=m_r-k}^{m_r} q_{in_r} A_{\substack{i+k' \\ n_r+l'}} + \sum_{j=n_r-l}^{n_r} q_{m_r j} A_{\substack{m_r+k' \\ j+l'}}\bigg) \\ & -\frac{1}{2}\bigg(\sum_{i=m_r-k}^{m_r} q_{\substack{i-k'+k \\ n_r-l'+l}} A_{\substack{i+k \\ n_r+l}} + \sum_{j=n_r-l}^{n_r} q_{\substack{m_r-k'+k \\ j-l'+l}} A_{\substack{m_r+k \\ j+l}} + 2A_{m_r n_r}\bigg), \end{aligned}$$

$$\begin{aligned}
&\geq cA_{\substack{m_r+k\\n+l}} - \frac{1}{2}\bigg(\sum_{i=m_r-k}^{m_r} q_{in_r} A_{\substack{i+k'\\n_r+l'}} + \sum_{j=n_r-l}^{n_r} q_{m_rj} A_{\substack{m_r+k'\\j+l'}}\bigg)\\
&\quad -\frac{1}{2}\bigg(\sum_{i=m_r-k}^{m_r} q_{\substack{i-k'+k\\n_r-l'+l}} A_{\substack{i+k\\n_r+l}} + \sum_{j=n_r-l}^{n_r} q_{\substack{m_r-k'+k\\j-l'+l}} A_{\substack{m_r+k\\j+l}} + 2A_{m_rn_r}\bigg)\\
&\geq A_{\substack{m_r+k\\n_r+l}}\bigg[c - \frac{1}{2}\bigg(\sum_{i=m_r-k}^{m_r} q_{in_r} + \sum_{j=n_r-l}^{n_r} q_{m_rj}\bigg)\\
&\quad -\frac{1}{2}\bigg(\sum_{i=m_r-k}^{m_r} q_{\substack{i-k'+k\\n_r-l'+l}} + \sum_{j=n_r-l}^{n_r} q_{\substack{m_r-k'+k\\j-l'+l}} + 2\bigg)\bigg]\\
&\geq \beta_0 A_{\substack{m_{r+k}\\n_{r+l}}} \to \infty \quad \text{as } r \to \infty,
\end{aligned}$$

which contradicts the fact that $\xi = 0$. Therefore, $\{A_{mn}\}$ must be bounded. To this end, set

$$\lambda = \limsup_{m,n\to\infty} A_{m,n}$$

and let $\{A_{m_s+k,n_s+l}\}$ be a subsequence of $\{A_{mn}\}$ such that

$$\lim_{s\to\infty} A_{\substack{m_s+k\\n_s+l}} = \lambda.$$

Then for sufficiently small $\varepsilon > 0$ and for a sufficiently large s, it follows from (2.10) and (4.2) that

$$\begin{aligned}
C_{m_sn_s} &= \bigg[cA_{\substack{m_s+k\\n_s+l}} + A_{m_sn_s}\\
&\quad - \bigg(\sum_{i=m_s-k}^{m_s} q_{in_s} A_{\substack{i+k'\\n_s+l'}} + \sum_{i=n_s-l}^{n_s} q_{m_sj} A_{\substack{m_s+k'\\j+l'}}\bigg)\\
&\quad -\frac{1}{2}\bigg(\sum_{i=m_s-k}^{m_s} q_{\substack{i-k'+k\\n_s-l'+l}} A_{\substack{i+k\\n_s+l}} + \sum_{j=n_s-l}^{n_s} q_{\substack{m_s-k'+k\\j-l'+l}} A_{\substack{m_s+k\\j+l}} + 2A_{m_sn_s}\bigg)\bigg]\\
&\geq \bigg[cA_{\substack{m_s+k\\n_s+l}} - \bigg(\sum_{i=m_s-k}^{m_s} q_{in_s} A_{\substack{i+k'\\n_s+l'}} + \sum_{i=n_s-l}^{n_s} q_{m_sj} A_{\substack{m_s+k'\\j+l'}}\bigg)\\
&\quad -\frac{1}{2}\bigg(\sum_{i=m_s-k}^{m_s} q_{\substack{i-k'+k\\n_s-l'+l}} A_{\substack{i+k\\n_s+l}} + \sum_{j=n_s-l}^{n_s} q_{\substack{m_s-k'+k\\j-l'+l}} A_{\substack{m_s+k\\j+l}} + 2A_{m_sn_s}\bigg)\bigg]\\
&\geq cA_{\substack{m_s+k\\n_s+l}} - (\lambda+\varepsilon)(c-\beta_0).
\end{aligned}$$

By taking limits as $s \to \infty$ and using the fact that $\xi = 0$, we finally obtain

$$0 \geq c\lambda - (\lambda + \varepsilon)(c - \beta_0).$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\lambda = 0$ and the proof is thus completed. $\square$

5. Oscillation of equation (1.2).

Theorem 5.1 *Assume that* (1.3), (2.10) *and* (2.11) *hold, and that one of the two conditions* (2.2) *and* (2.3) *is satisfied. Then, every solution of* (1.2) *oscillates.*

Proof. Assume, on the contrary, that (1.2) has an eventually positive solution $\{A_{mn}\}$.

By Lemmas 5 and 6, it follows that the sequence $\{C_{st}\}$ defined by (2.10) is eventually positive and that

$$C_{m-1,n} + C_{m,n-1} - C_{mn} + (p_{mn} - q_{\substack{m+k-k' \\ n+l-l'}})A_{\substack{m+k \\ n+l}} \leq 0. \tag{5.1}$$

Also, eventually,

$$0 < C_{mn} \leq cA_{mn}. \tag{5.2}$$

Consequently,

$$C_{m-1,n} + C_{m,n-1} - C_{mn} + (p_{mn} - q_{\substack{m+k-k' \\ n+l-l'}})\frac{1}{c}\, C_{\substack{m+k \\ n+l}} \leq 0 \tag{5.4}$$

which implies that every solution of the equation

$$C_{m-1,n} + C_{m,n-1} - C_{mn} + (p_{mn} - q_{\substack{m+k-k' \\ n+l-l'}})C_{\substack{m+k \\ n+l}} = 0 \tag{5.4}$$

oscillates. However, by Lemma 2, inequality (5.4) cannot have an eventually positive solution. This contradiction proves the theorem. $\square$

Example 2. Consider the partial difference equation

$$(5.6)\quad A_{m-1,n} + A_{m,n-1} - A_{mn} + \frac{4(16n+1)}{8n} A_{\substack{m+2\\n+2}} - \frac{1}{8n} A_{m+1,n} = 0,$$

where $m \geq 2$, $n \geq 2$, $p_{mn} = (16n+1)/(8n)$, $q_{mn} = 1/(8n)$, $k = l = 2$, $k' = 1$, $l' = 0$. Since $k = 2 > 1 = k'$, $l > l'$, and

1^0. $$p_{mn} - q_{m+k-k',n+l-l'} = \frac{4(16n+1)}{8n} - \frac{1}{8(n+2)} > 0$$
$$\text{for } m \geq 2,\ n \geq 2,$$

2^0. $$\sum_{i=k'}^{k} \sum_{j=l'+1}^{l} q_{m-i,n-j} = \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{1}{8(n+j)} = \frac{2n+3}{4(n+1)(n+2)} < 1$$
$$\text{for } m \geq 2,\ n \geq 2,$$

3^0. Taking $\xi = 2$, then

$$p_{mn} = \frac{4(16n+1)}{8n} > \xi > \frac{256}{3125} = \frac{(k+l)^{k+l}}{(k+l+1)^{k+l+1}}.$$

Since all of the hypotheses of Theorem 5.1 are satisfied, all solutions of (5.6) are oscillatory. In fact (5.6) has a unique oscillatory solution given by $\{A_{mn}\} = \{(-1)^n(1/2^n)\}$ for $m \geq 2$, $n \geq 2$.

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