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OSCILLATION CRITERIA FOR SECOND-ORDER HALF-LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH DAMPING

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Dedicated to Professor Wenyuan Chen on his seventieth birthday

ABSTRACT. By using averaging functions and an inequality due to Hardy, Littlewood and Polya, several new oscillation criteria are established for the half-linear damped differential equation

$$[r(t)|y'(t)|^{\alpha-1}y'(t)]' + p(t)|y'(t)|^{\alpha-1}y'(t) + q(t)|y(t)|^{\alpha-1}y(t) = 0,$$

where $r \in C^1([t_0,\infty);(0,\infty))$, $\alpha > 0$ and $p, q \in C[t_0,\infty)$. Our results extend and improve the oscillation criteria of Kamenev, Li and Philos for linear equations. Several examples are inserted in the text to illustrate our results.

1. Introduction. In this paper we consider the problem of oscillation of the second-order half-linear damped differential equation

$$(1.1)\quad [r(t)|y'(t)|^{\alpha-1}y'(t)]' + p(t)|y'(t)|^{\alpha-1}y'(t) + q(t)|y(t)|^{\alpha-1}y(t) = 0,$$

on the half-line $[t_0,\infty)$. In equation (1.1) we assume that $r \in C^1([t_0,\infty);(0,\infty))$, $p, q \in C[t_0,\infty)$ and $\alpha > 0$ is a constant.

We recall that a function $y : [t_0, t_1) \to (-\infty,\infty)$, $t_1 > t_0$ is called a solution of equation (1.1) if $y(t)$ satisfies equation (1.1) for all $t \in [t_0, t_1)$. In the sequel it will always be assumed that solutions

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of equation (1.1) exist for any $t_0 \geq 0$. A solution $y(t)$ of equation (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

In the last two decades there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions for different classes of second order differential equations [**1**–**15**, **17**–**45**]. In the absence of damping, there is a great number of papers (see for example, [**17**–**32**, **34**–**38**, **40**, **44**] and the references quoted therein) devoted to the particular cases of equation (1.1) such as the linear differential equations

$$y''(t) + q(t)y(t) = 0 \tag{1.2}$$

$$[r(t)y'(t)]' + q(t)y(t) = 0 \tag{1.3}$$

and the half-linear differential equation

$$[r(t)|y'(t)|^{\alpha-1}y'(t)]' + q(t)|y(t)|^{\alpha-1}y(t) = 0. \tag{1.4}$$

An important tool in the study of oscillatory behavior of solutions for the equations (1.2)–(1.4) is the averaging technique. This goes back as far as to the classical results of Wintner [**40**] giving a sufficient condition for oscillation of equation (1.2), namely,

$$\lim \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)\,d\tau\,ds = \infty$$

and Hartman [**18**] who showed that the above limit cannot be replaced by the super limit and proved that the condition

$$\begin{aligned} -\infty < \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)\,d\tau\,ds \\ < \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)\,d\tau\,ds \leq \infty \end{aligned}$$

implies that equation (1.2) is oscillatory.

The results of Wintner were improved by Kamenev [**21**] who proved that the condition

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}}\int_{t_0}^{t} (t-s)^{n-1} q(s)\,ds = \infty,$$

for some $n > 2$ is sufficient for the oscillation of equation (1.2).

In 1989, Philos [**37**] presented a new oscillation criterion for equation (1.2) involving the Kamenev's type condition.

Theorem A. *Let* $H : D = \{(t, s) \mid t \geq s \geq t_0\} \to R$ *be a continuous function, which is such that*

$$H(t,t) = 0, \quad \text{for } t \geq t_0, \quad H(t,s) > 0 \quad \text{for all } (t,s) \in D,$$

and has a continuous and nonpositive partial derivative on D *with respect to the second variable. Moreover, let* $h : D \to R$ *be a continuous function with*

$$-\frac{\partial H}{\partial s} = h(t,s)\sqrt{H(t,s)}, \quad (t,s) \in D.$$

Then equation (1.2) *is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[q(s)H(t,s) - \frac{1}{4}h^2(t,s) \right] ds = \infty.$$

Theorem B. *Let the functions* H *and* h *be defined as in Theorem* A*, and moreover, suppose that*

$$0 < \inf_{s \geq t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] \leq \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} h^2(t,s)\, ds < \infty.$$

Then equation (1.2) *is oscillation if there exists a continuous function* A *on* $[t_0, \infty)$ *with*

$$\int_{t_0}^{\infty} A_+^2(s)\, ds = \infty$$

and such that

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[q(s)H(t,s) - \frac{1}{4}h^2(t,s) \right] ds \geq A(T),$$
for every $T \geq t_0$.

The above results of Philos were extended further to equation (1.3) by Li [**25**], to equation (1.4) by Manojlovic [**35**] and to the nonlinear differential equation

$$[r(t)y'(t)]' + q(t)f(y(t)) = 0, \tag{1.5}$$

by Li [**28**], where $f'(y) \geq \mu > 0$. Other related oscillation results can be found in Pino et al. [**4**], Došlý [**5**], Elbert [**6**, **7**], Hong et al. [**19**], Hsu and Yeh [**20**], Kandelaki et al. [**22**], Kong [**23**, **24**], Li and Yeh [**26**, **27**], Li and Agarwal [**30**–**32**], Lian et al. [**34**], Mirzov [**36**], and Wong and Agarwal [**44**].

In the presence of damping, a number of oscillation criteria were obtained for differential classes of nonlinear equations by Baker [**1**], Bobisud [**2**], Butler [**3**], Grace [**8**–**11**], Grace and Lalli [**12**, **13**], Grace et al. [**14**, **15**], Li et al. [**33**], Rogovchenko [**39**], Yan [**41**, **42**] and Yeh [**43**]. For the half-linear equation (1.1), however, to the best of our knowledge, Wong and Agarwal [**45**] only obtained several existence theorems for nonoscillatory solutions, but for the oscillation of equation (1.1) it has not been considered.

Motivated by the idea of Li [**28**, **33**] and Manojlovic [**35**], in this paper, by using averaging functions and in inequality due to Hardy et al. [**16**], we obtain several new criteria for oscillation criteria of equation (1.1). Our results improve and extend the results of Kamenev [**21**], Manojlovic [**35**] and Philos [**37**] and others. Finally, several examples are inserted in the text to illustrate our results.

In order to prove our theorems we use the following well-known inequality due to Hardy et al. [**16**].

Lemma A. *If A, B are nonnegative, then*

$$A^\lambda + (\lambda - 1)B^\lambda \geq \lambda A B^{\lambda-1}, \quad \lambda > 1,$$

where equality holds if and only if $A = B$.

2. Oscillation results. In the sequel we say that a function $H = H(t,s)$ belongs to a function class X, denoted by $H \in X$, if $H \in C(D, R)$ where $D = \{(t,s) : -\infty < s \leq t < \infty\}$, which satisfies

$$H(t,t) = 0, \quad H(t,s) > 0 \quad \text{for } t > s, \tag{2.1}$$

and has partial derivative $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial s} = -h(t,s)H(t,s)^{1/2}, \tag{2.2}$$

where h is a nonnegative and continuous function on D.

Theorem 2.1. *If there exists $H \in X$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)q(s) - \frac{r(s)|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}|^{\alpha+1}}{(\alpha+1)^{\alpha+1} H^{\frac{\alpha-1}{2}}(t,s)} \right] ds = \infty,$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1.1). Assume that $y(t) \neq 0$ for $t \geq t_0$. We define

$$u(t) = \frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\alpha-1}y(t)}, \quad t \geq t_0. \tag{2.4}$$

Then, for every $t \geq t_0$, we have

$$u'(t) = -q(t) - \frac{p(t)}{r(t)}\, u(t) - \alpha\, \frac{|u(t)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(t)}, \tag{2.5}$$

and, consequently,

$$\int_{t_0}^{t} H(t,s)q(s)\, ds = -\int_{t_0}^{t} H(t,s)u'(s)\, ds - \int_{t_0}^{t} H(t,s)\, \frac{p(s)}{r(s)}\, u(s)\, ds - \alpha \int_{t_0}^{t} H(t,s) \frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)}\, ds.$$

Since

$$\int_{t_0}^{t} H(t,s)u'(s)\, ds = -H(t,t_0)u(t_0) - \int_{t_0}^{t} \frac{\partial H(t,s)}{\partial s}\, u(s)\, ds, \tag{2.6}$$

the previous equality becomes

$$\begin{aligned}\int_{t_0}^{t} H(t,s)q(s)\,ds \le H(t,t_0)u(t_0) \\ + \int_{t_0}^{t} \Big| h(t,s)\sqrt{H(t,s)} + \frac{p(s)}{r(s)}\,H(t,s)\Big| |u(s)|\,ds \\ - \alpha\int_{t_0}^{t} H(t,s)\,\frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)}\,ds.\end{aligned} \tag{2.7}$$

Taking

$$A = (\alpha H(t,s))^{\frac{\alpha}{\alpha+1}}\,\frac{|u(s)|}{r^{\frac{1}{\alpha+1}}(s)}, \quad \lambda = \frac{\alpha+1}{\alpha},$$

$$B = \frac{\alpha^{\frac{\alpha}{\alpha+1}}}{(\alpha+1)^{\alpha+1}}\left(\frac{r^{\frac{\alpha}{\alpha+1}}(s)|h(t,s)\sqrt{H(t,s)}}{H^{\frac{\alpha^2}{\alpha+1}}(t,s)} + \frac{\frac{p(s)}{r(s)}H(t,s)|^{\alpha}}{H^{\frac{\alpha^2}{\alpha+1}}(t,s)}\right).$$

In view of Lemma A, we obtain for $t > s \ge t_0$,

$$\begin{aligned}\Big|h(t,s)\sqrt{H(t,s)} + \frac{p(s)}{r(s)}H(t,s)\Big| |u(s)| - \alpha H(t,s)\,\frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)} \\ \le \frac{r(s)\big|h(t,s)\sqrt{H(t,s)} + \frac{p(s)}{r(s)}H(t,s)\big|^{\alpha+1}}{(\alpha+1)^{\alpha+1}H^{\alpha}(t,s)} \\ = \frac{r(s)\big|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\big|^{\alpha+1}}{(\alpha+1)^{\alpha+1}H^{\frac{\alpha-1}{2}}(t,s)}.\end{aligned}$$

Hence, equation (2.7) implies

$$\begin{aligned}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)q(s)\,ds \le u(t_0) + \frac{1}{(\alpha+1)^{\alpha+1}H(t,t_0)} \\ \cdot\int_{t_0}^{t}\frac{r(s)\big|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\big|^{\alpha+1}}{[H(t,s)]^{\frac{\alpha-1}{2}}}\,ds,\end{aligned} \tag{2.8}$$

for $t \ge t_0$. Consequently,

$$\begin{aligned}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Bigg[H(t,s)q(s) \\ - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)\big|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\big|^{\alpha+1}}{[H(t,s)]^{\frac{\alpha-1}{2}}}\Bigg]\,ds \le u(t_0),\end{aligned}$$

for $t \geq t_0$. Taking the super limit as $t \to \infty$ in the above, we obtain a contradiction, which completes the proof. $\square$

As immediate consequences of Theorem 2.1 we obtain the following corollaries.

Corollary 2.1. *If there exists* $H \in X$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[\frac{r(s)\left|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right|^{\alpha+1}}{(\alpha+1)^{\alpha+1} H^{\frac{\alpha-1}{2}}(t,s)}\right] ds < \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)q(s)\,ds = \infty,$$

then every solution of equation (1.1) *is oscillatory.*

Corollary 2.2 (cf. [**19**], Theorem 2.1). *Let* $\alpha = 1$ *and* $p(t) \equiv 0$, *and let the functions* h *and* H *be as in Theorem* 2.1. *If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)q(s) - \frac{r(s)}{4}\, h^2(t,s)\right] ds = \infty,$$

then every solution of equation (1.3) *is oscillatory.*

In the same way as was done in [**21**], with an appropriate choice of the functions H and h, we can derive from Theorem 2.1 a number of oscillation criteria for equations (1.2)–(1.4). Let us consider, for example, the function $H(t,s)$ defined by

$$H(t,s) = (t-s)^{\lambda}, \quad (t,s) \in D,$$

where $\lambda > \alpha$ is a constant. Clearly, H belongs to the class X. Furthermore, the function

$$h(t,s) = \lambda(t-s)^{\frac{\lambda-2}{2}}, \quad (t,s) \in D$$

is continuous on $[t_0,\infty)$ and satisfies condition equation (2.2). Then by Theorem 2.1, we obtain the following oscillation criteria.

Corollary 2.3. *If*

$$\limsup_{t\to\infty} \frac{1}{t^\lambda}\int_{t_0}^{t}\left[(t-s)^\lambda q(s) - \frac{\lambda^{\alpha+1} r(s)}{(\alpha+1)^{\alpha+1}}\,(t-s)^{\lambda-\alpha-1}\right] ds = \infty,$$

then every solution of equation (1.4) *is oscillatory.*

Corollary 2.4 (cf. [**10**], Corollary 3). *Suppose that there exists a function $b \in C([t_0,\infty),(0,\infty))$ such that, for some $\lambda > 1$,*

$$\limsup_{t\to\infty} \frac{1}{B(t)^\lambda}\int_{t_0}^{t}\Bigg[(B(t)-B(s))^\lambda q(s) - \frac{(b(s)\lambda)^{\alpha+1} r(s)(B(t)-B(s))^{\lambda-\alpha-1}}{(\alpha+1)^{\alpha+1}}\Bigg] ds = \infty$$

where $B(t) = \int_{t_0}^t b(s)\,ds$. Then every solution of (1.4) *is oscillatory.*

Proof. Let us put

$$H(t,s) = [B(t)-B(s)]^\lambda, \quad (t,s) \in D;$$

then with the choice

$$h(t,s) = \lambda b(t)[B(t)-B(s)]^{\frac{\lambda-2}{2}}, \quad (t,s) \in D,$$

the conclusion follows directly from Theorem 2.1.

Theorem 2.2. *Suppose that there exists $H \in X$ such that*

$$0 < \inf_{s\ge t_0}\left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right] \le \infty, \tag{2.9}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[\frac{r(s)\left|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right|^{\alpha+1}}{(\alpha+1)^{\alpha+1} H^{\frac{\alpha-1}{2}}(t,s)}\right] ds < \infty. \tag{2.10}$$

If there exists a function $\phi \in C[t_0,\infty)$ such that, for every $T \ge t_0$,

(2.11)

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Bigg[H(t,s)q(s) - \frac{r(s)\big|h(t,s)+\frac{p(s)}{r(s)}\sqrt{H(t,s)}\big|^{\alpha+1}}{(\alpha+1)^{\alpha+1}H^{\frac{\alpha-1}{2}}(t,s)} \Bigg] ds \ge \phi(T),$$

and

$$\int_{t_0}^{\infty} \frac{\phi_+^{(\alpha+1)/\alpha}(s)}{r^{1/\alpha}(s)}\, ds = \infty, \tag{2.12}$$

where $\phi_+(t) = \max\{\phi(t),0\}$, then every solution of equation (1.1) *is oscillatory.*

Proof. Suppose that there exists a solution $y(t)$ of equation (1.1) such that $y(t) \neq 0$ for $t \ge t_0$. Define $u(t)$ as in equation (2.4). As in the proof of Theorem 2.1, we can obtain equation (2.7). Then for $t > T \ge t_0$, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Bigg[H(t,s)q(s) - \frac{r(s)\big|h(t,s)+\frac{p(s)}{r(s)}\sqrt{H(t,s)}\big|^{\alpha+1}}{(\alpha+1)^{\alpha+1}H^{\frac{\alpha-1}{2}}(t,s)} \Bigg] ds \le u(T).$$

Therefore, by equation (2.11) we have

$$\phi(T) \le u(T), \quad T \ge t_0, \tag{2.13}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)q(s)\, ds \ge \phi(t_0). \tag{2.14}$$

Define

$$P(t) = \frac{1}{H(t,t_0)} \int_{t_0}^t \Big| h(t,s)\sqrt{H(t,s)} + \frac{p(s)}{r(s)}H(t,s)\Big| |u(s)|\, ds,$$

$$Q(t) = \frac{\alpha}{H(t,t_0)} \int_{t_0}^t H(t,s)\, \frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)}\, ds.$$

Then by equations (2.7) and (2.14), we see that

$$
\begin{aligned}
\liminf_{t\to\infty}[Q(t)-P(t)] &\leq u(t_0) - \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)q(s)\,ds \\
&\leq u(t_0) - \phi(t_0) < \infty.
\end{aligned}
\tag{2.15}
$$

Now we claim that

$$
\int_{t_0}^{\infty}\frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)}\,ds < \infty. \tag{2.16}
$$

Suppose, to the contrary,

$$
\int_{t_0}^{\infty}\frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)}\,ds = \infty. \tag{2.17}
$$

By equation (2.9), there exists a positive constant k_1 such that

$$
\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] > k_1. \tag{2.18}
$$

Let k_2 be an arbitrary positive number. Then it follows from equation (2.17) that there exists $t_1 \geq t_0$ such that

$$
\int_{t_0}^{t}\frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)}\,ds \geq \frac{k_2}{\alpha k_1}, \quad t \geq t_1. \tag{2.19}
$$

Therefore,

$$
\begin{aligned}
Q(t) &= \frac{\alpha}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\frac{d}{ds}\left(\int_{t_0}^{s}\frac{|u(\tau)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(\tau)}\,d\tau\right) \\
&= \frac{\alpha}{H(t,t_0)}\int_{t_0}^{t}\left(-\frac{\partial H}{\partial s}(t,s)\right)\frac{d}{ds}\left(\int_{t_0}^{s}\frac{|u(\tau)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(\tau)}\,d\tau\right) \\
&\geq \frac{\alpha}{H(t,t_0)}\int_{t_1}^{t}\left(-\frac{\partial H}{\partial s}(t,s)\right)\frac{d}{ds}\left(\int_{t_0}^{s}\frac{|u(\tau)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{7}\alpha}(\tau)}\,d\tau\right) \\
&\geq \frac{k_2}{k_1 H(t,t_0)}\int_{t_1}^{t}\left(-\frac{\partial H}{\partial s}(t,s)\right)ds = \frac{k_2}{k_1}\,\frac{H(t,t_1)}{H(t,t_0)}.
\end{aligned}
$$

By equation (2.18), there exists $t_2 \geq t_1$ such that

$$\frac{H(t,t_1)}{H(t,t_0)} \geq k_1, \quad t \geq t_2,$$

which implies that $Q(t) \geq k_2$. Since k_2 is arbitrary,

$$\lim_{t\to\infty} Q(t) = \infty. \tag{2.20}$$

Next, consider a sequence $\{T_n\}_{n=1}^{\infty}$ in (t_0, ∞) with $\lim_{n\to\infty} T_n = \infty$ satisfying

$$\lim_{n\to\infty} [Q(T_n) - P(T_n)] = \liminf_{t\to\infty} [Q(t) - P(t)] < \infty.$$

Then, there exists a constant M such that

$$Q(T_n) - P(T_n) \leq M, \tag{2.21}$$

for all sufficiently large n. Since equation (2.20) ensures that

$$\lim_{n\to\infty} Q(T_n) = \infty, \tag{2.22}$$

and thus equation (2.21) implies that

$$\lim_{n\to\infty} P(T_n) = \infty. \tag{2.23}$$

Furthermore, equations (2.22) and (2.23) lead to the inequality

$$\frac{P(T_n)}{Q(T_n)} - 1 \geq -\frac{M}{Q(T_n)} > -\frac{1}{2}$$

for n large enough. Thus,

$$\frac{P(T_n)}{Q(T_n)} > \frac{1}{2}$$

for n large enough, which together with equation (2.23) implies

$$\lim_{n\to\infty} \frac{P^{\alpha+1}(T_n)}{Q^{\alpha}(T_n)} = \infty. \tag{2.24}$$

On the other hand, by Holder's inequality, we have for every $n \in N$

$$
\begin{aligned}
&P(T_n)\\
&= \frac{1}{H(T_n,t_0)} \int_{t_0}^{T_n} \left| h(T_n,s)\sqrt{H(T_n,s)} + \frac{p(s)}{r(s)}\,H(T_n,s)\right| |u(s)|\,ds\\
&= \int_{t_0}^{T_n} \left(\frac{\alpha^{\frac{\alpha}{\alpha+1}}}{H^{\frac{\alpha}{\alpha+1}}(T_n,t_0)}\, \frac{|u(s)| H^{\frac{\alpha}{\alpha+1}}(T_n,t_0)}{r^{\frac{1}{\alpha+1}}(s)} \right)\\
&\cdot \left(\frac{\alpha^{\frac{-\alpha}{\alpha+1}}}{H^{\frac{1}{\alpha+1}}(T_n,t_0)}\, \frac{r^{\frac{1}{\alpha+1}}(s)\big[h(T_n,s)\sqrt{H(T_n,s)} + \frac{p(s)}{r(s)}H(T_n,s)\big]}{H^{\frac{\alpha}{(\alpha+1)}}(T_n,t_0)} \right) ds\\
&\le \left(\frac{\alpha}{H(T_n,t_0)} \int_{t_0}^{T_n} \frac{|u(s)|^{\frac{\alpha+1}{\alpha}} H(T_n,t_0)}{r^{\frac{1}{\alpha}}(s)}\,ds \right)^{\frac{\alpha}{\alpha+1}}\\
&\cdot \left(\frac{1}{\alpha^\alpha H(T_n,t_0)} \int_{t_0}^{T_n} \frac{r(s)\big|h(T_n,s)\sqrt{H(T_n,s)} + \frac{p(s)}{r(s)}H(T_n,s)\big|^{\alpha+1}}{H^\alpha(T_n,t_0)} ds \right)^{\frac{1}{\alpha+1}}
\end{aligned}
$$

and, accordingly,

$$
\begin{aligned}
&\frac{P^{\alpha+1}(T_n)}{Q^\alpha(T_n)}\\
&\le \frac{1}{\alpha^\alpha H(T_n,t_0)} \int_{t_0}^{T_n} \frac{r(s)\big|h(T_n,s)\sqrt{H(T_n,s)} + \frac{p(s)}{r(s)}H(T_n,s)\big|^{\alpha+1}}{H^\alpha(T_n,t_0)}\,ds\\
&= \frac{1}{\alpha^\alpha H(T_n,t_0)} \int_{t_0}^{T_n} \frac{r(s)\big|h(T_n,s) + \frac{p(s)}{r(s)}\sqrt{H(T_n,s)}\big|^{\alpha+1}}{H^{\frac{\alpha-1}{2}}(T_n,t_0)}\,ds.
\end{aligned}
$$

So, because of equation (2.24), we have

$$
\lim_{n\to\infty} \frac{1}{H(T_n,t_0)} \int_{t_0}^{T_n} \frac{r(s)\big|h(T_n,s) + \frac{p(s)}{r(s)}\sqrt{H(T_n,s)}\big|^{\alpha+1}}{H^{\frac{\alpha-1}{2}}(T_n,t_0)}\,ds = \infty,
$$

which gives

$$
\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{r(s)\big|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\big|^{\alpha+1}}{H^{\frac{\alpha-1}{2}}(t,t_0)}\,ds = \infty,
$$

contradicting condition equation (2.10). Therefore, equation (2.16) holds. Now, from equation (2.13) we obtain

$$\int_{t_0}^{\infty} \frac{\phi_+^{\frac{\alpha+1}{\alpha}}(s)}{r^{\frac{1}{\alpha}}(s)}\, ds \le \int_{t_0}^{\infty} \frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)}\, ds < \infty,$$

which contradicts equation (2.12). This completes the proof. □

The following result is the direct consequence of Theorem 2.2 and uses the same choice of the functions H and h as in Corollary 2.3 above.

Corollary 2.5. *Suppose that there exists a function* $\phi \in C[t_0, \infty)$ *such that equation* (2.12) *holds along with*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_{t_0}^{t} r(s)(t-s)^{\lambda-\alpha-1}\left|\lambda + \frac{p(s)}{r(s)}\,(t-s)\right|^{\alpha+1} ds < \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_{T}^{t} \left[(t-s)^{\lambda}q(s) - \frac{r(s)(t-s)^{\lambda-\alpha-1}\left|\lambda + \frac{p(s)}{r(s)}(t-s)\right|^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right] ds \ge \phi(T)$$

for all $T \ge t_0$ *and for some* $\lambda > \alpha$. *Then every solution of equation* (1.1) *is oscillatory.*

Proof. The only thing to be checked is condition equation (2.9). With the above choice of the functions H and h, this is fulfilled automatically since

$$\lim_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} = \lim_{t\to\infty} \frac{(t-s)^{\lambda}}{(t-t_0)^{\lambda}} = 1$$

for any $s \ge t_0$. □

Theorem 2.3. *Suppose that there exists a function* $H \in X$ *such that equation* (2.9) *holds and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)q(s)\, ds < \infty. \tag{2.25}$$

If there exists $\phi \in C[t_0, \infty)$ *such that for every* $T \geq t_0$,

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)q(s) - \frac{r(s)\left|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right|^{\alpha+1}}{(\alpha+1)^{\alpha+1} H^{\frac{\alpha-1}{2}}(t,s)} \right] ds \geq \phi(T), \tag{2.26}$$

and equation (2.12) *hold, then every solution of equation* (1.1) *is oscillatory.*

Proof. For the nonoscillatory solution $y(t)$ of equation (1.1), as in the proof of Theorem 2.1, (2.7) and (2.8) are satisfied. As in the proof of Theorem 2.2, (2.13) holds for $t \geq T \geq t_0$. Using equation (2.25), we conclude that

$$\limsup_{t\to\infty} [Q(t) - P(t)] \leq u(t_0) - \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)q(s)\,ds < \infty.$$

It follows from equation (2.26) that

$$\phi(t_0) \leq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)q(s)\,ds - \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \frac{r(s)\left|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right|^{\alpha+1}}{(\alpha+1)^{\alpha+1} H^{\frac{\alpha-1}{2}}(t,s)} \Bigg] ds,$$

so that equation (2.25) implies

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \frac{r(s)\left|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right|^{\alpha+1}}{(\alpha+1)^{\alpha+1} H^{\frac{\alpha-1}{2}}(t,s)}\,ds < \infty.$$

Considering a sequence $\{T_n\}_{n=1}^\infty$ in (t_0, ∞) with $\lim_{n\to\infty} T_n = \infty$ such that

$$\lim_{n\to\infty} |Q(T_n) - P(T_n)] = \limsup_{t\to\infty} [Q(t) - P(t)].$$

Then, using the procedure of the proof of Theorem 2.2, we conclude that equation (2.16) holds. The remainder of the proof proceeds as in the proof of Theorem 2.2. The proof is complete. □

In the following we will establish several more general interval oscillation theorems. The main method is to introduce a new transformation for equation (1.1).

Theorem 2.4. *Suppose that the functions* H *and* h *are defined as in Theorem* 2.1. *If there exists* $\rho \in C^1[t_0, \infty)$ *such that*

(2.27)
$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Bigg[H(t,s)\rho(s)q(s) - \frac{r(s)\big|h(t,s)+\big(\frac{p(s)}{r(s)}-\frac{\rho'(s)}{\rho(s)}\big)\sqrt{H(t,s)}\big|^{\alpha+1}}{(\alpha+1)^{\alpha+1}H^{\frac{\alpha-1}{2}}(t,s)} \Bigg] ds = \infty,$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1.1). Assume that $y(t) \neq 0$ for $t \geq t_0$. Define

$$u(t) = \rho(t) \frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\alpha-1}y(t)}, \quad t \geq t_0,$$

then for every $t \geq t_0$, we have

$$u'(t) = -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}\, u(t) - \frac{p(t)}{r(t)}\, u(t) - \alpha \frac{|u(t)|^{\frac{\alpha+1}{\alpha}}}{(r(t)\rho(t))^{\frac{1}{\alpha}}}, \tag{2.28}$$

and, consequently,

$$\begin{aligned}\int_{t_0}^{t} H(t,s)\rho(s)q(s)\,ds = &-\int_{t_0}^{t} H(t,s)u'(s)\,ds \\ &- \int_{t_0}^{t} H(t,s)\left(\frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)}\right) u(s)\,ds \\ &- \alpha \int_{t_0}^{t} H(t,s)\, \frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{(r(s)\rho(s))^{\frac{1}{\alpha}}}\,ds.\end{aligned}$$

By equation (2.6), we have

(2.29)
$$\begin{aligned}\int_{t_0}^{t} H(t,s)\rho(s)q(s)\,ds &\le H(t,t_0)u(t_0) - \alpha\int_{t_0}^{t} H(t,s)\frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{(r(s)\rho(s))^{\frac{1}{\gamma}\alpha}}\,ds \\ &\quad + \int_{t_0}^{t}\Big|h(t,s)\sqrt{H(t,s)} + \Big(\frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)}\Big)H(t,s)\Big||u(s)|\,ds.\end{aligned}$$

Therefore, in view of Lemma A, with

$$A = (\alpha H(t,s))^{\frac{\alpha}{\alpha+1}}\frac{|u(s)|}{(\rho(s)r(s))^{\frac{1}{\alpha}}}, \quad \lambda = \frac{\alpha+1}{\alpha},$$
$$B = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\left(\frac{r(s)\rho(s)}{(\alpha H(t,s))^{\alpha}}\right)^{\frac{\alpha}{\alpha+1}}\Big|h(t,s)+\Big(\frac{p(s)}{r(s)}-\frac{\rho'(s)}{\rho(s)}\Big)\sqrt{H(t,s)}\Big|^{\alpha+1},$$

we obtain for $t > s \ge t_0$,

$$\begin{aligned}&\Big|h(t,s)\sqrt{H(t,s)}+\Big(\frac{p(s)}{r(s)}-\frac{\rho'(s)}{\rho(s)}\Big)H(t,s)\Big||u(s)| \\ &\quad - \alpha H(t,s)\,\frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{(r(s)\rho(s))^{\frac{1}{\alpha}}} \\ &\le \frac{\rho(s)r(s)\big|h(t,s)\sqrt{H(t,s)}+\big(\frac{p(s)}{r(s)}-\frac{\rho'(s)}{\rho(s)}\big)H(t,s)\big|^{\alpha+1}}{(\alpha+1)^{\alpha+1}[H(t,s)]^{\frac{\alpha-1}{2}}}.\end{aligned} \tag{2.30}$$

From equation (2.29) and equation (2.30), we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Bigg[&H(t,s)\rho(s)q(s) \\ &-\frac{\rho(s)r(s)\big|h(t,s)+\big(\frac{p(s)}{r(s)}-\frac{\rho'(s)}{\rho(s)}\big)\sqrt{H(t,s)}\big|^{\alpha+1}}{(\alpha+1)^{\alpha+1}H^{\frac{\alpha-1}{2}}(t,s)}\Bigg]ds \\ &\le u(t_0),\end{aligned}$$

which contradicts equation (2.27). The proof is complete.

Corollary 2.6. *Let equation* (2.27) *in Theorem* 2.4 *be replaced by*

(2.31)
$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\rho(s)r(s)\big|h(t,s)+\big(\frac{(p(s)}{r(s)}-\frac{\rho'(s)}{\rho(s)}\big)\sqrt{H(t,s)}\big|^{\alpha+1}}{(\alpha+1)^{\alpha+1}H^{\frac{\alpha-1}{2}}(t,s)}\,ds < \infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)\,ds=\infty, \tag{2.32}$$

then every solution of equation (1.1) *is oscillatory.*

Following the procedure of the proof of Theorems 2.2 and 2.3, we can also prove the following two theorems.

Theorem 2.5. *Let the functions H and h be defined as in Theorem* 2.1 *such that equation* (2.9) *holds. If there exist two functions $\rho\in C^1[t_0,\infty)$ and $\phi\in C[t_0,\infty)$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\cdot\int_{t_0}^{t}\frac{\rho(s)r(s)\big|h(t,s)+\big(\frac{p(s)}{r(s)}-\frac{\rho'(s)}{\rho(s)}\big)\sqrt{H(t,s)}\big|^{\alpha+1}}{(\alpha+1)^{\alpha+1}H^{\frac{\alpha-1}{2}}(t,s)}\,ds<\infty,$$

and that for every $T>t_0$,

(2.33)
$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\Bigg[H(t,s)\rho(s)q(s)-\frac{\rho(s)r(s)\big|h(t,s)+\big(\frac{p(s)}{r(s)}-\frac{\rho'(s)}{\rho(s)}\big)\sqrt{H(t,s)}\big|^{\alpha+1}}{(\alpha+1)^{\alpha+1}H^{\frac{\alpha-1}{2}}(t,s)}\Bigg]ds \ge \phi(T),$$

and (2.12) *holds, then every solution of* (1.1) *is oscillatory.*

Theorem 2.6. *Let the functions H and h be defined as in Theorem* 2.1 *such that* (2.9) *holds. If there exist two functions $\rho \in C^1[t_0, \infty)$ and $\phi \in C[t_0, \infty)$ such that*

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)|q(s)|\,ds < \infty,$$

and that (2.12) *and* (2.33) *hold, then every solution of equation* (1.1) *is oscillatory.*

3. Asymptotics of the forced equation. In this section we study the asymptotic behavior of solutions of the forced half-linear differential equation with damping

(3.1)

$$[r(t)|y'(t)|^{\alpha-1}y'(t)]' + p(t)|y'(t)|^{\alpha-1}y'(t) + q(t)|y(t)|^{\alpha-1}y(t) = e(t),$$

where $\alpha > 0$ is a constant.

The main result of this section is the following.

Theorem 3.1. *Let the assumptions of Theorem* 2.1 *hold, and suppose that the function $e \in C[t_0, \infty)$ satisfies*

$$\int^{\infty} |e(t)|\,dt < \infty. \tag{3.2}$$

Then every solution of (3.1) *satisfies*

$$\liminf_{t\to\infty} |y(t)| = 0.$$

Proof. Let $y(t)$ be a solution of equation (3.1), and suppose that

$$\liminf_{t\to\infty} |y(t)| = c > 0,$$

so $y(t)$ is nonoscillatory. Without loss of generality, we may assume that $y(t) > c > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. Differentiating the function $u(t)$ defined by (2.4), we obtain

$$u'(t) = -q(t) - \frac{p(t)}{r(t)}u(t) - \alpha\frac{|u(t)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(t)} + \frac{e(t)}{|y(t)|^{\alpha-1}y(t)}$$

for all $t \geq T_0$, and thus

$$u'(t) \leq -q(t) - \frac{p(t)}{r(t)}\, u(t) - \alpha\, \frac{|u(t)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(t)} + \frac{|e(t)|}{c^{\alpha}}.$$

Hence, for $t \geq T \geq T_0$, we have

$$\begin{aligned}\int_T^t H(t,s)q(s)\,ds \leq -\int_T^t H(t,s)u'(s)\,ds - \int_T^t H(t,s)\,\frac{p(s)}{r(s)}\,u(s)\,ds \\ - \alpha \int_T^t H(t,s)\,\frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)}\,ds + \frac{1}{c^{\alpha}}\int_T^t H(t,s)|e(s)|\,ds,\end{aligned}$$

and, consequently,

$$\begin{aligned}&\int_T^t H(t,s)q(s)\,ds \\ &\leq H(t,T)u(T) + \int_T^t \left| h(t,s)\sqrt{H(t,s)} + \frac{p(s)}{r(s)}H(t,s)\right| |u(s)|\,ds \\ &\quad - \int_T^t H(t,s)\,\frac{|u(s)|^{\frac{\alpha+1}{\alpha}}}{r^{1/\alpha}(s)}\,ds + \frac{1}{c^{\alpha}}\int_T^t H(t,s)|e(s)|\,ds.\end{aligned} \tag{3.3}$$

Let A and B be as in the proof of Theorem 2.1. In view of Lemma A, (3.3) implies that

$$\begin{aligned}\int_T^t H(t,s)q(s)\,ds \leq H(t,T)u(T) + \frac{1}{c^{\alpha}}\int_T^t H(t,s)|e(s)|\,ds + \frac{1}{(\alpha+1)^{\alpha+1}} \\ \cdot \int_T^t \frac{r(s)\left|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right|^{\alpha+1}}{[H(t,s)]^{\frac{\alpha-1}{2}}}\,ds,\end{aligned}$$

for $t \geq t_0$. Consequently,

$$\begin{aligned}\int_T^t \left[H(t,s)q(s)\,ds - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)\left|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right|^{\alpha+1}}{[H(t,s)]^{\frac{\alpha-1}{2}}}\right] ds \\ \leq H(t,T)u(T) + \frac{H(t,T)}{c^{\alpha}}\int_T^t |e(s)|\,ds.\end{aligned}$$

Now the proof proceeds in the same way as in Theorem 2.1.

Following the procedure of the proof, Theorems 2.4 and 3.1, we can also prove the following more general result.

Theorem 3.2. *Let the assumptions of Theorem* 2.4 *hold, and suppose that the function* $e \in C[t_0, \infty)$ *satisfies*

$$\int^{\infty} |e(t)|\, dt < \infty.$$

Then every solution of equation (3.1) *satisfies*

$$\liminf_{t\to\infty} |y(t)| = 0.$$

4. Examples. In this section we will show the applications of our oscillation criteria by three examples. We will see that the equations in the examples are oscillatory based on the results in Section 2, though the oscillation cannot be demonstrated by most other known criteria.

Example 4.1. Consider the nonlinear differential equation

$$\begin{aligned}(4.1)\quad [t^{-\beta}|y'(t)|^{\alpha-1}y'(t)]' - t^{-\beta}|y'(t)|^{\alpha-1}y'(t)& \\ + t^{\gamma}\left(\gamma\frac{2-\cos t}{t} + \sin t\right)|y(t)|^{\alpha-1}y(t) &= 0,\end{aligned}$$

for $t \geq 1$, where α, β, γ are arbitrary positive constants and $\alpha \neq 2$. Then, for any $t \geq 1$, we have

$$\begin{aligned}\int_1^t q(s)\, ds &= \int_1^t d(s^{\gamma}(2-\cos s)) \\ &= t^{\gamma}(2-\cos t) - (2-\cos 1) \geq t^{\gamma} - k_0,\end{aligned}$$

where $k_0 = 2 - \cos 1$. Taking $H(t,s) = (t-s)^2$ for $t \geq s \geq 1$, we have

$$\begin{aligned}&\frac{1}{t^2}\int_1^t \left[(t{-}!s)^2 q(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\,\frac{|2-(t-s)|^{\alpha+1}}{s^{\beta}(t-s)^{\alpha-1}}\right] ds \\ &\quad = \frac{1}{t^2}\int_1^t \left[2(t-s)\left(\int_1^s q(\tau)\, d\tau\right) - \frac{1}{(\alpha+1)^{\alpha+1}}\,\frac{|2-(t-s)|^{\alpha+1}}{s^{\beta}(t-s)^{\alpha-1}}\right] ds \\ &\quad \geq \frac{2}{t^2}\int_1^t (t-s)(s^{\gamma}-k_0)\, ds - \frac{2^{\alpha+1}}{(\alpha+1)^{\alpha+1}t^2}\int_1^t (t-s)^{1-\alpha}\, ds \\ &\quad = \frac{2t^{\gamma}}{(\gamma+1)(\gamma+2)} + \frac{k_1}{t^2} + \frac{k_2}{t} - k_0 - \frac{k_3}{t^{\alpha}}\left(1-\frac{1}{t}\right)^{2-\alpha},\end{aligned}$$

where

$$k_1 = \frac{2}{\gamma+2} - k_0, \quad k_2 = 2k_0 - \frac{2}{\gamma+1}, \quad k_3 = \frac{2^{\alpha+1}}{(\alpha+1)^{\alpha+1}(2-\alpha)}.$$

Consequently, (2.3) holds. Hence equation (4.1) is oscillatory by Theorem 2.1.

We observe that Theorem 2.2 can be applied in some cases in which Theorem 2.1 is not applicable. Such a case is described in the following example.

Example 4.2. Consider the differential equation

(4.2)

$$[t^\beta |y'(t)|^{\alpha-1} y'(t)]' + t^\beta |y'(t)|^{\alpha-1} y'(t) + t^\gamma \cos t |y(t)|^{\alpha-1} y(t) = 0, \quad t \ge 1,$$

where α, β, γ are constants such that $-1 < \gamma \le 1$, $0 < \alpha \ne 2$, $\alpha > \beta$ and $\gamma(\alpha+1) \ge \beta - \alpha$. For example, $\alpha = 1$, $\beta = 1$, $\gamma = 1$ satisfy the above assumption. Taking $H(t,s) = (t-s)^2$ for $t \ge s \ge 1$,

$$\begin{aligned}
\frac{1}{t^2}\int_1^t s^\beta \frac{|2-(t-s)|^{\alpha+1}}{(t-s)^{\alpha-1}}\, ds &\le \frac{2^{\alpha+1}}{t^2}\int_1^t \frac{s^\beta}{(t-s)^{\alpha-1}}\, ds \\
&= \begin{cases} 2^{\alpha+1} t^{\beta-2} \dfrac{t-1^{2-\alpha}}{2-\alpha}, & \beta > 0 \\ \dfrac{2^{\alpha+1}}{t^2}\, \dfrac{t-1^{2-\alpha}}{2-\alpha}, & \beta < 0 \end{cases} \\
&= \begin{cases} \dfrac{2^{\alpha+1} t^{\beta-\alpha}}{2-\alpha}\left(1-\dfrac{1}{t}\right)^{2-\alpha}, & \beta > 0 \\ \dfrac{2^{\alpha+1}}{(2-\alpha)t^\alpha}\left(1-\dfrac{1}{t}\right)^{2-\alpha}, & \beta < 0. \end{cases}
\end{aligned}$$

Therefore, (2.10) holds and for arbitrary small constant $\varepsilon > 0$, there exists $t_1 \ge 1$ such that, for $T \ge t_1$,

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_1^t \left[(t-s)^2 s^\gamma \cos s - \frac{s^\beta |2-(t-s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(t-s)^{\alpha-1}}\right] ds \ge -T^\gamma \cos T - \varepsilon.$$

Now set $\phi(T) = -T^\gamma \cos T - \varepsilon$. Then there exists an integer N such that $(2N+1)\pi - (\pi/4) > t_1$, and, if $n \in N$,

$$(2n+1)\pi - \frac{\pi}{4} \le T \le (2n+1)\pi + \frac{\pi}{4}, \quad \phi(t) \ge \delta T^\gamma,$$

where δ is a small constant. Taking into account that $\gamma(\alpha+1) \le \beta - \alpha$, we obtain

$$\begin{aligned}\int_1^\infty \frac{\phi_+^{\frac{\alpha+1}{\alpha}}(s)}{r^{\frac{1}{\gamma}\alpha}(s)}\, ds &\ge \sum_{n=N}^\infty \delta^{\frac{\alpha+1}{\alpha}} \int_{(2n+1)\pi-\frac{\pi}{4}}^{(2n+1)\pi+\frac{\pi}{4}} s^{\frac{\gamma(\alpha+1)-\beta}{\alpha}}\, ds \\ &\ge \sum_{n=N}^\infty \delta^{\frac{\alpha+1}{\alpha}} \int_{(2n+1)\pi-\frac{\pi}{4}}^{(2n+1)\pi+\frac{\pi}{4}} \frac{1}{s}\, ds = \infty.\end{aligned}$$

Accordingly, all conditions of Theorem 2.2 are satisfied, and hence equation (4.2) is oscillatory.

Example 4.3. Consider the half-linear differential equation

$$\left[\frac{1}{5t}\,|y'(t)|y'(t)|\right]' + \frac{1}{t^2}\,|y'(t)|y'(t) + 2|y(t)|y(t) = \frac{2}{t^2}, \tag{4.3}$$

where $t \ge 1$. Now let $H(t,s) = (t-s)^3$, $h(t,s) = 3(t-s)^{1/2}$. Then, by straightforward computation, we obtain

$$\begin{aligned}&\frac{1}{t^3}\int_T^t \left[2(t-s)^3 - \frac{\left|3 + \frac{5s}{(}t-s)\right|^3}{135s}\right] ds \\ &\qquad = \frac{1}{2t^3}\,(t-T)^4 - \frac{1}{135t^3}\int_T^t \frac{1}{s}\left|\frac{5t}{s} - 2\right|^3 ds \\ &\qquad \ge \frac{1}{2t^3}\,(t-T)^4 - \frac{25}{27}\int_T^t \frac{1}{s^4}\, ds \\ &\qquad = \frac{1}{2t^3}\,(t-T)^4 + \frac{25}{81t^3} - \frac{25}{81T^3},\end{aligned}$$

and, hence,

$$\lim_{t\to\infty} \frac{1}{t^3}\int_T^t \left[2(t-s)^3 - \frac{\left|3 + \frac{5}{s}(t-s)\right|^3}{135s}\right] ds = \infty,$$

so assumption (2.3) holds.

Thus, by Theorem 3.1, we conclude that all solutions of (4.3) satisfy

$$\liminf_{t\to\infty} |y(t)| = 0.$$

Observe that $y(t) = 1/t$ is such a solution.

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REFERENCES

1. J.W. Baker, *Oscillation theorems for a second order damped nonlinear differential equation*, SIAM J. Math. Anal. **25** (1973), 37–40.

2. L.E. Bobisud, *Oscillation of solutions of damped nonlinear differential equations*, SIAM J. Math. Anal. **18** (1970), 601–606.

3. G.J. Butler, *The oscillatory behavior of a second order nonlinear differential equation with damping*, J. Math. Anal. Appl. **57** (1977), 273–289.

4. M. Del Pino, M. Elgueta and R. Manasevich, *Generalizing Hartman's oscillation result for* $(|x'|^{p-2}x')' + c(t)|x|^{p-2}x = 0$, $p > 1$, Houston J. Math. **17** (1991), 63–70.

5. O. Došlý, *A remark on half-linear extension of the Hartman-Wintner theorem*, Fourth Mississippi State Conf. on Differential Equations and Computation Simulations, Electronic J. Differential Equations, Conf. **03**, 1999, pp. 29–37.

6. Á. Elbert, *A half-linear second order differential equation*, in *Qualitative theory of differential equations*, Colloq. Math. Soc. János Bolyai, vol. 30, Szeged, 1979, pp. 153–180.

7. ———, *Oscillation and nonoscillation theorems for nonlinear ODE*, Lecture Notes in Math., vol. 964, Springer, New York, 1992.

8. S.R. Grace, *Oscillation theorems for second order nonlinear differential equations with damping*, Math. Nachr. **141** (1989), 117–127.

9. ———, *Oscillation criteria for second order nonlinear differential equations with damping*, J. Austral. Math. Soc. Ser. A **49** (1990), 43–54.

10. ———, *Oscillation theorems for nonlinear differential equations of second order*, J. Math. Anal. Appl. **171** (1992), 220–241.

11. ———, *Oscillation theorems for certain second order perturbed nonlinear differential equations*, J. Math. Anal. Appl. **77** (1980), 205–214.

12. S.R. Grace and B.S. Lalli, *Integral averaging technique for oscillation of second order nonlinear differential equations*, J. Math. Anal. Appl. **149** (1990), 277–311.

13. ———, *Oscillation theorems for second order superlinear differential equations with damping*, J. Austral. Math. Soc. Ser. A **53** (1992), 156–165.

14. S.R. Grace, B.S. Lalli and C.C. Yeh, *Oscillation theorems for nonlinear second order differential equations with a nonlinear damping term*, SIAM J. Math. Anal. **15** (1984), 1082—1093.

15. ———, *Addendum*: *Oscillation theorems for nonlinear second order differential equations with a nonlinear damping term*, SIAM J. Math. Anal. **19** (1988), 1252–1253.

16. G.H. Hardy, J.E. Litlewood and G. Polya, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 1988.

17. B.J. Harris, *On the oscillation of solutions of linear differential equations*, Mathematica **33** (1984), 214–226.

18. P. Hartman, *On nonoscillatory linear differential equations of second order*, Amer. J. Math. **74** (1952), 389–400.

19. H.L. Hong, W.C. Lian and C.C. Yeh, *The oscillation of half-linear differential equations with an oscillatory coefficient*, Math. Comput. Modelling **24** (1996), 77–86.

20. H.B. Hsu and C.C. Yeh, *Oscillation theorems for second order half-linear differential equations*, Appl. Math. Lett. **9** (1996), 71–77.

21. I.V. Kamenev, *Integral criteria of linear differential equations of second order*, Mat. Zametki **23** (1978), 249–251.

22. N. Kandelaki, A. Lomtatidze and D. Ugulava, *Oscillation and nonoscillation criteria of a second order half-linear equation*, Georgian Math. J. **7** (2000), 1–19.

23. Q. Kong, *Interval criteria for oscillation of second-order linear ordinary differential equations*, J. Math. Anal. Appl. **229** (1999), 258–270.

24. ———, *Oscillation criteria for second order half-linear differential equations*, Fields Institute Communications, vol. 21, 1999, pp. 317–323.

25. H.J. Li, *Oscillation criteria for second order linear differential equations*, J. Math. Anal. Appl. **194** (1995), 217–234.

26. H.J. Li and C.C. Yeh, *Sturmian comparison theorem for half-linear second-order differential equations*, Proc. Royal Soc. Edinburgh **125A** (1995), 1193–1204.

27. ———, *Oscillation of half-linear second order differential equations*, Hiroshima Math. J. **25** (1995), 584–596.

28. W.T. Li, *Oscillation of certain second-order nonlinear differential equations*, J. Math. Anal. Appl. **217** (1998), 1–14.

29. ———, *Positive solutions of second order nonlinear differential equations*, J. Math. Anal. Appl. **221** (1998), 326–337.

30. W.T. Li and R.P. Agarwal, *Interval oscillation criteria related to integral averaging technique for certain nonlinear differential equations*, J. Math. Anal. Appl. **245** (2000), 171–188.

31. ———, *Interval oscillation criteria for second order nonlinear differential equations with damping*, Comput. Math. Appl. **40** (2000), 217–230.

32. ———, *Interval oscillation criteria for a forced nonlinear ordinary differential equation*, Appl. Anal. **75** (2000), 341–347.

33. W.T. Li, M.Y. Zhang and X.L. Fei, *Oscillation criteria for second order nonlinear differential equations with damped term*, Indian J. Pure Appl. Math. **30** (1999), 1017–1029.

34. W.C. Lian, C.C. Yeh and H.J. Li, *The distance between zeros of an oscillatory solution to a half-linear differential equation*, Comput. Math. Appl. **29** (1995), 39–43.

35. J.V. Manojlovic, *Oscillation criteria for second-order half-linear differential equations*, Math. Comput. Modelling **30** (1999), 109–119.

36. J.D. Mirzov, *On some analogs of Sturm's and Kneser's theorems for nonlinear systems*, J. Math. Anal. Appl. **54** (1976), 418–425.

37. Ch.G. Philos, *Oscillation theorems for linear differential equations of second order*, Arch. Math. (Basel) **53** (1989), 482–492.

38. Yu.V. Rogovchenko, *Oscillation criteria for second order nonlinear perturbed differential equations*, J. Math. Anal. Appl. **215** (1997), 334–357.

39. ———, *Oscillation theorems for second order equations with damping*, Nonlinear Anal. **41** (2000), 1005–1028.

40. A. Wintner, *A criterion of oscillatory stability*, Quart. Appl. Math. **7** (1949), 115–117.

41. J.R. Yan, *A note on an oscillation criterion for an equation with damped term*, Proc. Amer. Math. Soc. **90** (1984), 277–280.

42. ———, *Oscillation theorems for second order linear differential equations with damping*, Proc. Amer. Math. Soc. **98** (1986), 276–282.

43. C.C. Yeh, *Oscillation theorems for nonlinear second order differential equations with damping term*, Proc. Amer. Math. Soc. **84** (1982), 397–402.

44. P.J.Y. Wong and R.P. Agarwal, *Oscillatory behavior of solutions of certain second order nonlinear differential equations*, J. Math. Anal. Appl. **198** (1996), 337–354.

45. ———, *Oscillation theorems and existence criteria of asymptotically monotone solutions for second order differential equations*, Dynamic Systems Appl. **4** (1995), 477–496.

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