ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 1, Spring 2004

A SURVEY ON RECENT ADVANCES ON THE NIKODÝM BOUNDEDNESS THEOREM AND SPACES OF SIMPLE FUNCTIONS

J.C. FERRANDO AND L.M. SÁNCHEZ RUIZ

ABSTRACT. In this paper we review the research about the barrelledness properties of the normed space of simple functions associated to a Boolean ring provided with the supremum-norm. We also exhibit some results concerning the barrelledness of certain closely related normed spaces of vector-valued functions. We have included an explanation of the strategy of some proofs and given account of the relevant techniques.

1. Introduction. This paper aims to survey the literature concerning normed spaces of simple functions associated to certain Boolean rings and study the barrelledness of some spaces of vector-valued bounded functions. Let us start by recalling some definitions that will be used throughout this paper. A subset A of a topological space X is said to be of *first category* if it is the union of a sequence of nowhere dense subsets of X, otherwise A is called of *second category*. A topological space is said to be *Baire* if each nonempty open subset is of second category. A subset A of a topological vector space, tvs for short, E over the field $\mathbf{K}$ of the real or complex numbers is *absolutely convex* if $\lambda x + \mu y \in A$ for each $x, y \in A$ and $\lambda, \mu \in \mathbf{K}$ such that $|\lambda| + |\mu| \leq 1$ and *absorbing* if for each $x \in E$ there exists $\lambda > 0$ such that $\lambda x \in A$. Each neighborhood of the origin of a tvs is absorbing. A tvs is called *locally convex*, lc, if there exists a base of neighborhoods of the origin formed by (closed) absolutely convex sets. Hereafter every tvs will be assumed to be Hausdorff. It can be shown that a tvs is metrizable if and only if it has a countable base of neighborhoods of the origin. A complete metrizable tvs E is called an (F)-space. If in addition E is lc, then E

2000 AMS *Mathematics Subject Classification.* Primary 46A08, 46E40, 46E27, 46G10. Secondary 28A33, 28B05.
Key words and phrases. Strong barrelledness properties, bounded measures, Boolean rings.
Supported by DGESIC PB97-0342 and Presidencia de la Generalitat Valenciana.
Received by the editors on July 17, 2001, and in revised form on February 8, 2002.

is called a *Fréchet* space. An lcs E is said to be *barrelled* [**59**] if each barrel, i.e., each absorbing closed absolutely convex subset, of E is a neighborhood of the origin; equivalently, if each linear mapping with closed graph from E into any Banach or any Fréchet space is continuous [**60**], holding Fréchet implies Baire lcs which implies barrelled. Let us mention that a tvs E is said to be *ultrabarrelled* [**71**], $\mathcal{L}$-barrelled in [**2**], if each linear mapping with closed graph from E into any (F)-space is continuous. Note that each Baire tvs is ultrabarrelled and each ultrabarrelled lcs is barrelled.

Given a nonempty set I, let us denote by $l_\infty(I)$ the Banach space over $\mathbf{K}$ consisting of all bounded scalar functions defined on I equipped with the supremum norm $\|f\| = \sup\{|f(t)| : t \in I\}$, whereas $l_0^\infty(I)$ will stand for the linear subspace of $l_\infty(I)$ formed by all those functions taking finitely many different values. The research on the space $l_0^\infty(I)$ starts in 1973 when Bennet and Kalton [**10**] noted that for $I = \mathbf{N}$ it becomes a barrelled space of first category. The barrelledness of this space was shown by Dieudonné that very year, cf. [**93**, p. 133], a fact which independently was also pointed out by Saxon [**79**] in 1974. The investigation of the topological properties of the space $l_0^\infty(I)$ brought in [**9**] and [**20**] to light the fact that it is not ultrabarrelled and contains no separable infinite-dimensional barrelled subspace; in particular, it does not contain a copy of c_0.

The space of the simple functions associated to a ring $\mathcal{A}$ of subsets of a nonempty set Ω is defined as follows. A function $f : \Omega \to \mathbf{K}$ is said to be $\mathcal{A}$-*simple* if it is a linear combination of characteristic functions χ_A, $A \in \mathcal{A}$. The vector space $l_0^\infty(\mathcal{A})$ of all $\mathcal{A}$-simple functions will be named the *space of simple functions* associated to the ring $\mathcal{A}$. Unless otherwise stated, we will assume that $l_0^\infty(\mathcal{A})$ is endowed with the supremum norm

$$\|f\| = \sup\{|f(\omega)| : \omega \in \Omega\}.$$

Setting $V = \operatorname{acx}\{\chi_A : A \in \mathcal{A}\}$, another natural norm is defined on $l_0^\infty(\mathcal{A})$ by the gauge of V,

$$\|f\|_V = \inf\{a > 0 : f \in aV\}.$$

Both norms are equivalent since if $f \in l_0^\infty(\mathcal{A})$, $\|f\| \leq 1$, it is not difficult to show by induction on the number of nonvanishing different values taken by f that $f \in 4V$, cf. [**41**, Proposition 5.1.1]. Hence,

$\|\cdot\| \leq \|\cdot\|_V \leq 4\|\cdot\|$. The completion of $l_0^\infty(\mathcal{A})$ is denoted by $l_\infty(\mathcal{A})$, so that $l_\infty(\mathcal{A})$ is the Banach space of all bounded $\mathcal{A}$-measurable functions equipped with the supremum norm. If E is a vector space, a function $f : \Omega \to E$ is called $\mathcal{A}$-simple if it is of the form $f = \sum_{i=1}^n x_i \chi_{A_i}$ for $x_i \in E$ and $A_i \in \mathcal{A}$, $1 \leq i \leq n$. The space of all E-valued $\mathcal{A}$-simple functions will be denoted by $l_0^\infty(\mathcal{A}, E)$. If E is an lcs whose topology is generated by a family $\mathcal{P}$ of semi-norms, then $l_0^\infty(\mathcal{A}, E)$ becomes an lcs when equipped with the semi-norms $\{q_p, p \in \mathcal{P}\}$ where $q_p(f) := \sup\{p(f(\omega)) : \omega \in \Omega\}$. We will denote by $ba\,(\mathcal{A})$ the vector space over $\mathbf{K}$ of the bounded finitely additive scalar measures defined on $\mathcal{A}$ equipped with the supremum norm. If Σ is an algebra of subsets of Ω and Π represents the family of all (finite) partitions of Ω by members of Σ, then a function $f : \Omega \to \mathbf{K}$ is Σ-simple if there exist n scalars $a_1, \ldots, a_n$ with $a_i \neq a_j$ if $i \neq j$ and $\{E_1, \ldots, E_n\} \in \Pi$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$ and the following two norms may be considered on $ba(\Sigma)$

(1) The variation norm $\|\mu\|_1 = |\mu|(\Omega) = \sup\{\sum_{E \in \pi} |\mu(E)| : \pi \in \Pi\}$.

(2) The supremum norm $\|\mu\|_\infty = \sup\{|\mu(E)| : E \in \Sigma\}$.

We may identify $ba(\Sigma)$ with $l_0^\infty(\Sigma)^*$ by means of the algebraic isomorphism T defined by $\langle T\mu, \chi_E \rangle = \mu(E)$ for each $\mu \in ba(\Sigma)$ and $E \in \Sigma$. Writing $\mu(f)$ instead of $\langle T\mu, f \rangle$, routine calculations give that $\|\mu\|_1 = \sup\{|\mu(f)| : \|f\| \leq 1\}$ and

$$\|\mu\|_\infty = \sup\left\{\left|\mu\left(\sum_{i=1}^n a_i \chi_{E_i}\right)\right| : \{E_i\}_{i=1}^n \in \Pi, \sum_{i=1}^n |a_i| \leq 1\right\}.$$

So we have $\|\cdot\|_1 = \|\cdot\|^*$ and $\|\cdot\|_\infty = \|\cdot\|_V^*$, providing the inequalities $\|\cdot\|_\infty \leq \|\cdot\|_1 \leq 4\|\cdot\|_\infty$. Moreover, the following linear isometries hold,

$$l_0^\infty(\Sigma)^* \cong (ba(\Sigma), \|\cdot\|_1), \quad (l_0^\infty(\Sigma), \|\cdot\|_V)^* \cong (ba(\Sigma), \|\cdot\|_\infty).$$

If Σ is a σ-algebra, the classical Nikodým boundedness theorem, cf. [**21, 23**], establishes that if $\{\mu_t : t \in I\}$ is a subset of $ba(\Sigma)$ such that $\sup\{|\mu_t(E)| : t \in I\} < \infty$ for each $E \in \Sigma$, then $\sup\{\|\mu_t\|_\infty : t \in I\} < \infty$. When $\mathcal{A}$ is a ring of subsets of a set Ω enjoying this property, $\mathcal{A}$ is said to have *property* (N). As noticed by Schachermayer [**81**], property (N) is equivalent to the barrelledness of the space $l_0^\infty(\mathcal{A})$.

In the forthcoming sections we review a number of results about the barrelledness properties of the space of simple functions, also covering the vector-valued case. We also include some results upon Boolean rings with property (N) and give room to the study of the barrelledness of some spaces of E-valued bounded functions (E usually normed) closely related with the space $l_0^\infty(\mathcal{A}, E)$.

2. Strong barrelledness properties. We require recalling several notions of barrelledness that are located between barrelledness itself and the Baire property, many of which were originally introduced in order to extend the classic Banach's closed graph theorem. In the previous section a closed graph theorem for barrelled spaces has been stated. The closed graph theorem for lc Baire spaces given in [**93**] demands the definition of quasi-Suslin spaces. A Hausdorff topological space E is called *quasi-Suslin* [**93**] if there exist a Polish space X, i.e., a complete metrizable separable topological space, and a map f from X into the power set of E such that $\cup\{f(x) : x \in X\} = E$ and if $\{x_n\}$ is a sequence in X converging to $x \in X$ and $z_n \in Tx_n$ for each $n \in \mathbf{N}$, then $\{z_n\}$ has a cluster point in Tx. If E is an lc Baire space, F a quasi-Suslin lcs and T a linear map from E into F with closed graph, then T is continuous. Current monographs covering strong barrelledness up to 1995 are [**41, 61, 69, 93**].

To start with, let us mention that in 1972 Saxon [**78**] introduced *Baire-like* (BL) spaces as those lc spaces E such that, given an increasing sequence of closed absolutely convex subsets of E that covers E, one of them is a neighborhood of 0. The closed graph theorem for BL spaces states that each linear mapping with closed graph from a BL space into an (LB)-space, i.e., a strict inductive limit of Banach spaces, is continuous [**78**]. Like barrelled spaces, BL spaces are stable under the formation of separated quotients, countable-codimensional subspaces, arbitrary products, completions and satisfy the three-space problem [**41**, Section 1.2]. Because of the Amemiya-Komura theorem [**6**], each metrizable lcs is barrelled if and only if it is BL. Hence, if the algebra Σ has property (N), the space $l_0^\infty(\Sigma)$ is BL. A deep result of Saxon [**78**], cf. [**61**, p. 91] and [**41**, p. 15], states that if E is barrelled and has no copy of φ (the vector space $\mathbf{K}^{(N)}$ endowed with its strongest locally convex topology), then E is BL. If E is a BL space and F is a dense subspace, Amemiya-Komura's theorem gives that if F is barrelled then F is BL. In 1973 Todd and Saxon [**86**] introduced *unordered Baire-like* (UBL) spaces as those lc spaces E such that, given an arbitrary sequence of closed absolutely convex subsets covering E, one of them is a neighborhood of 0. The closed graph theorem for UBL spaces asserts that each linear mapping with closed graph from a UBL space into an inductive limit of countably many Fréchet spaces is continuous [**86**]. It is obvious that lcs Baire implies UBL implies BL implies barrelled. UBL spaces have similar stability properties to BL spaces, cf. [**41**, Section 1.3]. The BL and UBL spaces have been extended in the realm of tvs by Ką̧kol [**56**], Pérez Carreras [**67**] and Kakol and Roelcke [**57, 58**]. *Baire-hyperplane* (BH) spaces were defined in 1979 by Valdiva as those lcs that cannot be covered by any sequence of closed hyperplanes [**89**]. BH spaces also enjoy good permanence properties, and it should be noted that a BH space need not be barrelled, though each UBL space is BH and barrelled. In 1981, Valdivia and Pérez Carreras [**94**] introduced *totally barrelled* (TB) spaces as those lc spaces E such that, given a sequence of vector subspaces of E that covers E, one of them is barrelled and its closure is finite-codimensional in E. Later on, in 1982, Pérez Carreras and Bonet [**68**] proved that an lcs E is TB if and only if, given a sequence of vector subspaces of E that covers E, one of them is BL. TB spaces have the same stability properties as BL and UBL spaces, cf. [**41**].

In order to state a closed graph theorem for TB spaces, we need the notion of $\mathcal{C}$-web [**16**]. Denoting by $W(\mathbf{N})$ the language defined by the infinite alphabet $\mathbf{N}$ including the empty word (the null length word that we will denote by λ), i.e., $W(\mathbf{N}) = \cup\{\mathbf{N}^k : k \in \mathbf{N} \cup \{0\}\}$, let us call a *web* in a set X to a family $\mathcal{W} = \{C_w : w \in W(\mathbf{N})\}$ of subsets of X such that $X = C_\lambda$ and for each $w \in W(\mathbf{N})$, if (w, n) denotes the word in $W(\mathbf{N})$ obtained by adding the letter n to the word w, then $C_w = \cup_{n=1}^{\infty} C_{w,n}$. The web $\mathcal{W}$ is called *increasing* [**41**] if $C_{w,n} \subseteq C_{w,n+1}$ for each $(w, n) \in W(\mathbf{N}) \times \mathbf{N}$ and *absolutely convex* [**60**] if each set $C \in \mathcal{W}$ is absolutely convex. A *strand* of $\mathcal{W}$ is a sequence of subsets $\{C_{n_1,\ldots,n_i}\}_i$ where $\{n_i\}$ is a sequence in $\mathbf{N}$. A $\mathcal{C}$-*web* is a web $\mathcal{W}$ in an lcs E with the property that, for each strand $\{C_{n_1,\ldots,n_i}\}_i$ of $\mathcal{W}$ there is a sequence $\{\rho_i\}$ of positive numbers such that, for all $0 \le \lambda_i \le \rho_i$ and all $x_i \in C_{n_1,\ldots,n_i}$ the series $\sum_{i=1}^{\infty} \lambda_i x_i$ converges in E. An lcs containing a $\mathcal{C}$-web is said to be a *webbed space*. The closed graph theorem for TB spaces given in [**69**] ensures that for each linear mapping T with closed graph from a TB space E into an lc space F containing an absolutely convex $\mathcal{C}$-web, there exists a vector subspace

H of F dominated by a Fréchet space $H(\tau)$ such that $T(E) \subseteq H$ and $T : E \to H(\tau)$ is continuous. It is easy to note that UBL implies TB implies BL. In 1981 Saxon and Narayanaswami [**80**] proved that each infinite-dimensional Fréchet space contains a dense TB subspace which is not UBL. In 1983, Arias de Reyna [**7**] proved the following result, solving thereby a problem posed in [**94**].

Theorem 2.1. *If Σ is an infinite σ-algebra, then the space $l_0^\infty(\Sigma)$ is not* TB.

The idea of the proof requires some knowledge of the Stone space of a Boolean ring, cf. [**25, 52, 83, 84**], and runs as follows. Let $\{\Omega_n\}$ be a tree of infinite subsets of Ω formed by elements of Σ and let Λ denote the algebra generated by the finite unions of elements of $\{\Omega_n\}$. If S_Σ and S_Λ stand for the Stone spaces of Σ and Λ, respectively, write $\overline{E} = \{\mu \in S_\Sigma : \mu(E) = 1\}$ if $E \in \Sigma$ and $\hat{A} = \{\lambda \in S_\Lambda : \lambda(A) = 1\}$ if $A \in \Lambda$. Since the transpose of the canonical injection from $l_0^\infty(\Lambda)$ into $l_0^\infty(\Sigma)$ is a continuous linear map from $l_0^\infty(\Sigma)^*$ (weak*) into $l_0^\infty(\Lambda)^*$ (weak*), its restriction J to S_Σ is a continuous map from S_Σ onto S_Λ. According to [**41**, Propositions 6.6.2 and 6.6.3], this implies that there is a closed subset Z of S_Σ with the property that if $G \in \Sigma$ verifies that $\overline{G} \cap Z \neq \varnothing$, there is a $p \in \mathbf{N}$ with $J^{-1}(\hat{\Omega}_p) \cap Z \subseteq \overline{G}$. The fact that $J^{-1}(\hat{\Omega}_n) \cap Z \neq \varnothing$ for each $n \in \mathbf{N}$ enables us to define a sequence $\{F_n\}$ of closed vector subspaces of $l_0^\infty(\Sigma)$ by setting $F_n := \{f \in l_0^\infty(\Sigma) : \langle \mu, f\rangle = \langle \lambda, f\rangle \ \forall\, \mu, \lambda \in J^{-1}(\hat{\Omega}_n) \cap Z\}$. By the property above the sequence $\{F_n\}$ covers $l_0^\infty(\Sigma)$. Moreover, each F_n is infinite-codimensional since for each positive integer k it is possible to choose k pairwise disjoint sets $A_1, \dots, A_k \in \Lambda$ such that $F_n \cap \operatorname{sp}(\{\chi_{A_i} : 1 \leq i \leq k\}) = \{0\}$. Hence $l_0^\infty(\Sigma)$ is not TB.

If E and F are metrizable barrelled spaces, then $E \otimes_\pi F$ is barrelled, cf. [**55**]. Quite similar results for other barrelledness properties hold [**92**], cf. [**41**]. Unlike barrelled spaces none of the classes mentioned so far are stable under the formation of locally convex hulls. A wide class of barrelled spaces enjoying this property is obtained by considering inductive limits of Banach spaces, the so called *ultrabornological* spaces (if we drop completeness and consider inductive limits of normed spaces we generate the class of *bornological* spaces, being well known that each sequentially complete bornological space is ultrabornological).

Recently the interest on ultrabornological spaces has been recovered since many useful noncomplete normed spaces of measurable functions enjoy this property [**18, 51**]. A *Banach disk* is a closed absolutely convex subset of a tvs whose linear span becomes a Banach space when provided with the norm defined by its Minkowski functional. An lcs is ultrabornological if and only if it is the locally convex hull of its Banach disks [**60**] and each linear mapping with sequentially closed graph from an ultrabornological space into a webbed space is continuous [**16**]. In [**31**] it is shown that if E is an infinite-dimensional Banach space with a Schauder basis, there is a dense subspace F of E which is TB and ultrabornological but not UBL. In the particular case that E coincides with l_1 this result was already known, since [**93**, p. 277] proved that the subspace F of l_1 formed by the sequences whose support is a subset of $\mathbf{N}$ of density zero is a TB ultrabornological space which is not UBL. The subsets of $\mathbf{N}$ of density zero will be paid more attention at the end of Section 4. Since each Banach disk of $l_0^\infty(\Sigma)$ happens to be finite-dimensional whenever Σ is an infinite algebra of subsets of Ω [**20**], we have

Theorem 2.2 [**20**]. *If Σ is an infinite algebra of subsets of Ω, then $l_0^\infty(\Sigma)$ is not ultrabornological.*

This result implies that if Σ is an infinite algebra with property (N), then $l_0^\infty(\Sigma)$ is a nonultrabornological barrelled space although $l_0^\infty(\Sigma)$ is bornological since metrizable. Let us remark that the first examples of bornological barrelled spaces which are not ultrabornological were given by Valdivia [**87**]. Despite the above, $l_0^\infty(\Sigma)$ still can be represented as a projective limit of ultrabornological spaces. Indeed if $\Re$ stands for the family of all the ultrafilters on Ω and for each $U \in \mathcal{U} \in \Re$ we write $L(U) = \{f \in l_\infty(\Omega) : f(t) = f(s) \ \forall\, t, s \in U\}$ and $L(\mathcal{U}) = \{L(U) : U \in \mathcal{U}\}$, then each $L(\mathcal{U})$ is a vector space such that $l_0^\infty(\Sigma)$ coincides with $\cap\{L(\mathcal{U}) : \mathcal{U} \in \Re\}$. Since each $L(\mathcal{U})$ is ultrabornological [**32**] and $l_0^\infty(\Sigma)$ is the locally convex kernel of the family $\{L(\mathcal{U}) : \mathcal{U} \in \Re\}$, we are done.

There exists an underlying relationship between strong barrelledness properties and vector-valued measure theory that has been highlighted in [**39, 41**], which adds new reasons for the interest in this kind of properties to the aforementioned closed graph theorems. Both reasons motivated the introduction of suprabarrelled (SB) spaces by Valdivia

[**88, 90**] in 1978 (called (*db*) in [**72, 80**]) as those lc spaces E such that, given an increasing sequence of vector subspaces of E that covers E, one of them is dense and barrelled. The closed graph theorem for SB spaces assures that each linear mapping from an SB space into an (LF)-space, i.e., a strict inductive limit of Fréchet spaces, is continuous. The above definition was generalized by Rodríguez Salinas [**73**] in 1980 by transfinite induction as follows. Calling barrelled of class 0 to the barrelled spaces, for each successor ordinal $\alpha+1$ a space E is said to be *barrelled of class* $\alpha+1$, given an increasing sequence of vector subspaces of E that cover E, one of them is dense and barrelled of class α and, for each limit ordinal $\alpha \neq 0$, a space E is said to be *barrelled of class* α if E is barrelled of class β for all $\beta < \alpha$, thus SB spaces becoming the class of barrelled spaces of class 1. The study of the properties of barrelled spaces of class $n \in \mathbf{N}$ and of class ω_0, called $\aleph_0$ in [**41**], as well as distinguishing examples may be found in [**38, 41**], setting that none of the following implications can be reversed

$$\begin{aligned} TB &\Longrightarrow \text{Barrelled of class } \aleph_0 \Longrightarrow \text{Barrelled of class } n+1 \Longrightarrow \\ &\Longrightarrow \text{Barrelled of class } n \Longrightarrow BL \Longrightarrow \text{Barrelled.} \end{aligned}$$

In order to state a closed graph theorem for barrelled spaces of class n, let us call *p-sequence* in a vector space L to a countable family $\{L_{n_1\cdots n_p} : n_1,\dots,n_p \in \mathbf{N}\}$ of vector subspaces such that $\{L_{n_1} : n_1 \in \mathbf{N}\}$ is an increasing sequence covering L and, for each $(n_1,\dots,n_{q-1}) \in \mathbf{N}^{q-1}$ with $1 < q \le p$ the sequence $\{L_{n_1\cdots n_{p-1}n_p} : n_p \in \mathbf{N}\}$ is increasing and verifies that $\bigcup_{n_p=1}^{\infty} L_{n_1\cdots n_{p-1}n_p} = L_{n_1\cdots n_{p-1}}$. If T is a linear mapping with closed graph from a barrelled space E of class p into an lcs F containing a p-sequence $\mathcal{W}_p$ such that each $L \in \mathcal{W}_p$ has a dominating Fréchet space $L(\tau_L)$, there is an $H \in \mathcal{W}_p$ such that $T(E) \subseteq H$ and $T : E \to H(\tau_H)$ is continuous [**41**].

In 1992 the authors introduced *baireled* spaces [**42**] as a class $\mathcal{A}$ of BL spaces that is maximal in the sense that if $E \in \mathcal{A}$ and $\{E_n\}$ is an increasing sequence of vector subspaces of E covering E, one of them belongs to $\mathcal{A}$. Its definition is facilitated by calling *linear web* [**42**] in an lcs E to any increasing web of E formed by vector subspaces. Then *baireled* spaces [**41**] are those lcs E such that each linear web in E contains a strand formed by BL spaces. Baireledness is transmitted by dense subspaces and inherited by closed quotients, countable-codimensional subspaces and finite products, holding that the topological product of arbitrarily many baireled spaces is baireled whenever each product of countably many of them is [**42**]. In *op. cit.* it is shown that baireled spaces are strictly located between TB spaces and barrelled spaces of class $\aleph_0$ giving distinguishing examples. Non-TB spaces which are baireled are obtained by showing that if E denotes a baireled metrizable space and F is a UBL space, then $E \otimes_\pi F$ is baireled [**42**, Proposition 4] and recalling that if E is the subspace of l_1 formed by all the scalar sequences whose support has density zero, then E is a TB space [**94**, Example 1] such that $E \otimes_\pi l_2$ is not TB [**94**, Corollary 8.1], cf. [**41**, Lemma 4.5.1]. Non-baireled spaces which are barrelled of class $\aleph_0$ are obtained in each nonnormable Fréchet space [**42**, Theorem 2] by using an old result of Eidelheit, cf. [**41**, Lemma 3.3.2], after showing that the space ω, product of $\aleph_0$ copies of $\mathbf{K}$, contains a dense subspace E with the above property. This subspace E is built up as the union of an increasing sequence of subspaces $\{E_r\}$, where $E_r := \omega^r \times \prod_{i=r+1}^{\infty} H_i$, $r \in \mathbf{N}$, and each H_n is a dense subspace of $\omega^{\mathbf{N}} \cong \omega$ which is barrelled of class n but not barrelled of class $n+1$ [**41**, Proposition 3.3.1]. Including baireled spaces in, the scheme above is simplified by taking into account that every dense barrelled subspace of a BL space is BL. Thus, if B_0 denotes the class of BL spaces, then a space E is barrelled of class n, or briefly $E \in B_n$, if and only if, given an increasing sequence of vector subspaces of E covering E, one of them belongs to B_{n-1}. So, for each n, we have

$$TB \Rightarrow \text{Baireled} \Rightarrow \text{Barrelled of class } \aleph_0 \Rightarrow B_n \Rightarrow B_{n-1} \Rightarrow \text{Barrelled.}$$

Baireled spaces admit several interesting characterizations. Mimicking the above notation, let us denote by B_α the class of barrelled spaces of class α for each ordinal $\alpha \geq \omega_0$, then we have:

Theorem 2.3. *Given a space E, the following assertions are equivalent*:

(i) [**42**] *E is baireled.*

(ii) [**75**] *Each linear web in E contains a strand formed by dense barrelled spaces.*

(iii) [**42**] *Each increasing sequence of subspaces of E covering E contains a baireled space.*

(iv) [**42**] *Each linear web in E contains a strand formed by baireled spaces.*

(v) [**42**] *E belongs to a maximal class of BL spaces in the sense explained above.*

(vi) [**75**] *$E \in B_\alpha$ for each ordinal number α.*

(vii) *$E \in B_\alpha$ for some ordinal $\alpha \geq \omega_1$.*

(viii) *Each linear web in E contains a strand formed by elements of $\cup\{B_\alpha : \alpha \geq 0\}$.*

The above theorem gathers and extends previous results of [**42, 75**], implying that baireledness cannot be strengthened by any barrelledness property of class $\alpha > \omega_1$. As a simple consequence of [**91**, Theorems 1 and 2], we have the following closed graph theorem.

Theorem 2.4 [**42**]. *Given a linear mapping T with closed graph from a baireled space E into an lcs F containing an increasing absolutely convex $\mathcal{C}$-web, there exists a vector subspace H of F dominated by a Fréchet space $H(\tau)$ such that $T(E) \subseteq H$ and $T : E \to H(\tau)$ is continuous.*

Baireled spaces have proven to be useful since López Pellicer [**62**] has shown that $l_0^\infty(\Sigma)$ is baireled whenever Σ is a σ-algebra of subsets of a set Ω. This result answers in the positive a question posed in [**42**] and improves previous results concerning strong barrelledness properties of the space of simple functions, see [**36, 37, 43, 74, 88**] and [**41**]. Another class of lc spaces with strong barrelledness properties not mentioned above having nice closed graph theorems, as for instance Valdivia's convex-Baire spaces [**93**], have been introduced elsewhere. An account of some of these may be found in [**41**]. Barrelled spaces of class n and baireled spaces have been defined in the realm of tvs in [**76, 77**].

3. Baireledness of the space of simple functions. In this section we are going to review the proof [**62**] of the baireledness of the space $l_0^\infty(\Sigma)$ whenever Σ is a σ-algebra of subsets of a nonempty set Ω. The argument, which is highly technical, uses an original combinatorial instrument along with some sliding hump methods inspired on Valdivia's seminal paper [**88**], going beyond the techniques previously developed in [**36, 43**] in order to prove the barrelledness of class $\aleph_0$ of the space $l_0^\infty(\Sigma)$. Firstly we look at the main ideas of [**43**]. These are split into an algebraic part where the sliding hump arguments prevail and another part where the topological structure of the space $l_0^\infty(\Sigma)$ comes into play, as in Theorem 3.2 below. Given $A \in \Sigma$, $l_0^\infty(\Sigma|_A)$ will stand for the vector subspace of $l_0^\infty(\Sigma)$ spanned by $\{\chi_B : B \in \Sigma, B \subseteq A\}$, that is, the vector space formed by all the $\Sigma|_A$-simple functions where $\Sigma|_A$ denotes the σ-algebra generated by restricting Σ to A, and $\mathcal{F}(\Sigma)$ will stand for the family of finite-dimensional vector subspaces of $l_0^\infty(\Sigma)$.

It is easy to show that if $A \in \Sigma$ and no $F \in \mathcal{F}(\Sigma)$ satisfies $l_0^\infty(\Sigma|_A) \subseteq E+F$ for a given vector subspace E of $l_0^\infty(\Sigma)$, the following properties are true:

(1) Given a partition of A by means of q members $Q_1, Q_2, \dots, Q_q$ of Σ, there is some $i \in \{1, \dots, q\}$ such that there is no $F \in \mathcal{F}(\Sigma)$ with $l_0^\infty(\Sigma|_{Q_i}) \subseteq E+F$.

(2) Given any $q \in \mathbf{N}$ and $x_1, x_2, \dots, x_r \in l_0^\infty(\Sigma)$, there exists a partition of A by means of q members $Q_1, Q_2, \dots, Q_q$ of Σ such that $\chi_{Q_i}, \notin \operatorname{sp}(E \cup \{x_1, x_2, \dots, x_r\})$, for $1 \le i \le q$.

Given $p \in \mathbf{N}$ it is called a *p-net* any subset of $\mathbf{N}^p$ whose elements $(m_1, m_2, \dots, m_p)$ are such that m_1 takes infinitely many values; for each of these m_1, m_2 takes infinitely many values; and for each of these $m_1, m_2, \dots, m_{p-1}$, the pth coordinate takes infinitely many values m_p. $\mathbf{N}^p$ is a simple example of a p-net. An exhaustive use of properties 1 and 2 above enables to show that if $\mathcal{N}(\mathcal{S})$ is a p-net containing a finite set $\mathcal{S} \subset \mathbf{N}^p$, $\{E_s : s \in \mathcal{N}(\mathcal{S})\}$ is a family of vector subspaces of $l_0^\infty(\Sigma)$ and $A \in \Sigma$ is such that no $F \in \mathcal{F}(\Sigma)$ satisfies $l_0^\infty(\Sigma|_A) \subseteq E_s + F$, $s \in \mathcal{N}(\mathcal{S})$, then given $x_1, x_2, \dots, x_r \in l_0^\infty(\Sigma)$ there are $|\mathcal{S}|$ pairwise disjoint elements $\{M_s : s \in \mathcal{S}\}$ of Σ, contained in A, such that $\chi_{M_s} \notin \operatorname{sp}(E_s \cup \{x_1, x_2, \dots, x_r\})$, $s \in \mathcal{S}$. Moreover, there is no $F \in \mathcal{F}(\Sigma)$ with $l_0^\infty(\Sigma|_{A \setminus \cup_{w \in \mathcal{S}} M_w}) \subseteq E_s + F$ for each $s \in \mathcal{N}^*(\mathcal{S})$, where $\mathcal{N}^*(\mathcal{S})$ is some p-net contained in $\mathcal{N}(\mathcal{S})$ such that $\mathcal{S} \subset \mathcal{N}^*(\mathcal{S})$. Setting $^\sharp w = \sum_{i=1}^k w_i$ for each $w = (w_1, \dots, w_k) \in W(\mathbf{N})$, then the following lemma holds, implying Theorem 3.2 below.

Lemma 3.1 [**43**]. *Let* $\{E_s : s \in \mathbf{N}^p\}$, $p \in \mathbf{N}$, *be a family of infinite-codimensional vector subspaces of* $l_0^\infty(\Sigma)$. *Then there is a p-net*

$$\{(m(i_1), m(i_1, i_2), \dots , m(i_1, i_2, \dots , i_p)) \in \mathbf{N}^p : i_1, i_2, \dots , i_p \in \mathbf{N}\}$$

and a sequence $\{M_t : t \in \mathbf{N}^{p+1}\}$ *of pairwise disjoint elements of* Σ *such that, setting* $p(t) := (m(i_1), m(i_1, i_2), \dots , m(i_1, i_2, \dots , i_p))$ *for each* $t = (i_1, \dots , i_{p+1}) \in \mathbf{N}^{p+1}$, *we have that* $\chi_{M_t} \notin sp\,(E_{p(t)} \cup \{\chi_{M_r} : r \in \mathbf{N}^{p+1}, \sharp r < \sharp t\})$.

Theorem 3.2 [**43**]. $l_0^\infty(\Sigma)$ *is barrelled of class* $\aleph_0$.

For the baireledness, some additional notation will be helpful. If $w = (n_1, \dots , n_i, \dots , n_q) \in W(\mathbf{N})$, set $|w| = q$ for the *length* of w and $P_i w := (n_1, \dots , n_i)$, for $1 \le i \le |w|$ and, given $T \subseteq W(\mathbf{N})$, write $P_i T := \{P_i w : w \in T, |w| \ge i\}$.

A nonempty subset $T \subset W(\mathbf{N})$ is said to be a *v-web* if it verifies the following three conditions:

(1) For each word $w \in T$ and for each $1 \le i \le |w|$ there are infinitely many words in T whose first $i-1$ letters coincide with those of w and whose ith letter is different in each one of these words.

(2) Given any word $w \in T$ there is no longer word in T whose first $|w|-1$ letters coincide with those of w.

(3) For each sequence $\{w_n\}$ of words in T such that $|w_n| \ge n$ for each $n \in \mathbf{N}$ there are two consecutive words w_p and w_{p+1} in the sequence whose first p letters do not coincide.

If $t \in W(\mathbf{N})$, then $b(t) := \{P_1 t, P_2 t, \dots , P_{|t|} t\}$ is the *branch* of t. The set $\mathcal{B}_T = \cup_{t \in T} b(t)$ formed by the branches of the elements of the v-web T is called the *v-tree* determined by T. One may see $\mathcal{B}_T$ as a tree with infinitely many vertices, each labeled by some word of $W(\mathbf{N})$, and a root (the empty word), i.e., an infinite arborescence, such that each of his infinitely many branches has finite length and each father vertex $s \in \mathcal{B}_T \setminus T$ has infinitely many sons (s, k) belonging to $\mathcal{B}_T$, but if a son belongs to T, then all his siblings also belong to T. Each p-net in $\mathbf{N}^p$ and $\cup_{i=1}^\infty \{i\} \times \mathbf{N}^i$ are examples of v-webs. Infinite subsets of $\mathbf{N}$ are also examples of v-webs which we will call *trivial*. Note that any subset T^* of a v-web that satisfies condition 1 in the above definition is a v-web, too. Given a v-web T its v-tree $\mathcal{B}_T$ satisfies that, for each $(n_1, n_2, \ldots, n_i) \in \mathcal{B}_T$ there are infinitely many $p \in \mathbf{N}$ such that $(n_1, n_2, \ldots, n_{i-1}, p) \in \mathcal{B}_T$ but if $(n_1, n_2, \ldots, n_i) \in T$ no $(n_1, \ldots, n_{i-1}, p, q) \in \mathcal{B}_T$.

If $\{E_t : t \in W(\mathbf{N})\}$ is a linear web in an lcs E and T is a v-web, then $E = \cup\{E_n : n \in P_1T\}$ and for each $s \in P_pT \setminus T$, $E_s = \cup\{E_{s,n} : (s,n) \in P_{p+1}T\}$. So, using condition 3, we deduce that $E = \cup\{E_t : t \in T\}$. If the v-web $T = T_1 \cup T_2$ is trivial and T_1 does not contain any v-web, then T_2 must contain some v-web. This can be extended to any v-web by means of the following combinatorial lemma, from which it follows that if a v-web T satisfies that $T = T_1 \cup T_2 \cup \cdots \cup T_p$, some T_i must contain a v-web.

Lemma 3.3 [**62**]. *Let T be a v-web. If $T_0 \subseteq T$ does not contain any v-web, then $T \setminus T_0$ does.*

The argument works as follows. When T is trivial, we set $I_1 := T \setminus T_0$, thus I_1 becoming a v-web contained in $T \setminus T_0$, and $J_1 := \varnothing$. If T is not trivial, then $P_1T \cap T = \varnothing$ and there is some $a_0 \in \mathbf{N}$ such that for each $a_1 \in J_1 := \{m \in P_1T : m > a_0\}$ there is no v-web U_{a_1} with $\{a_1\} \times U_{a_1} \subseteq T_0$. In this case we set $I_1 := \varnothing$. Given $a_1 \in J_1$, if there exists a maximal infinite subset $M_{a_1} \subseteq \mathbf{N}$ such that $\{a_1\} \times M_{a_1} \subseteq T \setminus T_0$, then we represent the set formed by the sons of a_1 that belong to $T \setminus T_0$ by $T_{a_1} := \{a_1\} \times M_{a_1}$ and write $G_{a_1} := \varnothing$ (sons of a_1 that generate grandsons of a_1 in $\mathcal{B}_T$). Otherwise, since $J_1 \subseteq P_2T \setminus T$, the vertex a_1 has infinitely many sons, hence there is a maximal infinite subset $N(a_1)$ of $\mathbf{N}$ such that $(a_1, m) \in P_2T$ for each $m \in N(a_1)$. By the definition of J_1, $(a_1, m) \notin T_0$ for infinitely many $m \in N(a_1)$ and, since the previous case does not happen, $(a_1, m) \notin T \setminus T_0$ for infinitely many $m \in N(a_1)$. Hence $(a_1, m) \notin T$ for any $m \in N(a_1)$. Let $a_0(a_1) \in N(a_1)$ be such that for each $a_2 \in N(a_1)$, $a_2 > a_0(a_1)$, there is no v-web U_{a_1,a_2} with

$$\{(a_1, a_2)\} \times U_{a_1a_2} = \{a_1\} \times (\{a_2\} \times U_{a_1a_2}) \subseteq T_0.$$

Then we take $G_{a_1} := \{(a_1, a_2) \in P_2T : a_2 \in N(a_1), a_2 > a_0(a_1)\}$ and $T_{a_1} := \varnothing$. Note that now G_{a_1} stands for an infinite set of sons of a_1, none of which belongs to T and from all of which a_1 has got grandsons in $\mathcal{B}_T$, and T_{a_1} stands for the set formed by the sons of a_1 that belong to T, none in this case.

Writing $I_2 = \cup\{T_{a_1} : a_1 \in J_1\}$ and $J_2 = \cup\{G_{a_1} : a_1 \in J_1\}$, we proceed again with each vertex $(a_1, a_2) \in J_2$. Then either there exists an infinite $M_{a_1a_2} \subseteq \mathbf{N}$ such that $\{(a_1, a_2)\} \times M_{a_1a_2} \subseteq T \setminus T_0$, in which case we write $T_{a_1a_2} := \{(a_1, a_2)\} \times M_{a_1a_2}$ and $G_{a_1a_2} := \varnothing$, or there is a maximal infinite subset $N(a_1, a_2)$ of $\mathbf{N}$ such that $(a_1, a_2, m) \in P_3T \setminus T$ for each $m \in N(a_1, a_2)$. In this case there is some $a_0(a_1, a_2) \in N(a_1, a_2)$ such that for each $a_3 \in N(a_1, a_2)$, $a_3 > a_0(a_1, a_2)$, there is not any v-web $U_{a_1a_2a_3}$ with $\{(a_1, a_2, a_3)\} \times U_{a_1a_2a_3} \subseteq T_0$. Then we set $G_{a_1a_2} := \{(a_1, a_2, a_3) \in P_3T : a_3 \in N(a_1, a_2), a_3 > a_0(a_1, a_2)\}$ which is infinite and $T_{a_1a_2} := \varnothing$. Take $I_3 = \cup\{T_{a_1a_2} : (a_1, a_2) \in J_2\}$ and $J_3 = \cup\{G_{a_1a_2} : (a_1, a_2) \in J_2\}$. Continuing in this way, if some J_k were empty, the inductive process would end and $I_1 \cup I_2 \cup \cdots \cup I_k$ would be a v-web contained in $T \setminus T_0$. If $I = \cup_{j=1}^{\infty} I_j$ were empty, then we would be able to determine a sequence $\{\alpha_n\}$ in $\mathbf{N}$ such that each $(\alpha_1, \alpha_2, \dots, \alpha_n) \in J_n$; therefore there exists a sequence $\{t_n\}$ in T such that $P_n t_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$, contradicting condition 3 for T to be a v-web. Hence $I \neq \varnothing$, and I is a v-web contained in $T \setminus T_0$.

Using the previous result together with some techniques of sliding hump resembling those of the first stage, one gets the key Proposition 3.4 below, from which the baireledness of the space $l_0^{\infty}(\Sigma)$ is, as we are going to see, an almost straightforward consequence.

Proposition 3.4 [**62**]. *Let $\{E_w : w \in W(\mathbf{N})\}$ be a linear web in $\ell_0^{\infty}(\Sigma)$, and let T be a v-web. Then there exists some $t \in T$ such that E_t is barrelled.*

Theorem 3.5. *The space $l_0^{\infty}(\Sigma)$ is baireled.*

The proof runs by contradiction, assuming that there is a linear web $\{E_w : w \in W(\mathbf{N})\}$ in $l_0^{\infty}(\Sigma)$ none of whose strands is entirely formed by BL subspaces. Since $l_0^{\infty}(\Sigma) \in B_1$, there exists some $b_1 \in \mathbf{N}$ such that, for each $n_1 \geq b_1$, E_{n_1} is barrelled (and therefore BL because of metrizability) and dense. For each fixed $a_1 \geq b_1$ either there is no E_w barrelled and dense for $w \in \{a_1\} \times \mathbf{N}$, in which case we set $C_{a_1} := \{a_1\} \times N_{a_1}$ (where N_{a_1} is a cofinite subset of $\mathbf{N}$ such that E_w is dense for each $w \in C_{a_1}$) and $D_{a_1} := \varnothing$, or there is a cofinite subset M_{a_1} in $\mathbf{N}$ such that E_w is barrelled (hence BL) and dense for each

$w \in \{a_1\} \times M_{a_1}$ in which case we set $C_{a_1} := \varnothing$ and $D_{a_1} := \{a_1\} \times M_{a_1}$. Setting $I_1 := \cup\{C_{a_1} : a_1 \geq b_1\}$ and $J_1 := \cup\{D_{a_1} : a_1 \geq b_1\}$, note that $J_1 \neq \varnothing$ since $l_0^\infty(\Sigma) \in B_2$. We proceed in the same way with each $(a_1, a_2) \in J_1$ to build up C_{a_1,a_2} and D_{a_1,a_2}, then I_2 and J_2. Going on, we are able to build up in this way the sets I_n and J_n for each $n \in \mathbf{N}$, noting that each $J_n \neq \varnothing$ since $l_0^\infty(\Sigma) \in B_{\aleph_0} \subset B_n$. Now set $I = \cup_{n=1}^\infty I_n$. If $I = \varnothing$, we may find a sequence $\{a_n\}$ in $\mathbf{N}$ such that $E_{a_1,a_2,\dots,a_n}$ is barrelled and dense in $l_0^\infty(\Sigma)$ for each $n \in \mathbf{N}$, a contradiction. If $I \neq \varnothing$, it is easy to check that I fulfills conditions 1 and 2 of the definition of v-web, condition 3 being consequence of the fact that there is no strand of barrelled subspaces in $l_0^\infty(\Sigma)$. According to Proposition 3.4, there exists $t \in I$ such that E_t is barrelled. This contradicts the definition of I.

4. Rings with property (N)**.** There are two common ways of extending the classic Nikodým boundedness theorem. The first one consists of determining a subfamily $\mathcal{M}$ of a given σ-algebra Σ with the property that pointwise boundedness on $\mathcal{M}$ of a set in $ba(\Sigma)$ assures that this set is uniformly bounded on Σ. The second alternative, far from working into a σ-algebra of sets, is to investigate what Boolean rings satisfy property (N), that is, what Boolean rings have an associated space of simple functions which is barrelled. Concerning the first line of research, let us mention that in 1951 Dieudonné [**24**] shows that if X is a Hausdorff topological compact space, $\mathcal{B}$ is the σ-algebra of Borel sets of $X, \mathcal{U}$ is the family of open sets of X and M is a set of scalarly-valued measures on $\mathcal{B}$, then $\{\mu(A) : \mu \in M, A \in \mathcal{B}\}$ is a bounded set provided that $\{\mu(U) : \mu \in M\}$ is bounded for each $U \in \mathcal{U}$. If X is a Hausdorff regular space and $\mathcal{U}$ is the family of all regular open sets of X, a similar result was obtained by Gänssler in 1971 [**49**]. This last result also holds for group-valued measures [**13**]. Other extensions of Nikodým boundedness theorem for semigroup-valued measures can be found in [**15**]. The baireledness of the space of simple functions provides the following remarkable extension of the Nikodým boundedness theorem.

Theorem 4.1 [**62**]. *If* $\mathcal{V} = \{\Sigma_w : w \in W(\mathbf{N})\}$ *is an increasing web in a* σ*-algebra* Σ*, there exists a strand* $\{\Sigma_{n_1 n_2 \dots n_i}\}_i$ *in* $\mathcal{V}$ *such that any family* $\{\mu_s : s \in S\} \subseteq ba(\Sigma)$ *which is pointwise bounded in some*

$\Sigma_{n_1 n_2 \ldots n_i}$ *is uniformly bounded in* Σ.

Now we will review some results about rings whose associated spaces of simple functions are barrelled. Indeed, in 1981, Moltó [**64**] introduced a class of Boolean rings, namely, algebras with property (f), which satisfy the so-called Vitali-Hahn-Saks (VHS) property, containing some classes of algebras previously considered by Seever [**82**] and Faires [**30**], named algebras with the interpolation (I) property. In 1984, Freniche introduced Boolean algebras with the subsequential interpolation (SI) property [**48**], a new class of algebras containing algebras with property (f), that also satisfy property (VHS). Afterwards, in 1992, Gassó [**50**] defined a class of Boolean algebras, namely algebras with the local interpolation (LI) property, which included those algebras having property (f) as well as all known examples of algebras with property (N) which failed to have property (VHS) [**53, 81**]. Other classes of rings extending Nikodým's boundedness theorem have been introduced by Haydon [**54**], Aizpuru [**4, 5**] and Drewnowski et al. [**26, 28**]. Some examples of algebras of sets which fail to have property (N) may be found in [**81**] and [**41**, pp. 131–135]. An intrinsic characterization of those Boolean rings with property (N) is still unknown.

According to the Stone representation theorem each Boolean algebra $(\mathcal{A}, +, \cdot)$ is isomorphic to the algebra $(\mathcal{C}, \Delta, \cap)$ of all clopen sets of the Stone space $S_{\mathcal{A}}$ of the algebra, hence there is no loss of generality considering only algebras of open and compact sets of some topological space on which such a family of sets is a base of the topology. A Boolean algebra $\mathcal{A}$ has *property* (VHS) [**81**] if given a sequence $\{\mu_n\}$ of scalar bounded finitely additive measures on $\mathcal{A}$ such that $\{\mu_n(A)\}$ converges for every $A \in \mathcal{A}$, then $\{\mu_n\}$ is uniformly exhaustive, i.e., for each sequence $\{A_n\}$ of pairwise disjoint elements of $\mathcal{A}$, given $\varepsilon > 0$ there is a $k(\varepsilon) \in \mathbf{N}$ such that $\sup_n |\mu_n(A_i)| \leq \varepsilon$ for each $i \geq k(\varepsilon)$. A Boolean algebra $\mathcal{A}$ is said to have the *property* (I) if for each pair of sequences $\{A_n\}$ and $\{B_n\}$ of pairwise disjoint elements in $\mathcal{A}$ and such that $A_i \cap B_j = \varnothing$ for each $i, j \in \mathbf{N}$, there exists a $B \in \mathcal{A}$ such that $A_n \subseteq B$ for each $n \in \mathbf{N}$ and $B \cap B_n = \varnothing$ for every $n \in \mathbf{N}$. Clearly, each σ-algebra has property (I). A Boolean algebra $\mathcal{A}$ is said to have *property* (f) if for each pair of sequences $\{A_n\}$ and $\{B_n\}$ of pairwise disjoint elements in $\mathcal{A}$ and such that $A_i \cap B_j = \varnothing$ for each $i, j \in \mathbf{N}$, there exists a subsequence $\{B_{n_k}\}$ of $\{B_n\}$ which satisfies the following conditions:

(1) There exists some $A \in \mathcal{A}$ such that

$$(B_{n_k} \subseteq A \wedge A_k \cap A = \varnothing), \quad \text{for all } k \in \mathbf{N}.$$

(2) For each set $J \subseteq \mathbf{N}$ there is some $A_J \in \mathcal{A}$ with

$$(B_{n_k} \subseteq A_J \forall\, k \in J) \wedge (B_{n_k} \cap A_J = \varnothing \forall\, k \in \mathbf{N} \setminus J).$$

A Boolean algebra $\mathcal{A}$ is said to have the *property* (SI) if for every sequence $\{A_n\}$ of pairwise disjoint elements in $\mathcal{A}$ and for each infinite set $M \subseteq \mathbf{N}$, there exists $A \in \mathcal{A}$ and an infinite set $N \subseteq M$ such that $A_n \subseteq A$ if $n \in N$ and $A_n \cap A = \varnothing$ if $n \in \mathbf{N} \setminus N$. If $\mathcal{A}$ is a Boolean algebra with property (f), it is shown in [**48**] that $\mathcal{A}$ has property (SI). The latter property implies property (VHS) and there exist algebras with property (SI) which do not have property (f). If $\mathcal{A}$ is a Boolean algebra which has property (VHS), then $\mathcal{A}$ also has property (N). This may be seen as follows. Assume $\mathcal{A}$ is a Boolean algebra with property (VHS) which does not have property (N). Then there is a pointwise bounded family $\mathcal{M}$ of scalar bounded finitely additive measures on $\mathcal{A}$ which are not uniformly bounded. This allows us to obtain a pairwise disjoint sequence $\{E_n\}$ of elements in $\mathcal{A}$ and a sequence $\{\lambda_n\} \subseteq \mathcal{M}$ such that $|\lambda_n(E_n)| > n$ for each $n \in \mathbf{N}$. Setting $\mu_n := \lambda_n/n$, it is clear that $\mu_n \in ba(\mathcal{A})$ for each $n \in \mathbf{N}$ and $\lim_n \mu_n(A) = 0$ for each $A \in \mathcal{A}$. As $\mathcal{A}$ has property (VHS) the sequence $\{\mu_n\}$ is uniformly exhaustive, so for $\varepsilon = 1/2$ there exists some $k \in \mathbf{N}$ such that $\sup_n |\mu_n(E_i)| < 1/2$ for each $i \geq k$. Hence $1/2 > |\mu_k(E_k)| > 1$, a contradiction. This shows that Freniche algebras have property (N) and hence their associated spaces of simple functions are barrelled. If $\mathcal{A}$ is a Boolean algebra with property (f), then $l_0^\infty(\mathcal{A})$ is an SB space [**65**]. Ferrer et al. showed in [**46**] that if $\mathcal{A}$ is a Boolean algebra with property (I), then $l_0^\infty(\mathcal{A})$ is barrelled of class $\aleph_0$. Abraham introduced in [**1**] a class of Boolean algebras enjoying property (VHS) that he called algebras with the Vitali-Hahn-Saks-Nikodým-Saeki (VHSNS) property and showed that each Boolean algebra with the property (SI) has property (VHSNS). Therefore, the following scheme holds

$$\text{(I)} \Longrightarrow \text{(f)} \Longrightarrow \text{(SI)} \Longrightarrow \text{(VHSNS)} \Longrightarrow \text{(VHS)} \Longrightarrow \text{(N)}.$$

As stated, a wide class of Boolean algebras with property (N) that may fail to have property (VHS) was introduced in [**50**]. Given a Boolean algebra $\mathcal{A}$, denote by $S_{\mathcal{A}}$ its Stone space, $\operatorname{co}(S_{\mathcal{A}})$ the algebra, isomorphic to $\mathcal{A}$, of all clopen subsets of $S_{\mathcal{A}}$ and identify $\mathcal{A}$ and $\operatorname{co}(S_{\mathcal{A}})$. A Boolean algebra $\mathcal{A}$ is said to have *property* (LI) if for each $s \in S_{\mathcal{A}}$ there exists a decreasing sequence $(T_n(s))$ of clopen neighborhoods of s in $S_{\mathcal{A}}$ in such a way that if $\{A_n\}$ and $\{B_n\}$ are two sequences of pairwise disjoint elements in $\mathcal{A}$ with $A_i \cap B_j = \varnothing$ for each $i, j \in \mathbf{N}$ and $A_n, B_n \subseteq T_n(s)$ for each $n \in \mathbf{N}$, then there exists a subsequence $\{B_{n_k}\}$ of $\{B_n\}$ which satisfies conditions 1 and 2 above. It is obvious that each Boolean algebra with property (f) has property (LI). It is more difficult to show that property (LI) implies property (N). Let us glance at the argument. If $T_n(s)$ is as in the definition of property (LI) and $\{E_n\}$ is a sequence of pairwise disjoint elements in $\mathcal{A}$ such that $E_n \subseteq T_n(s)$ for each $n \in \mathbf{N}$, adapting some techniques of [**64**] it may be shown that for each sequence $\{\mu_n\}$ of bounded finitely additive scalar measures defined on $\mathcal{A}$ there exists a subsequence $\{F_n\}$ of $\{E_n\}$ such that, if $\mathcal{B}$ denotes the σ-algebra $\{\cup_{n\in N} F_n : N \subseteq \mathbf{N}\}$, for each positive integer n there exists a countably additive scalar measure λ_n on $\mathcal{B}$ satisfying: (a) $\lambda_n(E) = \mu_n(E)$ for each $E \in \mathcal{A} \cap \mathcal{B}$ and (b) for all $B \in \mathcal{B}$ there exists some $A_B \in \mathcal{A}$ with $\mu_n(A_B) = \lambda_n(B)$. Then, assuming by contradiction that there exists a Boolean algebra $\mathcal{A}$ with property (LI) which fails to have property (N), a point $s \in S_{\mathcal{A}}$ may be obtained, together with a sequence $\{T_n(s)\}$ of clopen neighborhoods of s in $S_{\mathcal{A}}$, two strictly increasing sequences of positive integers $\{n_i\}$ and $\{k_i\}$ and a sequence $\{E_i\}$ of pairwise disjoint elements of $\mathcal{A}$ such that $E_i \subseteq T_{k_i}(s)$ and $|\mu_{n_i}(E_i)| > i$ for each $i \in \mathbf{N}$. The statement above provides a subsequence $\{E_{p_i}\}$ of $\{E_p\}$ and a sequence $\{\lambda_i\}$ in $\operatorname{ca}(\mathcal{B})$, where $\mathcal{B} = \{\cup_{i\in N} E_{p_i} : N \subseteq \mathbf{N}\}$, such that $\lambda_i(E) = \mu_{n_i}(E)$ for each $E \in \mathcal{A} \cap \mathcal{B}$ and, given $B \in \mathcal{B}$, there exists some $A_B \in \mathcal{A}$ with $\lambda_i(B) = \mu_{n_i}(A_B)$ for each $i \in \mathbf{N}$. These facts imply that $\{\lambda_i\}$ is pointwise bounded on $\mathcal{B}$ and $|\lambda_{p_i}(E_{p_i})| > p_i$ for each $i \in \mathbf{N}$. But, according to Nikodým's convergence theorem [**21**, p. 90], $\{\lambda_{p_i}\}$ is uniformly exhaustive on $\mathcal{B}$. Hence there is a k such that $\sup_j |\lambda_j(E_{p_i})| < 1$ for each $i \geq k$, which gives the contradiction $p_k < |\lambda_{p_k}(E_{p_k})| < 1$. So we have that $(f) \Rightarrow (\mathrm{LI}) \Rightarrow (N)$.

A Boolean algebra $\mathcal{A}$ is said to have *property* (G) if each weak* convergent sequence in $C(S_{\mathcal{A}})^*$ is weak convergent, where $C(S_{\mathcal{A}})$ denotes the Banach space of all continuous functions on $S_{\mathcal{A}}$ provided with the supremum norm. Let us remark that $(N) \not\Rightarrow (G)$ [**53, 81**]. The classic theorem of Dieudonné-Grothendieck which characterizes the relatively compact sets of the Banach space $rca(\text{bo}\,(S_{\mathcal{E}}))$ of all regular Borel measures defined on the σ-algebra $\text{bo}\,(S_\Sigma)$ of the Borel sets in S_Σ for a σ-algebra Σ, see [**21**, p. 98], implies the well-known Groethendieck's theorem which establishes that each σ-algebra Σ has property (G). The same argument shows that each Boolean algebra with property (VHS) has property (G). Consequently, (VHS) $\Rightarrow$ (N) $\wedge$ (G). The converse is also true thanks to a theorem of Diestel, Faires and Huff [**22**]. So one has the following.

Theorem 4.2. *For each Boolean algebra,* (VHS) $\Leftrightarrow$ (N) $\wedge$ (G).

A Boolean algebra $\mathcal{A}$ of clopen subsets in a compact totally disconnected space S is said to be *up-down semi-complete* [**14**] if, for each sequence $\{A_n\}$ of pairwise disjoint elements in $\mathcal{A}$ such that $\sup_n\{A_n\}$ exists, in $\mathcal{A}$, then $\sup_k\{A_{n_k}\}$ exists for each strictly increasing subsequence $\{n_k\} \subseteq \mathbf{N}$. Although a Boolean up-down semi-complete algebra need not have property (N), if $\mathcal{A}$ is a Boolean up-down semi-complete algebra such that for each countably additive non-negative measure μ defined on $\mathcal{A}$ and each sequence $\{A_n\}$ of μ-null elements in $\mathcal{A}$ there exists $B \in \mathcal{A}$ such that $\mu(B) > 0$ and $B \cap A_n = \varnothing$, for each $n \in \mathbf{N}$, then $\mathcal{A}$ has property (VHS) [**14**, Theorem 2.6]. Let us remark that Haydon's Boolean algebras [**54**] do have property (VHS) and that a wide class of Boolean algebras with property (G) containing Haydon's class was introduced in [**3**].

It should be mentioned that the algebra $\mathcal{J}$ of Jordan subsets of the interval $[0,1]$ has property (LI) which, as we know, implies that $\mathcal{J}$ has property (N). However, $\mathcal{J}$ does not have property (VHS). Indeed, if λ denotes the Lebesgue measure on $[0,1]$,

$$A_n := \{(2i-1)/2^n : 1 \le i \le 2^{n-1}\} \quad \text{and} \quad \mu_n := 1/2^{n-1} \sum_{i=1}^{2^{n-1}} \delta_{\frac{2i-1}{2^n}}$$

for each $n \in N$, then $\{A_n\}$ is a sequence of pairwise disjoint elements of $\mathcal{J}$ and $\mu_n(E) \to \lambda(E)$ for each $E \in \mathcal{J}$. Since $\mu_n(A_n) = 1$, then $\{\mu_n\}$ is a pointwise convergent sequence of bounded scalar additive measures which is not uniformly exhaustive. Hence, according to the Diestel-Faires-Huff theorem, $\mathcal{J}$ cannot have property (G). Historically, this one was the first example, due to Schachermayer [**81**], of a Boolean up-down semi-complete algebra with property (N) which does not have property (G), answering in the negative the question (N) $\Rightarrow$ (G) raised by Seever [**82**]. Another classical example of a Boolean algebra with property (G) that fails to have property (N) is due to Talagrand [**85**]. In [**34**] it is shown that the space $l_0^\infty(\mathcal{J})$ is even barrelled of class $\aleph_0$. Other examples of Boolean algebras with property (LI) that fail to have property (VHS) may be found in [**50**] and [**34**]. However, there are classes of rings of sets with property (N) which seem not to be included in any of the preceding classes. As stated in the theorem below, one of these is the ring $\mathcal{Z}$ of the subsets of $\mathbf{N}$ of density zero, that we are going to consider briefly. Recall that a subset A of $\mathbf{N}$ is said to be of *density zero* if

$$\lim_{n\to\infty} \frac{|\{m \in A : m \leq n\}|}{n} = 0.$$

Theorem 4.3 [**26**]. *The space $l_0^\infty(\mathcal{Z})$ is barrelled.*

The proof of this theorem is based upon the fact that the vector subspace $\mathcal{Z}(c_0)$ of c_0 of all those elements whose support is a member of $\mathcal{Z}$ is barrelled. In fact, if M is a subset of $ba(\mathcal{Z})$ which is pointwise bounded on $\mathcal{Z}$ and for each $\mu \in M$, we denote by μ^c the countably additive component of μ, i.e., $\mu^c(A) = \sum_{n\in A} \mu(\{n\})$ for each $A \in \mathbf{N}$, we claim that the set $M^c = \{\mu^c : \mu \in M\}$ is uniformly bounded on $\mathcal{Z}$. The proof of this claim is carried out by identifying M^c with a $\sigma(l_1, l_0^\infty(\mathcal{Z}))$-bounded subset of l_1 and showing that M^c is $\beta(l_1, l_0^\infty(\mathcal{Z}))$-bounded. But if $\mathcal{Z}(c_0)$ is barrelled, one only needs to show that M^c is $\sigma(l_1, \mathcal{Z}(c_0))$-bounded, a fact that may be easily verified. Finally, if $\mu \in M$ and we write $\mu^p = \mu - \mu^c$ for the finitely additive part of μ, it is not difficult to establish that the set $M^p = \{\mu^p : \mu \in M\}$ is uniformly bounded over $\mathcal{Z}$, which completes the proof. The barrelledness of $\mathcal{Z}(c_0)$ may be derived as a consequence of Auerbach's lemma below, which is interesting for itself since it provides a way to recognize the elements of l_1 based on absolute subseries convergence of coordinates whose support has density zero. There are many generalizations of Auerbach's lemma, see [**28**] and references therein. One of them, due to Pastéka [**66**], has been used in [**28**] to extend Theorem 4.3 to the ideal $\mathcal{R}$ of all η-null sets of any strongly nonatomic submeasure η defined on a σ-algebra of subsets.

Lemma 4.4 [**8**]. *Let $\xi = (\xi_n) \in \omega$. If $\sum_{n\in A} |\xi_n| < \infty$ for each $A \in \mathcal{Z}$, then $\xi \in l_1$.*

Note that if the lemma does not hold, there is a sequence of positive integers $\{n_i\}_i$ with $2n_i \leq n_{i+1}$ and $n_1 = 1$ such that $\sum_{j=n_i}^{n_{i+1}-1} |\xi_j| > 2^i$ for each $i \in \mathbf{N}$. Setting $A_1 := [1, n_1)$ and splitting the subsequent intervals $[n_k, n_{k+1})$, $k \in \mathbf{N}$, one may obtain a sequence $\{A_i\}$ of pairwise disjoint finite subsets of $\mathbf{N}$ such that $A = \cup_{i=1}^{\infty} A_i \in \mathcal{Z}$ and $\sum_{j\in A_i} |\xi_j| > 1$ for each $i \in \mathbf{N}$. Hence $\sum_{n\in A} |\xi_n| = +\infty$, a contradiction. We are ready for the final step.

Theorem 4.5. *The space $\mathcal{Z}(c_0)$ is ultrabornological.*

Let us provide a sketch of the proof. Given $A \in \mathcal{Z}$, denote by $c_0(A)$ the vector subspace of c_0 formed by those elements $\xi \in c_0$ with support contained in A. Since $\mathcal{Z}(c_0) = \cup\{c_0(A) : A \in \mathcal{Z}\}$, endowing $\mathcal{Z}(c_0)$ with the locally convex hull topology τ of $\{c_0(A) : A \in \mathcal{Z}\}$, then $\mathcal{Z}(c_0)$ is ultrabornological. In order to see that τ coincides with the topology induced by c_0 on $\mathcal{Z}(c_0)$, it suffices to see that $(\mathcal{Z}(c_0)(\tau))^* = l_1$. Let u be a continuous linear form on $\mathcal{Z}(c_0)(\tau)$ and write $\xi_n = \langle u, e_n\rangle$ for each $n \in \mathbf{N}$, where e_n denotes the nth unit vector of c_0. Continuity of $u|_{c_0(A)}$ on $c_0(A)$ implies that $\sum_{n\in A} |\xi_n| < \infty$ for each $A \in \mathcal{Z}$, so Lemma 4.4 gives that $\xi = (\xi_n) \in l_1$. Hence the one-to-one linear map $T : (\mathcal{Z}(c_0)(\tau))^* \to l_1$ such that $Tu = \xi$ is well defined. This map is also onto since, given $\xi = (\xi_n) \in l_1$, the linear form v on $\mathcal{Z}(c_0)$ defined by $v(x) = \sum_{n=1}^{\infty} \xi_n x_n$ for $x = (x_n) \in \mathcal{Z}(c_0)$ is τ-continuous (because of $\|v|_{c_0(A)}\|_\infty \leq \|\xi\|_1$ for each $A \in \mathcal{Z}$) and satisfies that $Tv = \xi$.

Let us say that, using different ideas, it can be shown that $l_0^\infty(\mathcal{Z})$ is a barrelled space of class $\aleph_0$ [**40**]. We do not know whether $l_0^\infty(\mathcal{Z})$ has property (VHS). On the other hand, Nikodým boundedness theorem may also be considered for families of countably additive scalar measures. So a Boolean ring $\mathcal{R}$ is said to have the σ-Nikodým property [**29**] if every pointwise bounded subset M of $ca(\mathcal{R})$ is uniformly bounded. Clearly, property (N) implies σ-Nikodým property, there exist rings with σ-Nikodým property that lack the property (N) [**29**, Examples 4.1 and 4.2].

5. The space of vector-valued simple functions. Throughout this section Σ will stand for an algebra of subsets of a set Ω and, given an lcs E, as usual $l_0^\infty(\Sigma, E)$ will denote the linear space of all E-valued Σ-simple functions defined on Ω endowed with the uniform convergence topology. An lcs F is said to be *nuclear* if $E \otimes_\pi F = E \otimes_\varepsilon F$ for each lcs E. Using the so-called property (B) of Pietsch [**70**], Freniche [**47**] gave in 1984 necessary and sufficient conditions for $l_0^\infty(\Sigma, E)$ to be barrelled by means of the following result.

Theorem 5.1. *$l_0^\infty(\Sigma, E)$ is barrelled if and only if both $l_0^\infty(\Sigma)$ and E are barrelled and E is nuclear.*

When E is a normed space, this is tantamount to saying that the space $l_0^\infty(\Sigma, E)$ is barrelled if and only if $l_0^\infty(\Sigma)$ is barrelled and E is finite-dimensional. Let us just illustrate the necessity of these conditions in the particular case $E = l_1$. Afterwards we will give a simple argument when E is an arbitrary Banach space.

Suppose that X is a Banach space and set $Q := \operatorname{acx}\{\chi_E x : x \in X, \|x\| \leq 1, E \in \Sigma\}$. Let $\|\cdot\|_Q$ be the norm defined by the gauge of Q on $l_0^\infty(\Sigma, X)$ (note that Q is bounded and absorbing). Let us denote by $B(l_0^\infty(\Sigma), X)$ the Banach space formed by the continuous bilinear forms on $l_0^\infty(\Sigma) \times X$ endowed with the uniform convergence norm, which is linearly isometric to the space $\mathcal{L}(l_0^\infty(\Sigma), X^*)$ of the continuous linear mappings of $l_0^\infty(\Sigma)$ into X^* equipped with the topology of uniform convergence of operators, and consider the one-to-one linear mapping $T : B(l_0^\infty(\Sigma), X) \to (l_0^\infty(\Sigma, X), \|\cdot\|_Q)^*$ defined by $T\varphi(\chi_E x) = \varphi(\chi_E, x)$, $E \in \Sigma$, $x \in X$ and linearly extend to the whole of $l_0^\infty(\Sigma, X)$. Let us see that T is onto. In fact, given a continuous linear form u on $(l_0^\infty(\Sigma, X), \|\cdot\|_Q)$, then $\varphi_u(\chi_E, x) := u(\chi_E x)$ defines a bilinear form on $l_0^\infty(\Sigma) \times X$. This φ_u is continuous since, according to Section 1, each $f \in l_0^\infty(\Sigma)$ with $\|f\| \leq 1$ may be represented as $f = \sum_{i=1}^k a_i \chi_{A_i}$ with $\sum_{i=1}^k |a_i| \leq 4$ and $A_i \in \Sigma$, $1 \leq i \leq k$, leading to $|\varphi_u(f, x)| \leq 4\|u\|_Q^*$ for each $x \in X$

with $\|x\| \leq 1$. Indeed, T is an isomorphism for if $h = \sum_{i=1}^{n} \chi_{E_i} x_i \in Q$ then $|T\varphi(h)| \leq \sum_{i=1}^{n} |\varphi(\chi_{E_i}, x_i)| \leq \sum_{i=1}^{n} \|\varphi\| \cdot \|\chi_{E_i}\| \cdot \|x_i\| \leq \|\varphi\|$.

Note that if X is an infinite-dimensional Banach space with a Schauder basis and $l_0^\infty(\Sigma, X)$ is barrelled, then $l_0^\infty(\Sigma, X)^* = (l_0^\infty(\Sigma, X), \|\cdot\|_Q)^*$ and therefore the norms $\|\cdot\|$ and $\|\cdot\|_Q$ are equivalent. In fact, since the identity mapping from $(l_0^\infty(\Sigma, X), \|\cdot\|_Q)$ onto $l_0^\infty(\Sigma, X)$ is one-to-one and continuous, its transpose has a weak*-dense range and hence $l_0^\infty(\Sigma, X)^*$ is algebraically isomorphic to a weak*-dense subspace of $(l_0^\infty(\Sigma, X), \|\cdot\|_Q)^*$. Identifying topologically $(l_0^\infty(\Sigma, X), \|\cdot\|_Q)^*$ and $B(l_0^\infty(\Sigma), X)$, select $\varphi \in (l_0^\infty(\Sigma, X), \|\cdot\|_Q)^*$. Let $\{x_n\}$ be a Schauder basis of X, and let P_n be the canonical projection on the first n coordinates for each $n \in \mathbf{N}$. Clearly $\{P_n x\}$ converges to x for each $x \in X$. So if we set $\varphi_n(\cdot,\cdot) := \varphi(\cdot, P_n(\cdot))$ for each $n \in \mathbf{N}$, then $\{\varphi_n(f,x)\}$ converges to $\varphi(f,x)$ for each $f \in l_0^\infty(\Sigma)$, $x \in X$, holding that $\varphi_n \in l_0^\infty(\Sigma, X)^*$ since both norms, $\|\cdot\|$ and $\|\cdot\|_Q$, are equivalent on each $l_0^\infty(\Sigma, P_n(X)) = l_0^\infty(\Sigma, \mathrm{sp}(\{x_1, \ldots, x_n\})) \cong l_0^\infty(\Sigma)^n$. As $l_0^\infty(\Sigma, X)$ is a barrelled space, $l_0^\infty(\Sigma, X)^*$ is weak*-sequentially complete and so $\varphi \in l_0^\infty(\Sigma, X)^*$.

The previous statement gives that, assuming that Σ is an infinite algebra, then $l_0^\infty(\Sigma, l_1)$ cannot be barrelled. Indeed, if $\{e_n\}$ denotes the canonical basis of l_1 and $\{A_n\}$ is an infinite partition of Ω by nonempty elements of Σ, let us take some $\omega_i \in A_i$ for each $i \in \mathbf{N}$, and let $\{e_i^*\}$ be the sequence of functional coefficients of the basis. Writing $H_i(f,\xi) = e_i^* \xi \cdot f(\omega_i)$, $f \in l_0^\infty(\Sigma)$, $\xi \in l_1$, $i \in \mathbf{N}$, and $\psi_n = \sum_{i=1}^{n} H_i$ for each $n \in \mathbf{N}$, if $l_0^\infty(\Sigma, l_1)$ were barrelled, given that $\psi_n \in B(l_0^\infty(\Sigma), l_1)$ for each $n \in \mathbf{N}$, the sequence $\{\psi_n\}$ considered in $l_0^\infty(\Sigma, l_1)^*$, would be uniformly bounded. This is a contradiction since, choosing $h_n = \sum_{i=1}^{n} e_i \chi_{A_i}$, then $\psi_n(h_n) = n$ for each $n \in \mathbf{N}$.

If X is an arbitrary Banach space, then $\ell_0^\infty(\Sigma, X) \cong \ell_0^\infty(\Sigma) \otimes_\varepsilon X$ whilst $(\ell_0^\infty(\Sigma, X), \|\cdot\|_Q) \cong \ell_0^\infty(\Sigma) \otimes_\pi X$. If $\ell_0^\infty(\Sigma, X)$ is barrelled, then $\ell_\infty(\Sigma) \otimes_\varepsilon X$ is barrelled and, since $\ell_\infty(\Sigma)$ has the approximation property, $\ell_\infty(\Sigma) \otimes_\varepsilon X = \ell_\infty(\Sigma) \otimes_\pi X$ [**55**, p. 486]. So if Σ is an infinite σ-algebra, it follows that $\ell_\infty \otimes_\varepsilon X = \ell_\infty \otimes_\pi X$. Hence $c_0 \otimes_\varepsilon X = c_0 \otimes_\pi X$ and, c_0 being an S_p-space, $1 \leq p < \infty$, X is nuclear (*op. cit.*).

If Ω is a set, $\mathcal{A}$ a ring of subsets of Ω, $\mu : \mathcal{A} \to [0, +\infty]$, a nontrivial finitely additive measure defined on $\mathcal{A}$ and E an lcs, let us denote by $\mathcal{A}_0$ the ideal of all μ-null sets in $\mathcal{A}$ and by $l_0^\infty(\mu, E)$ the quotient space

$l_0^\infty(\mathcal{A}, E)/l_0^\infty(\mathcal{A}_0, E)$. Then one has the following results, analogous to Theorems 5.1 and 2.2, where we write $l_0^\infty(\mu)$ instead of $l_0^\infty(\mu, \mathbf{K})$.

Theorem 5.2 [**17**]. *$l_0^\infty(\mu, E)$ is barrelled if and only if $l_0^\infty(\mu)$ is barrelled and E is barrelled and nuclear.*

Theorem 5.3 [**17**]. *If there exists an infinite sequence $\{\Omega_n\}$ of pairwise disjoint sets in $\mathcal{A}$ such that $\mu(\Omega_n) > 0$, then $l_0^\infty(\mu, E)$ is not ultrabornological.*

6. Spaces of bounded vector-valued Σ-measurable functions. Given an lcs space E and an algebra Σ of subsets of a nonempty set Ω, Mendoza [**63**] shows that the vector space $B(\Sigma, E)$ of all the functions of Ω into E that are uniform limits of a sequence of Σ-simple E-valued functions defined over Ω, endowed with uniform convergence topology, is barrelled if and only if E is barrelled. If X is a normed space that enjoys properties stronger than barrelledness; for instance, if X is BH, the following holds.

Theorem 6.1 [**35**]. *$B(\Sigma, X)$ is a* BH *space if and only if X is a* BH *space.*

Hereafter we will work with a normed space X and a σ-algebra Σ and will exhibit results similar to Mendoza's concerning other related spaces of bounded vector-valued Σ-measurable functions. We will denote by $K(\Sigma, X)$ the vector space over $\mathbf{K}$ of all bounded X-valued functions defined on Ω for which there exists a countable partition $\{A_n\}$ of Ω by nonempty elements of Σ such that f is constant on each A_n endowed with the supremum-norm, whereas $l_\infty(\Sigma, X)$ will stand for the vector space over $\mathbf{K}$ of all the X-valued functions defined over Ω that are uniform limit of a sequence of elements of $K(\Sigma, X)$, also endowed with the supremum-norm. Let us point out that $l_\infty(\Sigma, \hat{X})$ is the completion of $K(\Sigma, X)$, where $\hat{X}$ stands for the completion of X. As in Section 5, Q will stand for $\operatorname{acx}\{\chi_A x : x \in X, \|x\| \leq 1, A \in \Sigma\}$. Due to the fact that $(l_0^\infty(\Sigma, X), \|\cdot\|_Q) \cong l_0^\infty(\Sigma) \otimes_\pi X$, if X is barrelled then the space $(l_0^\infty(\Sigma, X), \|\cdot\|_Q)$ is barrelled. On the other hand, working with a certain Banach disk, it can be shown that if $\{A_n\}$ is a pairwise disjoint sequence of elements of Σ and T is a barrel in $K(\Sigma, X)$, then there exists $m \in \mathbf{N}$ such that T absorbs the closed unit ball of $K(\Sigma|_{\cup_{n>m} A_n}, X)$. These two observations allow to conclude

Theorem 6.2 [**33**]. *The space $K(\Sigma, X)$ is barrelled if and only if X is barrelled.*

The strategy of the proof is as follows. The *only if* part is trivial since, fixed any $\omega \in \Omega$, the linear mapping ϕ defined of $K(\Sigma, X)$ onto X by $\phi(f) = f(\omega)$ is bounded. The converse proceeds by contradiction. So, assuming that $K(\Sigma, X)$ is not barrelled though X is, there is a barrel T in $K(\Sigma, X)$ which is not a neighborhood of 0 in $K(\Sigma, X)$. By recurrence we obtain a normalized sequence $\{\Omega_n\}$ in Σ such that (a) $\operatorname{supp} f_{n+1} \subseteq \Omega_n$, (b) $f_n \notin 2nT$ and (c) $f_n(t) = x_n$ for all $t \in \Omega_n$. Defining $g_n = f_n - \chi_{\Omega_n} x_n$ for each $n \in \mathbf{N}$, there must exist $k \in \mathbf{N}$ such that $\chi_{\Omega_n} x_n \in nT$ for each $n \geq k$. So, $g_n \notin nT$ for each $n \geq k$. The sequence $\{g_n\}$ generates a copy of c_0 in the completion of $K(\Sigma, X)$ since its elements have pairwise disjoint supports. Noticing that this copy is contained in $K(\Sigma, X)$, there is an $m \geq k$ such that $g_m \in mT$, a contradiction.

If we consider the vector space $l_\infty(X) = K(2^N, X)$ of all bounded sequences in X provided with the supremum-norm, the previous theorem shows that this space is barrelled if and only if X is. On the other hand, since $K(\Sigma, X)$ is a dense subspace of $l_\infty(\Sigma, X)$, we obtain

Corollary 6.3 [**33**]. *The space $l_\infty(\Sigma, X)$ is barrelled if and only if X is barrelled.*

If $l_\infty(\Omega, X)$ stands for the vector space of all bounded X-valued functions defined on Ω equipped with the supremum-norm, then $l_\infty(\Omega, X)$ coincides with $l_\infty(2^\Omega, X)$ whenever X is separable (the nonseparable case will be considered in the next section). Consequently, assuming X separable, $l_\infty(\Omega, X)$ is barrelled if and only if X does.

The above techniques also work with $B(\Sigma, X)$ using the subspace $K(\Sigma, X) \cap B(\Sigma, X)$ instead of $K(\Sigma, X)$, thus obtaining Mendoza's result for a σ-algebra (for an algebra some appropriate modifications are needed [**35**]). Furthermore, if X is a normed barrelled space of class $n \in \mathbf{N}$, exploiting the previous techniques it can be shown that $B(\Sigma, X)$ is barrelled of class n [**41, 45**]. We do not know if $B(\Sigma, X)$ is baireled whenever X is. If $I = (0, 1]$, let $\mathrm{St}\,(I, X)$ be the vector subspace of $l_\infty(I, X)$ formed by the *step functions* and let $\mathrm{Reg}\,(I, X)$ be the (closed) vector subspace of $l_\infty(I, X)$ formed by the *regulated functions*, i.e., those that are uniform limit of a sequence of step functions. Identifying $\mathrm{St}\,(I, X)$ with a subspace of $B(\Sigma, X)$, Σ being the algebra of all finite unions of right semi-closed intervals contained in I, then $\mathrm{St}\,(I, X) = l_0^\infty(\Sigma, X)$ and $\mathrm{Reg}\,(I, X)$ coincides with $B(\Sigma, X)$. According to Mendoza's theorem, $\mathrm{Reg}\,(I, X)$ is barrelled if and only if X does, whereas according to Freniche's theorem, $\mathrm{St}\,(I, X)$ is not barrelled if X is infinite-dimensional. A bit more can be stated about the space of step functions, since

Theorem 6.4 [**35**]. *No* $\mathrm{St}\,(I, X)$ *space is barrelled.*

Given a measure space (Ω, Σ, μ) and an lcs E, one may consider the quotient space $K(\mu, E)$ of the vector space $K(\Sigma, E)$, provided with the uniform convergence topology, obtained by identifying those functions which are equal almost everywhere with respect to μ. This space has been studied in [**19**], assuming E is a DF-space. In particular, if X is a normed space and the measure μ is σ-finite and atomless, then $K(\mu, X)$ is barrelled. Finally, let us point out that some results given in this section are valid for algebras with property (N) [**44**].

7. The space of bounded vector-valued functions. As in the previous section, if Ω is some nonempty set and X a normed space, we will denote by $l_\infty(\Omega, X)$ the vector space over $\mathbf{K}$ of all bounded X-valued functions defined on Ω equipped with the supremum-norm. In this section we are going to show that, assuming X is barrelled, then $l_\infty(\Omega, X)$ is barrelled whenever $|\Omega|$ or $|X|$ is nonmeasurable. This result has been obtained by Drewnowski, et al. [**27**] and the proof described here adapts their methods. Let us recall that a nonempty set E is said to have measurable cardinal if there exists a free ultrafilter $\mathcal{U}$ on E, i.e., such that $\cap\mathcal{U} = \varnothing$, which is closed under the formation of countable intersections. It is not known whether there is a set with measurable cardinal. Extensive information about measurable cardinals and ultrafilter theory may be found in [**11, 12**]. The key point of [**27**] is the following result, whose proof we sketch below.

Lemma 7.1. *Let Σ be a σ-algebra of subsets of Ω and H a weak* bounded countable subset of $l_\infty(\Omega, X)^*$. The mapping $\eta : \Sigma \to [0, +\infty]$ defined by*

$$\eta(A) = \sup\{|\langle u, f\rangle| : u \in H, f \in l_\infty(\Omega, X), \|f\| \leq 1, \ \operatorname{supp} f \subseteq A\}$$

is a submeasure on Σ which satisfies the following:

(1) *For each decreasing sequence $\{A_n\}$ in Σ with $\eta(A_n) = +\infty$ for each $n \in \mathbf{N}$, one has that $\eta(\cap_{n=1}^{\infty} A_n) = +\infty$.*

(2) *Each family $\mathcal{A}$ of pairwise disjoint elements of Σ such that $\eta(A) > 0$ for each $A \in \mathcal{A}$ is countable.*

(3) *If $\eta(\Omega) = +\infty$, there exists a subset S of Ω such that the collection $\mathcal{U} = \{A \in \Sigma : A \subseteq S \wedge \eta(A) = +\infty\}$ is a Σ-ultrafilter on S closed under the formation of countable intersections.*

It is obvious that $\eta(\varnothing) = 0$, $\eta(A) \leq \eta(B)$ for each $A, B \in \Sigma$ with $A \subseteq B$ and $\eta(A \cup B) \leq \eta(A) + \eta(B)$ for each $A, B \in \Sigma$. Thus η is a submeasure on Σ.

1. Assume there is a decreasing sequence $\{A_n\}$ in Σ with $\eta(A_n) = +\infty$ for each $n \in \mathbf{N}$ and $\eta(\cap_{n=1}^{\infty} A_n) < \infty$. Setting $C_n := A_n \backslash \cap_{i=1}^{\infty} A_i$ for $n \in \mathbf{N}$; due to the subadditivity of η one has that $\eta(C_n) = +\infty$ for each $n \in \mathbf{N}$. Since $\eta(C_n) > n$ there exists some $u_n \in H$ and $f_n \in l_\infty(\Omega, X)$ with $\|f_n\| \leq 1$ and $\operatorname{supp} f_n \subseteq C_n$ such that $|\langle u_n, f_n\rangle| > n$. The linear map $\varphi : l_1 \to l_\infty(\Omega, \hat{X})$ defined by $\varphi(\xi) = \sum_{n=1}^{\infty} \xi_n f_n$ is continuous since $\|\varphi(\xi)\| \leq \|\xi\|_1$ for each $\xi \in l_1$. Given that $\sum_{n=1}^{\infty} \xi_n f_n(\omega)$ is a finite sum for each $\omega \in \Omega$ since $\operatorname{supp} f_n \subseteq C_n$ and $\cap_{n=1}^{\infty} C_n = \varnothing$, then φ is a continuous linear mapping from l_1 into $l_\infty(\Omega, X)$. Consequently, $M := \{u \circ \varphi : u \in H\}$ is a weak* bounded set in $l_1^* \cong l_\infty$ and hence $\sup\{\|u \circ \varphi\|_\infty : u \in H\} < \infty$. But

$$\|u_n \circ \varphi\|_\infty = \sup_{\substack{\xi \in l_1 \\ \|\xi\|_1 \leq 1}} |\langle u_n \circ \varphi, \xi\rangle| \geq |\langle u_n \circ \varphi, e_n\rangle| = |\langle u_n, f_n\rangle| > n,$$

and consequently $\sup\{\|u \circ \varphi\|_\infty : u \in H\} = +\infty$, a contradiction.

2. Assume that there exists an uncountable family $\mathcal{A}$ of pairwise disjoint elements of Σ such that $\eta(A) > 0$ for each $A \in \mathcal{A}$. Then for

each $A \in \mathcal{A}$ there is an $f_A \in l_\infty(\Omega, X)$ with $\|f_A\| \leq 1$, $\operatorname{supp} f_A \subseteq A$ and $u_A \in H$ such that $\langle u_A, f_A \rangle > 0$. Since $|\mathcal{A}| > \aleph_0$ there is a $\delta > 0$ and an uncountable subfamily $\mathcal{A}_1 \subseteq \mathcal{A}$ such that $\langle u_A, f_A \rangle > \delta$ for each $A \in \mathcal{A}_1$ and, H being countable, there must exist $u \in H$, $u \neq 0$ and an infinite subfamily $\mathcal{A}_2 \subseteq \mathcal{A}_1$ such that $u = u_A$ for each $A \in \mathcal{A}_2$. Let $k \in \mathbf{N}$ be such that $k\delta > \|u\|$ and choose k different sets $\{A_1, \ldots, A_k\}$ in $\mathcal{A}_2$. Then $\| \sum_{i=1}^k f_{A_i} \| = \max_{1 \leq i \leq k} \|f_{A_i}\| \leq 1$, while on the other hand

$$\Big\| \sum_{i=1}^k f_{A_i} \Big\| \geq \Big| \Big\langle \frac{u}{\|u\|}, \sum_{i=1}^k f_{A_i} \Big\rangle \Big| = \frac{1}{\|u\|} \sum_{i=1}^k \langle u_{A_i}, f_{A_i} \rangle = \frac{k\delta}{\|u\|} > 1,$$

a contradiction.

3. Let $\mathcal{G}$ be the class of all the families $\mathcal{A}$ of pairwise disjoint elements of Σ such that $0 < \eta(A) < \infty$ for each $A \in \mathcal{A}$. Consider in $\mathcal{G}$ the ordering $\mathcal{A} \leq \mathcal{B} \Leftrightarrow (A \in \mathcal{A} \Rightarrow A \in \mathcal{B})$. Then $(\mathcal{G}, \leq)$ becomes an inductive set and, according to Zorn's lemma, there must exist a maximal element $\mathcal{D}$ in $\mathcal{G}$. Then, by virtue of 2, $|\mathcal{D}| \leq \aleph_0$. Let $P := \cup \mathcal{D}$ and note that $P \in \Sigma$. If $\eta(P) = +\infty$, then $\mathcal{D}$ must be an infinite family since η is subadditive and $0 < \eta(A) < \infty$ for each $A \in \mathcal{D}$. Assuming that $\mathcal{D} = \{D_n : n \in \mathbf{N}\}$ and, setting $A_n := \cup_{i=n}^\infty D_i$ for each $n \in \mathbf{N}$, then $\{A_n\}$ is a decreasing sequence in Σ such that $\eta(A_n) = +\infty$ for each $n \in \mathbf{N}$. Consequently, 1 implies that $\eta(\varnothing) = \eta(\cap_{n=1}^\infty A_n) = +\infty$, a contradiction. Therefore, $\eta(P) < \infty$.

Due to the fact that $\eta(\Omega) = +\infty$, then $\eta(\Omega \setminus P) = +\infty$. Set $Q := \Omega \setminus P$ and note that, if $A \subseteq Q$ verifies that $A \in \Sigma$, then either $\eta(A) = 0$ or $\eta(A) = +\infty$. If there is no $A \in \Sigma$ with $A \subseteq Q$ such that $\eta(A) = \eta(Q \setminus A) = +\infty$ we set $S := Q$. Otherwise, we set $A_1 = A$. If there is no $A' \in \Sigma$ with $A' \subseteq A_1$ such that $\eta(A') = \eta(A_1 \setminus A') = +\infty$ we set $S := A_1$. Otherwise we set $A_2 = A'$. According to 1, this process must stop with some $A_p \in \Sigma$, $A_p \subseteq Q$, such that for each partition $\{B, C\}$ of A_p with elements of Σ, then either $\eta(B) = 0$ and $\eta(C) = +\infty$ or $\eta(B) = +\infty$ and $\eta(C) = 0$. We conclude by taking $S = A_p$ and noting that $\mathcal{U}$ is a Σ-filter on S. Moreover, $\mathcal{U}$ is a Σ-ultrafilter on S since, for each partition $\{B, C\}$ of S with elements of Σ, then either $B \in \mathcal{U}$ or $C \in \mathcal{U}$. According to 1, $\mathcal{U}$ is closed under the formation of countable intersections.

Theorem 7.2 [**27**]. *Assume that Ω is a nonempty set and X is a normed barrelled space. If $|\Omega|$ or $|X|$ is nonmeasurable, then $l_\infty(\Omega, X)$ is barrelled.*

In fact, if $l_\infty(\Omega, X)$ is not barrelled there exists a weak* bounded countable set $H \subseteq l_\infty(\Omega, X)^*$ which is not uniformly bounded. This means that $\eta(\Omega) = +\infty$. If S and $\mathcal{U}$ are, respectively the subset of Ω and the ultrafilter on S determined in part 3 of the preceding lemma with $\Sigma = 2^\Omega$, then for each $\omega \in \Omega$ we have $\eta(\{\omega\}) = \sup\{|\langle u, x\chi_{\{\omega\}}\rangle| : x \in X, \|x\| \leq 1, u \in H\}$. Setting $h_u(x) := \langle u, x\chi_{\{\omega\}}\rangle$ for each $x \in X$, then $h_u \in X^*$ for each $u \in H$. Barrelledness of X and the fact that $\sup\{|h_u(x)| : u \in H\} < \infty$ for each $x \in X$ imply that $\sup\{\|h_u\|_{X^*} : u \in H\} < \infty$. So $\eta(\{\omega\}) < \infty$ for each $\omega \in \Omega$. Therefore $\mathcal{U}$ is a free ultrafilter on S. Since $\mathcal{U}$ is closed under countable intersections, $|S|$ need be measurable, which implies that $|\Omega|$ must be measurable.

If $|X|$ were nonmeasurable, $|f(S)|$ would be nonmeasurable for each $f \in l_\infty(\Omega, X)$. If $f \in l_\infty(\Omega, X)$ with $\|f\| \leq 1$ and $\operatorname{supp} f \subseteq S$, it may be seen that there is an $A \in \mathcal{U}$ such that $f|_A$ is constant. Since $|\langle u, f\chi_{S\setminus A}\rangle| \leq \eta(S \setminus A) = 0$, then $\langle u, f\rangle = \langle u, f\chi_A\rangle$ for each $u \in H$. But f being constant in A, there is an $x_f \in X$ with $\|x_f\| \leq 1$ such that $f\chi_A = x_f\chi_A$. So $\langle u, f\rangle = \langle u, x_f\chi_A\rangle$ for each $u \in H$. Now the linear map $T : X \to l_\infty(\Omega, X)$ defined by $Tx = x\chi_S$ is an isomorphic embedding of X into $l_\infty(\Omega, X)$, so if T^* denotes the adjoint map of T, the previous discussion implies that

$$\sup\{|\langle T^*u, x\rangle| : u \in H, x \in X, \|x\| \leq 1\} = \eta(S).$$

Then $\sup\{|\langle T^*u, x\rangle| : u \in H, x \in X, \|x\| \leq 1\} = +\infty$, a contradiction since $T^*(H)$ being a weak* bounded subset of X^*, is uniformly bounded.

If $|\Omega|$ is measurable and X is a Banach space with $|X|$ measurable, then $l_\infty(\Omega, X)$ is barrelled, since it is a Banach space. Therefore, the conditions required in Theorem 7.2 are not necessary. We do not know if the previous theorem is true without any restriction on the cardinality of the set Ω or the (barrelled) space X.

REFERENCES

1. P. Abraham, *On the Vitali-Hahn-Saks-Nikodým theorem*, Quaestiones Math. **19** (1996), 397–407.

2. N. Adasch, B. Ernst and D. Keim, *Topological vector spaces. The theory without convexity conditions*, Springer-Verlag, Berlin, 1978.

3. A. Aizpuru, *Relaciones entre propiedades de supremo y propiedades de interpolación en álgebras de Boole*, Collect. Math. **39** (1988), 115–125.

4. ———, *On the Grothendieck and Nikodým properties of Boolean algebras*, Rocky Mountain J. Math. **22** (1992), 1–10.

5. ———, *The Nikodým property and local properties of Boolean algebras*, Colloq. Math. **71** (1996), 79–85.

6. I. Amemiya and Y. Komura, *Über nicht vollständige Montel-räume*, Math. Ann. **177** (1968), 273–277.

7. J. Arias de Reyna, $\ell_0^\infty(\Sigma)$ *no es totalmente tonelado*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **79** (1983), 77–78.

8. H. Auerbach, *Über die Vorzeichenverteilung in unendlichen Reihen*, Stud. Math. **2** (1930), 228–230.

9. J. Batt, P. Dierolf and J. Voigt, *Summable sequences and topological properties of* $m_0(I)$, Arch. Math. **28** (1977), 86–90.

10. G. Bennet and N. Kalton, *Inclusion theorems for k-spaces*, Canad. J. Math. **25** (1973), 511–524.

11. W.W. Comfort, *Ultrafilters*: *Some old and some new results*, Bull. Amer. Math. Soc. **83** (1977), 417–455.

12. W.W. Comfort and G. Negrepontis, *The theory of ultrafilters*, Grundlehren Math. Wiss. Bd. **211**, Springer-Verlag, Berlin, 1975.

13. C. Constantinescu, *On Nikodým's boundedness theorem*, Libertas Math. **1** (1981), 51–73.

14. F.K. Dashiell, *Nonweakly compact operators from order-Cauchy complete* $C(S)$ *lattices with application to Baire classes*, Trans. Amer. Math. Soc. **266** (1981), 397–413.

15. J.L. De María and P. Morales, *A non-commutative version of the Nikodým boundedness theorem*, Atti Sem. Mat. Fis. Univ. Modena **42** (1994), 505–517.

16. M. De Wilde, *Closed graph theorems and webbed spaces*, Pitman Res. Notes Math. Ser. **19**, Longman Sci. Tech., Harlow, 1978.

17. S. Díaz, L. Drewnowski, A. Fernández, M. Florencio and P.J. Paúl, *Barrelledness and bornological conditions on spaces of vector-valued* μ*-simple functions*, Resultate Math. **71**, Birkhäuser, Basel, 1992, pp. 289–298.

18. S. Díaz, A. Fernández, M. Florencio and P.J. Paúl, *A wide class of ultrabornological spaces of measurable functions*, J. Math. Anal. Appl. **190** (1995), 697–713.

19. ———, *The space of countably simple bounded functions with values in a* DF*-space*, Rev. Matemática U. Compl. Madrid **7** (1994), 219–232.

20. P. Dierolf, S. Dierolf and L. Drewnowski, *Remarks and examples concerning unordered Baire-like and ultra-barrelled spaces*, Colloq. Math. **39** (1978), 109–116.

21. J. Diestel, *Sequences and series in Banach spaces*, Springer-Verlag, New York, 1984.

22. J. Diestel, B. Faires and R. Huff, *Convergence and boundedness of measures in non sigma complete algebras*, unpublished.

23. J. Diestel and J. Uhl, *Vector measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, 1977.

24. J. Dieudonné, *Sur la convergence des suites de mesures de Radon*, Ann. Acad. Brasil. Ciencias **23** (1951), 21–38.

25. N. Dinculeanu, *Vector measures*, Pergamon Press, Oxford, 1967.

26. L. Drewnowski, M. Florencio and P.J. Paúl, *Barrelled subspaces of spaces with subseries decompositions or Boolean rings of projections*, Glasgow Math. J. **36** (1994), 57–69.

27. ———, *On the barrelledness of spaces of bounded vector functions*, Arch. Math. **63** (1994), 449–458.

28. ———, *Some new classes of rings of sets with the Nikodým property*, in Proc. Trier Conf. on Functional Analysis (Deirolf, Dineen and Domański, eds.), Walter de Gruyter & Co., Berlin, 1996, pp. 143–152.

29. L. Drewnowski and P. Paúl, *The Nikodým property for ideals of sets defined by matrix summability methods*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **94** (2000), 485–503.

30. B.T. Faires, *On Vitali-Hahn-Saks-Nikodým type theorems*, Ann. Inst. Fourier (Grenoble) **26** (1976), 99–114.

31. J.C. Ferrando, *Subespacios totalmente tonelados en espacios de Banach con una base*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **86** (1992), 211–216.

32. ———, *A projective description of the simple scalar function space*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **86** (1992), 231–236.

33. ———, *On the barrelledness of the vector-valued bounded function space*, J. Math. Anal. Appl. **184** (1994), 437–440.

34. ———, *Strong barrelledness properties in certain $\ell_0^\infty(\mathcal{A})$ spaces*, J. Math. Anal. Appl. **190** (1995), 194–202.

35. ———, *On some spaces of vector-valued bounded functions*, Math. Scand. **84** (1999), 71–80.

36. J.C. Ferrando and M. López Pellicer, *Strong barrelledness properties in $\ell_0^\infty(X,\mathcal{A})$ and bounded finite additive measures*, Math. Ann. **287** (1990), 727–736.

37. ———, *An ordered suprabarrelled space*, J. Austral. Math. Soc. Ser. A **51** (1991), 106–117.

38. ———, *Barrelled spaces of class n and of class $\aleph_0$*, Sem. Mat. Fund. UNED, Fasc. **4** (1992).

39. ———, *A note on a theorem of J. Diestel and B. Faires*, Proc. Amer. Math. Soc. **115** (1992), 1077–1081.

40. ———, *On the ideal of all subsets of* **N** *of density zero*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **92** (1998), 21–25.

41. J.C. Ferrando, M. López Pellicer and L.M. Sánchez Ruiz, *Metrizable barrelled spaces*, Pitman Res. Notes Math. Ser. **332**, Longman Sci. Tech., Harlow, 1995.

42. J.C. Ferrando and L.M. Sánchez Ruiz, *A maximal class of spaces with strong barrelledness conditions*, Proc. Roy. Irish Acad. Sect. A **92** (1992), 69–75.

43. ———, *On a barrelled space of class* $\aleph_0$ *and measure theory*, Math. Scand. **71** (1992), 96–104.

44. ———, *A note on vector-valued bounded function space*, in *Topological vector spaces, algebras and related areas* (Lau and Tweddle, eds.), Pitman Res. Notes Math. Ser. **316**, Longman Sci. Tech., Harlow, 1994, 20–23.

45. ———, *Strong barrelledness properties in* $B(\Sigma, X)$, Bull. Austral. Math. Soc. **52** (1995), 207–214.

46. J.R. Ferrer, M. López Pellicer and L.M. Sánchez Ruiz, *A strongly barrelled space*, Math. Japon. **39** (1994), 89–94.

47. F.J. Freniche, *Barrelledness conditions of the space of vector valued and simple functions*, Math. Ann. **267** (1984), 479–489.

48. ———, *The Vitali-Hahn-Saks theorem for Boolean algebras with the subsequential interpolation property*, Proc. Amer. Math. Soc. **92** (1984), 362–366.

49. P. Gänssler, *A convergence theorem for measures in regular Hausdorff spaces*, Math. Scand. **29** (1971), 237–244.

50. M.T. Gassó, *Algebras with the local interpolation property*, Rocky Mountain J. Math. **22** (1992), 181–196.

51. A. Gilioli, *Natural ultrabornological, non-complete, normed function spaces*, Arch. Math. **61** (1993), 465–477.

52. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand-Reinhold Co., New York, 1960.

53. W.H. Graves and R.F. Wheeler, *The Grothendieck and Nikodým properties*, Rocky Mountain J. Math. **13** (1983), 333–353.

54. R. Haydon, *A non reflexive Grothendieck space that does not contain* ℓ_∞, Israel J. Math. **40** (1981), 65–73.

55. H. Jarchow, *Locally convex spaces*, B.G. Teubner, Stuttgart, 1981.

56. J. Kąkol, *Topological linear spaces with some Baire-like properties*, Functiones et Approx. **13** (1982), 109–116.

57. J. Kąkol and W. Roelcke, *Unordered Baire-like spaces without local convexity*, Collect. Math. **43** (1992), 43–53.

58. ———, *Topological vector spaces with some Baire-type properties*, Note Mat. **12** (1992), 77–92.

59. G. Köthe, *Topological vector spaces* I, Springer-Verlag, New York, 1969.

60. ———, *Topological vector spaces* II, Springer-Verlag, New York, 1979.

61. M. Kunzinger, *Barrelledness, Baire-like and* (LF)*-spaces*, Pitman Res. Notes Math. Ser. **298**, Longman Sci. Tech., Harlow, 1993.

62. M. López Pellicer, *Webs and bounded finitely additive measures*, J. Math. Anal. Appl. **210** (1997), 257–267.

63. J. Mendoza, *Barrelledness conditions on* $S(\Sigma, E)$ *and* $B(\Sigma, E)$, Math. Ann. **261** (1982), 11–22.

64. A. Moltó, *On the Vitali-Hahn-Saks theorem*, Proc. Roy. Soc. Edinburgh Sect. A **90** (1981), 163–173.

65. ———, *On uniform properties in exhausting additive set function spaces*, Proc. Roy. Soc. Edinburgh Sect. A **90** (1981), 175–184.

66. M. Paštéka, *Convergence of series and submeasures on the set of positive integers*, Math. Slovaca **40** (1990), 273–278.

67. P. Pérez Carreras, *Sobre ciertas clases de espacios vectoriales topológicos*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **76** (1982), 585–594.

68. P. Pérez Carreras and J. Bonet, *Remarks and examples concerning suprabarrelled and totally barrelled spaces*, Arch. Math. **34** (1982), 340–347.

69. ———, *Barrelled locally convex spaces*, North-Holland Math. Stud. **131**, North-Holland, Amsterdam, 1987.

70. A. Pietsch, *Nuclear locally convex spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete **66**, Springer, 1972.

71. W. Robertson, *Completions of topological vector spaces*, Proc. London Math. Soc. **8** (1958), 242–257.

72. W.J. Robertson, I. Tweddle and F.E. Yeomans, *On the stability of barrelled topologies* III, Bull. Austral. Math. Soc. **22** (1980), 99–112.

73. R. Rodríguez Salinas, *Sobre el teorema de la gráfica cerrada. Aplicaciones lineales subcontinuas*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **74** (1980), 811–825.

74. ———, *Sobre la clase del espacio tonelado* $\ell_0^\infty(\Sigma)$, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **74** (1980), 827–829.

75. ———, *On superbarrelled spaces. Closed graph theorems*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **89** (1995), 7–10.

76. L.M. Sánchez Ruiz, *A countable number of* $\mathcal{L}$*-barrelled spaces*, Bull. Soc. Math. Belg. **45** (1993), 59–67.

77. L.M. Sánchez Ruiz and F. Mínguez, *Ultrabarrelledness increasingly inherited*, in *Function spaces* (Hudzik and Skrzypczak, eds.), Lecture Notes in Pure and Appl. Math **213**, Dekker, New York, 2000, pp. 469–474.

78. S.A. Saxon, *Nuclear and product spaces, Baire-like spaces and the strongest locally convex topology*, Math. Ann. **197** (1972), 87–106.

79. ———, *Some normed barrelled spaces which are not Baire*, Math. Ann. **209** (1974), 153–160.

80. S.A. Saxon and P.P. Narayanaswami, *Metrizable* (LF)*-spaces,* (db)*-spaces and the separable quotient problem*, Bull. Austral. Math. Soc. **23** (1981), 65–80.

81. W. Schachermayer, *On some classical measure-theoretic theorems for non* σ*-complete Boolean algebras*, Dissertationes Math. **214** (1982), 1–36.

82. G.L. Seever, *Measures on* F*-spaces*, Trans. Amer. Math. Soc. **133** (1968), 267–280.

83. Z. Semadeni, *Banach spaces of continuous functions* I, Polish Scient. Publishers, Warsaw, 1971.

84. R. Sikorski, *Boolean algebras*, Springer-Verlag, New York, 1964.

85. M. Talagrand, *Propieté de Nikodým et propieté de Grothendiek*, Studia Math. **68** (1984), 165–171.

86. A.R. Todd and S. Saxon, *A property of locally convex Baire spaces*, Math. Ann. **206** (1973), 23–34.

87. M. Valdiva, *A class of bornological barrelled spaces which are not ultrabornological*, Math. Ann. **194** (1971), 43–51.

88. ———, *On certain barrelled normed spaces*, Ann. Inst. Fourier (Grenoble) **29** (1979), 39–56.

89. ———, *On Baire-hyperplane spaces*, Proc. Edinburgh Math. Soc. Sect. A **22** (1979), 247–255.

90. ———, *On suprabarrelled spaces*, in Conf. on Functional Analysis Holomorphy and Approximation Theory (Rio de Janeiro 1978), Lecture Notes in Math. **843**, Springer-Verlag, New York, 1981, pp. 572–580.

91. ———, *Localization theorems in webbed spaces*, in *Semesterberich Functionanalysis*, Tübingen, Sommersemester, 1982, pp. 49–57.

92. ———, *A class of locally convex spaces without $\mathcal{C}$-webs*, Ann. Inst. Fourier **32** (1982), 261–269.

93. ———, *Topics in locally convex spaces*, North-Holland Math. Stud. **67**, North-Holland, Amsterdam, 1982.

94. M. Valdivia and P. Pérez Carreras, *On totally barrelled spaces*, Math. Z. **178** (1981), 263–269.

Centro de Investigación Operativa, Universidad Miguel Hernández, E-03202 Elche (Alicante), Spain
E-mail address: `jc.ferrando@umh.es`

EUITI-Depto. de Matemática Aplicada, Universidad Politécnica de Valencia, E-46022, Valencia, Spain
E-mail address: `lmsr@mat.upv.es`