ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 3, Fall 2003

ON SOME PROPERTIES OF DEDDENS ALGEBRAS

M.T. KARAEV AND H.S. MUSTAFAYEV

ABSTRACT. Deddens algebras and Shulman subspaces are introduced and their properties are studied. The descriptions of Deddens algebras associated with nilpotent and idempotent elements are given.

1. Introduction. Let H be a Hilbert space, $B(H)$ be an algebra of all bounded linear operators in H. In [**1**], Deddens determined for any invertible operator A from $B(H)$ the following algebra:

$$B_A \stackrel{\text{def}}{=} \Big\{X \in B(H) : \sup_{n\geq 0} \Big\|A^n X A^{-n}\Big\| \stackrel{\text{def}}{=} C_X < +\infty\Big\}.$$

It was proved in [**1**] that, for $A \geq 0$, B_A coincides with the nest algebra generated by the nest $\{E_A([0,\lambda]) : \lambda \geq 0\}$ (where E_A is the spectral measure of A) that gives a suitable characterization of nest algebras in all respects. Recently Todorov [**7**] has extended this result to weakly or strongly closed bimodules of a nest algebra. In [**2**] Deddens and Wong have proved that if $A = \lambda I + N$, where $\lambda \in \mathbf{C} \setminus \{0\}$ is a complex number, and N is a nilpotent operator, then the algebra B_A coincides with the commutant $\{A\}'$ of A. In their proof of the last statement the Hilbert property of the space H is essentially used.

The main aim of this paper is to show that the result of Deddens and Wong is valid in any unital Banach algebra.

2. Deddens algebras. Let B be a Banach algebra with the unit e. For any invertible element $a \in B$ put

$$B_a \stackrel{\text{def}}{=} \Big\{x \in B : \sup_{n\geq 0} \|a^n x a^{-n}\| \stackrel{\text{def}}{=} C_x < +\infty\Big\}.$$

We call the algebra B_a the Deddens algebra.

Received by the editors on December 15, 2000, and in revised form on March 13, 2001.

Our main result is the following.

Theorem 1. *Let B be a Banach algebra with a unit e. If $a = e + b$, where b is a nilpotent element of the algebra B, then the Deddens algebra B_a coincides with the commutant $\{a\}'$, i.e., $B_a = \{a\}'$.*

Before passing to the proof of the theorem, we prove the following general lemma.

Lemma 2. *Let B, a, b be the same as in Theorem 1. Let $a_n \in B$, $n = 0, 1, 2, \ldots,$ be such that*

1) $\|a_n\| = O(n^\alpha)$, $n \to +\infty$, *for some α, $0 \le \alpha < 1$;*

2) *for some $c \in B$*

$$a_n a = a a_{n-1} + c, \quad n = 1, 2, \ldots .$$

Then $a_0 = a_1 = a_2 = \ldots$.

Proof. It is sufficient to prove the lemma in the case $c = 0$. Indeed, it follows from the equality

$$a_n a = a a_{n-1} + c, \quad n \ge 1,$$

that

$$d_n a = a d_{n-1}, \quad n \ge 1, \tag{1}$$

where $d_n \overset{\text{def}}{=} a_n - a_{n-1}$. It is clear that $\|d_n\| = O(n^a)$ for $n \to +\infty$. Assume that the lemma is valid for $c = 0$. Taking into account (1) and applying our hypothesis to the sequence (d_n), we obtain the equality

$$d_0 = d_1 = d_2 = \ldots ,$$

that is,

$$a_1 - a_0 = a_2 - a_1 = \cdots = x.$$

Hence

$$a_n = a_0 + nx, \quad n \ge 1,$$

whence it follows from condition 1) that

$$\|x\| \leq \frac{\|a_n - a_0\|}{n} \longrightarrow 0 \quad \text{for } n \to +\infty,$$

i.e., $x = 0$. Therefore, $a_0 = a_1 = a_2 = \cdots$.

So it is sufficient to prove the statement of the lemma for $c = 0$.

Let $k \geq 2$ be the nilpotency degree of the element $b \in B$, that is, $b^k = 0$, but $b^{k-1} \neq 0$. Then for any $n \geq k$,

$$(e+b)^n = e + \alpha_1 b + \alpha_2 b^2 + \cdots + \alpha_{k-1} b^{k-1},$$

where $\alpha_m \overset{\text{def}}{=} C_n^m = \dfrac{n!}{m!(n-m)!}$, $m = 1, 2, \dots, k-1$. The inverse element of $(e+b)^n$ has the form

$$(e+b)^{-n} = e + \beta_1 b + \beta_2 b^2 + \cdots + \beta_{k-1} b^{k-1}$$

for some numbers $\beta_1, \beta_2, \dots, \beta_{k-1}$. Taking the equality

$$(e + \beta_1 b + \cdots + \beta_{k-1} b^{k-1})(e + \alpha_1 b + \cdots + \alpha_{k-1} b^{k-1}) = e,$$

then removing the parentheses and identifying the coefficients, we obtain the system that connects the numbers $\alpha_1, \dots, \alpha_{k-1}$ with the numbers $\beta_1, \dots, \beta_{k-1}$:

$$\text{(2)} \qquad \begin{cases} \beta_1 + \alpha_1 = 0 \\ \alpha_1\beta_1 + \beta_2 + \alpha_2 = 0 \\ \alpha_2\beta_1 + \alpha_1\beta_2 + \beta_3 + \alpha_3 = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \alpha_{k-2}\beta_1 + \alpha_{k-3}\beta_2 + \cdots + \beta_{k-1} + \alpha_{k-1} = 0. \end{cases}$$

From the equality

$$a_n a = a a_{n-1}$$

we have

$$a_n (e+b)^n = (e+b)^n a_0,$$

that is,

$$a_n = (e+b)^n a_0 (e+b)^{-n}$$

or

$$a_n = (e + \alpha_1 b + \cdots + \alpha_{k-1} b^{k-1}) a_0 (e + \beta_1 b + \cdots + \beta_{k-1} b^{k-1}),$$

for all $n \geq k$. Hence we have

$$\begin{aligned}
a_n - a_0 = {} & (\beta_1 a_0 b + \beta_2 a_0 b^2 + \cdots + \beta_{k-1} a_0 b^{k-1}) \\
& + (\alpha_1 b a_0 + \alpha_1 \beta_1 b a_0 b + \alpha_1 \beta_2 b a_0 b^2 + \cdots \\
& + \alpha_1 \beta_{k-2} b a_0 b^{k-2} + \alpha_1 \beta_{k-1} b a_0 b^{k-1}) \\
& + (\alpha_2 b^2 a_0 + \alpha_2 \beta_1 b^2 a_0 b + \alpha_2 \beta_2 b^2 a_0 b + \cdots \\
& + \alpha_2 \beta_{k-1} b^2 a_0 b^{k-1}) + \cdots \\
& + (\alpha_{k-2} b^{k-2} a_0 + \alpha_{k-2} \beta_1 b^{k-2} a_0 b \\
& + \alpha_{k-2} \beta_2 b^{k-2} a_0 b^2 + \cdots \\
& + \alpha_{k-2} \beta_{k-2} b^{k-2} a_0 b^{k-2} + \alpha_{k-2} \beta_{k-1} b^{k-2} a_0 b^{k-1}) \\
& + (\alpha_{k-1} b^{k-1} a_0 + \alpha_{k-1} \beta_1 b^{k-1} a_0 b \\
& + \alpha_{k-1} \beta_2 b^{k-1} a_0 b^2 + \cdots + \alpha_{k-1} \beta_{k-2} b^{k-1} a_0 b^{k-2} \\
& + \alpha_{k-1} \beta_{k-1} b^{k-1} a_0 b^{k-1}).
\end{aligned} \tag{3}$$

Since

$$\begin{aligned}
\alpha_m = \alpha_m(n) &= \frac{1}{m!} [n(n-1)(n-2) \cdots (n-m+1)] \\
&= n\varphi_1(m) + n^2 \varphi_2(m) + \cdots + n^m \varphi_m(m)
\end{aligned}$$

$(m = 1, 2, \dots, k-1)$ where φ_i, $i = 1, 2, \dots, m$, do not depend on n, then as we see from system (2), β_m is also calculated by the formula

$$\beta_m = n\psi_1(m) + n^2 \psi_2(m) + \cdots + n^p \psi_p(m),$$

where $p \leq k-1$, and coefficients ψ_i, $i = 1, 2, \dots, p$, do not depend on n. Therefore, after some simple calculations we can write the equality (3) as follows:

$$\begin{aligned}
a_n - a_0 = {} & n f_1(k, a_0, b) + n^2 f_2(k, a_0, b) + \cdots \\
& + n^{2(k-1)} f_{2(k-1)}(k, a_0, b)
\end{aligned} \tag{4}$$

where $f_j(k, a_0, b) \in B$, $j = 1, 2, \dots, 2(k-1)$, do not depend on n. For convenience we determine

$$J_{2(k-1)} \stackrel{\text{def}}{=} (a_n - a_0) - n^{2(k-1)} f_{2(k-1)}(k, a_0, b).$$

We have from equality (4) by virtue of condition 1)

$$\|f_{2(k-1)}(k, a_0, b)\| \leq \frac{\|a_n - a_0\|}{n^{2(k-1)}} + \frac{\|J_{2(k-1)}\|}{n^{2(k-1)}} \longrightarrow 0$$

for $n \to +\infty$. Hence we conclude that

$$f_{2(k-1)}(k, a_0, b) = 0.$$

The sequential repetition of this argument shows us that all summands in equality (4) equal zero, and thus we obtain

$$a_n - a_0 = 0$$

for any $n \geq k$, that is, $a_n = a_0$, $n \geq k$. It remains to show that

$$a_{k-1} = a_{k-2} = \cdots = a_1 = a_0.$$

Since

$$a_n = (e+b)^{n-m} a_m (e+b)^{-(n-m)}, \quad m = 1, 2 \ldots, k-1,$$

then, by using similar arguments, we see that

$$a_n - a_m = 0,$$

and so $a_n = a_m$ for each $n \geq k$ and m, $1 \leq m \leq k-1$.

Thus

$$a_0 = a_1 = a_2 = \ldots .$$

The lemma is proved. □

Now we prove the theorem.

Proof of Theorem 1. Let $x \in B_a$ be any element. Put

$$c_n \overset{\text{def}}{=} a^n x a^{-n}, \quad n \geq 0.$$

Then

$$c_n a = a^n x a^{-n} a = a(a^{n-1} x a^{-(n-1)}) = a c_{n-1},$$

that is,

$$c_n a = a c_{n-1}, \quad n \geq 1. \tag{5}$$

Since $x \in B_a$, there exists a constant $c_x > 0$ such that $\|c_n\| \leq c_x$, $n \geq 0$. Taking into account these inequalities and equality (5), we state by means of Lemma 2 that $c_n = c_{n-1}$, $n \geq 1$, and in particular, $c_1 = c_0$, and thus

$$axa^{-1} = x.$$

Hence, $ax = xa$, $x \in \{a\}'$. Consequently, $B_a \subset \{a\}'$. The inclusion $\{a\}' \subset B_a$ is obvious, and therefore $B_a = \{a\}'$. The theorem is proved. □

Let B be a Banach algebra with the idempotent p (i.e., $p^2 = p$) and with the unit e. We introduce the following notation

$$S_p \stackrel{\text{def}}{=} \{x \in B : px(e-p) = 0\}.$$

Our next theorem describes the Deddens algebra associated with the idempotent.

Theorem 3. *Let B be a Banach algebra with an idempotent p and with the unit e. Then*

$$B_{e+p} = S_p.$$

Proof. Think of the algebra B as having a (2×2)-matrix decomposition relative to the decomposition of the identity $e = p + (e-p)$; thus elements of B have the form $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Relative to this decomposition, p takes the form $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $e + p = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. An easy calculation then shows that $(e+p)^n \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} (e+p)^{-n}$ is a bounded sequence if and only if $b_{12} = 0$, which is equivalent to the desired result. □

Corollary 4. *Let $a \in B$ be regular by a von Neumann element* (*that is, there exists an element $b \in B$ satisfying the condition $a = aba$*). *Let*

$$B^a \stackrel{\text{def}}{=} \{x \in B : xa = ay \text{ for some } y \stackrel{\text{def}}{=} y_x\}.$$

Then B^a is an algebra and $B^a \subset B_{e+p_a}$, where $p_a \overset{\text{def}}{=} ab$ is an idempotent element of B.

Proof. It is clear that B^a is an algebra and p_a is an idempotent. Now we show that $B^a \subset B_{e+p_a}$. We will let $x \in B$. Then it is clear from the equality $xa = ay$ that

$$bxa = bay.$$

We have

$$abxa = abay = ay = xa,$$

whence

$$(e - ab)xa = 0,$$

i.e.,

$$(e - ab)xab = 0;$$

consequently,

$$(e - p_a)xp_a = 0.$$

This equality means by virtue of Theorem 3 that $x \in S_{p_a} = B_{e+p_a}$. The proof is completed. □

In the remainder of this section we are concerned with Deddens operator algebras.

For two arbitrarily chosen operators $L, M \in B(H)$, we introduce for consideration the following subspace of the algebra $B(H)$:

$$\mathcal{U}(L, M) \overset{\text{def}}{=} \{L\}' + \{L\}'M.$$

Such subspaces have been studied in detail by Shulman (see [**5**], [**6**]) for the integration operator V, $(Vf)(x) = \int_0^x f(t)\,dt$ and multiplication operator T, $(Tf)(x) = xf(x)$ in the space $L^2[0,1]$ in relation with nontransitivity of root algebras. We call the subspace $\mathcal{U}(L, M)$ the Shulman subspace.

The relation between Deddens algebras and Shulman subspaces is established in the next theorem. Below, the number $\lambda \in C$ is assumed to be such that $L_\lambda \overset{\text{def}}{=} \lambda I + L$ is an invertible operator.

Theorem 5. *Let the operators $L, M \in B(H)$ satisfy the Kleinecke-Shirokov condition, i.e., $X \overset{\text{def}}{=} [M, L] \in \{L\}'$. Then the intersection of Deddens algebra $B_{\lambda I+L}$ and the weak closure of Shulman subspace $\mathcal{U}(L, M)$ coincide with a commutant of the operator L, that is,*

$$B_{\lambda I+L} \cap \overline{\mathcal{U}(L, M)}^{w} = \{L\}'.$$

Proof. Let $A \in B_{L_\lambda} \cap \overline{\mathcal{U}(L, M)}^{w}$ be any operator. Then there exist the sequences of operators X_n and Y_n from $\{L\}'$ such that

$$\lim_{n\to\infty} \langle (X_n + Y_n M)x, y\rangle = \langle Ax, y\rangle$$

for all $x, y \in H$. Then it is clear that

$$\lim_{n\to\infty} \langle (X_n + Y_n M)Lx, y\rangle = \langle ALx, y\rangle$$

and

$$\lim_{n\to\infty} \langle L(X_n + Y_n M)x, y\rangle = \langle LAx, y\rangle.$$

Since, by the condition of the theorem,

$$ML - LM = X \in \{L\}',$$

it follows that

$$\langle (AL - LA)x, y\rangle = \lim_{n\to\infty} \langle Y_n Xx, y\rangle$$

for all $x, y \in H$. Therefore

$$AL - LA \in \overline{\{L\}'X}^{w} \subset \{L\}',$$

or,

$$AL - LA = Y, \tag{6}$$

where $Y \in \{L\}'$. Therefore,

$$AL_\lambda - L_\lambda A = Y.$$

Hence
$$A - L_\lambda A L_\lambda^{-1} = Y L_\lambda^{-1},$$
that is,
$$L_\lambda A L_\lambda^{-1} = A - Y L_\lambda^{-1}. \tag{7}$$
By multiplying both sides of equality (7) from the left by L_λ, and from the right by L_λ^{-1}, and again considering (7), we get that
$$\begin{aligned} L_\lambda^2 A L_\lambda^{-2} &= L_\lambda A L_\lambda^{-1} - L_\lambda Y L_\lambda^{-2} \\ &= A - Y L_\lambda^{-1} - Y L_\lambda^{-1} = A - 2Y L_\lambda^{-1}, \end{aligned}$$
or simply,
$$L_\lambda^2 A L_\lambda^{-2} = A - 2Y L_\lambda^{-1}.$$
Thus we prove by induction that
$$L_\lambda^n A L_\lambda^{-n} = A - nY L_\lambda^{-1}, \quad n \geq 0.$$
Since $A \in B_{L_\lambda}$, we have
$$\|Y L^{-1}\| \leq \frac{\|A\| + C_A}{n} \longrightarrow 0, \quad n \to +\infty,$$
i.e.,
$$Y L_\lambda^{-1} = 0,$$
and therefore $Y = 0$. This means by virtue of (6) that $A \in \{L\}'$. Consequently,
$$B_{L_\lambda} \cap \overline{\mathcal{U}(L, M)}^w \subset \{L\}'.$$
The inverse inclusion is obvious and so
$$B_{L_\lambda} \cap \overline{\mathcal{U}(L, M)}^w = \{L\}'.$$
The theorem is proved. □

Example. Let V be the Volterra integration operator $f \to \int_0^x f(t)\,dt$ and T be the multiplication operator $f \to xf(x)$ in $L^2[0,1]$. It is easy to verify that
$$TV - VT = V^2.$$

Hence, the operators V and T satisfy the condition of Theorem 5 and therefore

$$B_{1+V} \cap \overline{\mathcal{U}(V,T)}^{w} = \{V\}'.$$

Before passing to the next result, we note the following:

The radical R of any complex normed algebra $\mathcal{D}$ with identity is defined as

$$R(\mathcal{D}) \overset{\text{def}}{=} \{x \in \mathcal{D} : xy \text{ is quasinilpotent for all } y \in \mathcal{D}\}.$$

For an invertible operator A, let

$$R_A \overset{\text{def}}{=} \{X \in B(H) : \lim_{k\to\infty} \|A^k X A^{-k}\| = 0\}.$$

It is known [**1**] that R_A is a bilateral ideal in algebra B_A contained in the radical $R(B_A)$.

Proposition 6. *Let $L, M \in B(H)$, and let $[M, L] \in \{L\}'$. Then*

$$R_{L_\lambda} \cap \overline{\mathcal{U}(L,M)}^{w} = \{0\}.$$

Proof. As we already proved in Theorem 5, for each $A \in \overline{\mathcal{U}(L,M)}^{w}$, there exists $Y \in \{L\}'$ such that

$$AL - LA = Y. \tag{8}$$

Then for each $n \geq 0$, we have

$$\begin{aligned}\|Y\| &= \|L_\lambda^n Y L_\lambda^{-n}\| = \|L_\lambda^n (AL - LA) L_\lambda^{-n}\| \\ &= \|L_\lambda^n A L L_\lambda^{-n} - L_\lambda^n L A L_\lambda^{-n}\| \\ &= \|L_\lambda^n A L_\lambda^{-n} L - L L_\lambda^n A L_\lambda^{-n}\| \leq 2\|L\| \|L_\lambda^n A L_\lambda^{-n}\|.\end{aligned}$$

Consequently,

$$\|L_\lambda^n A L_\lambda^{-n}\| \geq \frac{1}{2} \frac{\|Y\|}{\|L\|}, \tag{9}$$

$n \geq 0$. The statement of the proposition directly follows from (9). The proof is completed. □

Corollary 7. *Let $\Delta = \Delta_V$ be the inner derivation $X \to [V, X]$ of $B(L^2[0,1])$ and p_n be the polynomial of the form $p_n(z) = (1+z)^n$. Then*

$$\inf_{n\geq 0} \|p_n(V)\| \geq \frac{\pi}{4} \|\Delta_V | \ker \Delta_V^2\|,$$

where V is the Volterra integration operator in $L^2[0,1]$.

Proof. As Sarason [**4**] proved, $\{V\}' = \mathrm{alg}\,(V)$, the weak closed algebra generated by the operators V and I. Therefore, it follows from the results of Shulman (see [**6**, Theorem 1.1]) that

$$\ker \Delta_V^2 = \overline{\mathcal{U}(V,T)}^w, \tag{10}$$

where T is an operator of multiplication by independent variable in $L^2[0,1]$. Now by setting in Proposition 6 $L = V$, $M = T$ and taking into account the equality $\|V\| = \frac{2}{\pi}$ (see [**3**, Problem 188]), (8) and (10), we get from (9) that for any $A \in \ker \Delta_V^2$,

$$\frac{\pi}{4}\|AV - VA\| \leq \|(I+V)^n A (I+V)^{-n}\|, \quad n \geq 0.$$

Hence, taking into account the known equality $\|(I+V)^{-1}\| = 1$ (see [**3**, Problem 190]), we have

$$\frac{\pi}{4}\|AV - VA\| \leq \|(I+V)^n\|\|A\|,$$

that is,

$$\frac{\pi}{4}\|\Delta_V(A)\| \leq \|(I+V)^n\|\|A\|.$$

We have from this

$$\inf_{n\geq 0}\|p_n(V)\| \geq \frac{\pi}{4}\|\Delta_V | \ker \Delta_V^2\|.$$

The proof is completed. □

Acknowledgments. The authors wish to express their gratitude to Professor Yu.V. Turovskii for motivating discussions, reading the manuscript and making valuable remarks. We also wish to thank the referee for his valuable suggestions and comments, and in particular, for improving the proof of Theorem 3.

REFERENCES

1. J.A. Deddens, *Another description of nest algebras*, Lecture Notes in Math., vol. 693, Springer, New York, 1978, pp. 77–86.

2. J.A. Deddens and T.K. Wong, *The commutant of analytic Toeplitz operators*, Trans. Amer. Math. Soc. **184** (1973), 261–273.

3. P.R. Halmos, *A Hilbert space problem book*, 2nd. ed., 1982.

4. D. Sarason, *A remark on the Volterra operator*, J. Math. Anal. Appl. **12** (1965), 244–246.

5. V.S. Shulman, *On transitivity of some space of operators*, Functional Anal. Appl. **16** (1982), 91–92.

6. ———, *Invariant subspace and spectral mapping theorem*, Banach Center Publ. vol. 30, PWN, Warsaw, 1994, pp. 313–325.

7. I. Todorov, *Bimodules over nest algebras and Deddens theorem*, Proc. Amer. Math. Soc. **127** (1999), 1771–1780.

Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, F. Agaev str. 9, Baku AZ141, Azerbaijan

Current address: Suleyman Demirel University, Department of Mathematics, Faculty of Arts and Sciences, 32260, Isparta, Turkey
E-mail address: garayev@fef.sdu.edu.tr

Yuzuncu Yil University, Faculty of Arts and Sciences, Department of Mathematics, 65080 Van, Turkey
E-mail address: hmustafayev@yahoo.com