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# ON THE STRONG LAW FOR ASYMPTOTICALLY ALMOST NEGATIVELY ASSOCIATED RANDOM VARIABLES

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ABSTRACT. In this paper the Hajeck-Renyi type inequality for asymptotically almost negatively associated (AANA) random variables is derived and the strong law of large numbers is obtained by applying this inequality. The strong laws of large numbers for weighted sums of AANA random variables are also considered.

**1. Introduction.** Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $\{X_1, \dots, X_n\}$ a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathcal{P})$. A finite family $\{X_i,\, 1 \le i \le n\}$ is said to be negatively associated (NA) if for any disjoint subsets $A, B \subset \{1, \dots, n\}$ and any real coordinatewise nondecreasing functions $f : R^A \to R$ and $g : R^B \to R$,

$$\operatorname{Cov}(f(X_i;\, i \in A), g(X_j;\, j \in B)) \le 0.$$

Infinite family of random variables is negatively associated (NA) if every finite subfamily is negatively associated (NA). This concept was introduced by Joag-Dev and Proschan [**8**]. A sequence $\{X_n,\, n \ge 1\}$ of random variables is called asymptotically almost negatively associated (AANA) if there is a nonnegative sequence $q(m) \to 0$ such that

$$(1)\quad \operatorname{Cov}(f(X_m), g(X_{m+1}, \dots, X_{m+k})) \le q(m)(\operatorname{Var}(f(X_m))\operatorname{Var}(g(X_{m+1}, \dots, X_{m+k})))^{1/2}$$

for all $m, k \ge 1$ and for all coordinatewise increasing continuous functions $f$ and $g$ whenever the righthand side of (1) is finite. This definition was introduced by Chandra and Ghosal [**2**, **3**].

The family of AANA sequences contains negatively associated (in particular, independent) sequences (with $q(m) = 0$ for all $m \ge 1$)

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and also some sequences of random variables which do not deviate much from being negatively associated. Condition (1) is clearly satisfied if the $R_{2,2}$-measure of dependence (see [**1**]) between $\sigma(X_m)$ and $\sigma(X_{m+1}, X_{m+2}, \dots)$ converges to zero. The following is a nontrivial example of an AANA sequence. It is possible to construct similar examples, but we shall not discuss this topic any more here. Let $\{Y_n\}$ be i.i.d. $N(0,1)$ variables, and define $X_n = (1+a_n^2)^{-1/2}(Y_n + a_n Y_{n+1})$ where $a_n > 0$ and $a_n \to 0$. Note that $\{X_n\}$ is not NA (indeed, it is associated and 1-dependent). We shall show that the correlation coefficient between $U = f(X_m)$ and $V = g(X_{m+1}, \dots, X_{m+k})$ is dominated in absolute value by $a_m$. It suffices to prove this under the additional hypotheses $EU = 0 = EV$, $EU^2 = 1 = EV^2$. Then

$$\begin{aligned}
(\operatorname{Cov}(U,V))^2 &\le (\operatorname{Cov}(U, E(U \mid X_{m+1}, \dots, X_{m+k})))^2 \\
&\le E(E(U \mid X_{m+1}, \dots, X_{m+k}))^2 \\
&\le E(E(U \mid Y_{m+1}, \dots, Y_{m+k+1}))^2 \\
&\le E(E(U \mid Y_{m+1}))^2 \\
&= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x) \left( \frac{\psi_m(x,y)}{\phi(x)} - 1 \right) \phi(x)\, dx \right)^2 \phi(y)\, dy
\end{aligned}$$

where $\psi_m(x,y)$ is the conditional density of $X_m$ given by $Y_{m+1} = y$ and $\phi(x)$ is the density of $N(0,1)$. By the Cauchy-Schwarz inequality, the last integral is at most

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\psi_m(x,y)}{\phi(x)} - 1 \right)^2 \phi(x)\, dx \phi(y)\, dy = a_m^2,$$

(see [**2**]).

Hajeck-Renyi [**7**] proved that if $\{X_n,\, n \ge 1\}$ is a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 < \infty$, $n \ge 1$ and $\{b_n,\, n \ge 1\}$ is a positive nondecreasing real sequence, then for any $\varepsilon > 0$ and for any positive integer $m < n$,

$$\text{(2)} \qquad P\left( \max_{m \le j \le n} \left| \frac{\sum_{i=1}^{j} X_i}{b_j} \right| \ge \varepsilon \right) \le \varepsilon^{-2} \left( \sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{EX_j^2}{b_m^2} \right).$$

Since then, this inequality has been the concern of many authors (e.g., [**4, 6**] for martingales, [**9**] for negatively associated random variables, [**5**] for associated random variables).

In this paper we will extend (2) to AANA random variables and use this inequality to prove the strong law of large numbers. We also consider almost sure convergences for weighted sums of AANA random variables.

## 2. The Hajeck-Renyi type inequality for AANA random variables.

From the definition of AANA random variables (see (1)) we easily obtain the following lemma.

**Lemma 2.1.** *Let $\{X_n,\, n \geq 1\}$ be a sequence of asymptotically almost negatively associated* (AANA) *random variables and $\{a_n,\, n \geq 1\}$ a sequence of positive numbers. Then $\{a_n X_n,\, n \geq 1\}$ is also a sequence of* AANA *random variables.*

**Lemma 2.2** [**2, 3**]. *Let $\{X_1, \dots, X_n\}$ be a sequence of mean zero, square integrable random variables such that* (1) *holds for $1 \leq m < k+m \leq n$ and for all coordinatewise increasing continuous functions $f$ and $g$ whenever the righthand side of* (1) *is finite. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and $\sigma_k^2 = EX_k^2$, $k \geq 1$. Then, for $\varepsilon > 0$,*

$$P\left\{ \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i \right| \geq \varepsilon \right\} \leq 2\varepsilon^{-2}(A + (1+A^2)^{1/2})^2 \sum_{k=1}^{n} \sigma_k^2. \tag{3}$$

**Theorem 2.3.** *Let $\{b_n,\, n \geq 1\}$ be a sequence of positive nondecreasing real numbers and $\{X_1, \dots, X_n\}$ a sequence of mean zero, square integrable random variables such that* (1) *holds for $1 \leq m < k+m \leq n$ and for all coordinatewise increasing continuous functions $f$ and $g$ whenever the righthand side of* (1) *is finite. Let $A^2 = \sum_{k=1}^{n-1} q^2(k)$ and $\sigma_k^2 = EX_k^2$, $k \geq 1$. Then, for $\varepsilon > 0$,*

$$P\left\{ \max_{1 \leq k \leq n} \left| \frac{\sum_{i=1}^{k} X_i}{b_k} \right| \geq \varepsilon \right\} \leq 8\varepsilon^{-2}(A + (1+A^2)^{1/2})^2 \sum_{k=1}^{n} \frac{\sigma_k^2}{b_k^2}. \tag{4}$$

*Proof.* First note that $\{X_1/b_1, \dots, X_n/b_n\}$ is a sequence of mean zero, square integrable AANA random variables by Lemma 2.1. Thus

$\{X_1/b_1, X_2/b_2, \dots\}$ satisfies

$$\begin{aligned}(5)\quad & \mathrm{Cov}\,(f(b_m^{-1}X_m), g(b_{m+1}^{-1}X_{m+1}, \dots, b_{m+k}^{-1}X_{m+k})) \\ & \le q(m)[\mathrm{Var}\,(f(b_m^{-1}X_m))\mathrm{Var}\,(g(b_{m+1}^{-1}X_{m+1}, \dots, b_{m+k}^{-1}X_{m+k}))]^{1/2}\end{aligned}$$

for $1 \le m < k+m \le n$ and for all coordinatewise increasing continuous functions $f$ and $g$ whenever the righthand side of (5) is finite. Without loss of generality, setting $b_0 = 0$, we get

$$\begin{aligned}\sum_{j=1}^{k} X_j &= \sum_{j=1}^{k}\left(\sum_{i=1}^{j}(b_i - b_{i-1})\frac{X_j}{b_j}\right) \\ &= \sum_{i=1}^{k}(b_i - b_{i-1})\sum_{i\le j\le k}\frac{X_j}{b_j}.\end{aligned}$$

Since

$$b_k^{-1}\sum_{j=1}^{k}(b_j - b_{j-1}) = 1$$

$$(6)\qquad \left\{\left|\frac{\sum_{j=1}^{k} X_j}{b_k}\right| \ge \varepsilon\right\} \subset \left\{\max_{1\le i\le k}\left|\sum_{i\le j\le k}\frac{X_j}{b_j}\right| \ge \varepsilon\right\}.$$

From (6) we have

$$\begin{aligned}\left\{\max_{1\le k\le n}\frac{|\sum_{j=1}^{k} X_j|}{b_k} \ge \varepsilon\right\} &\subset \left\{\max_{1\le k\le n}\max_{1\le i\le k}\left|\sum_{i\le j\le k}\frac{X_j}{b_j}\right| \ge \varepsilon\right\} \\ &= \left\{\max_{1\le i\le k\le n}\left|\sum_{j\le k}\frac{X_j}{b_j} - \sum_{j<i}\frac{X_j}{b_j}\right| \ge \varepsilon\right\} \\ &\subset \left\{\max_{1\le k\le n}\left|\sum_{1\le j\le k}\frac{X_j}{b_j}\right| \ge \frac{\varepsilon}{2}\right\}.\end{aligned}$$

Hence by Lemma 2.2 the desired result (4) follows.

From Theorem 2.3, we can get the following more generalized Hajeck-Renyi type inequality.

**Theorem 2.4.** *Let* $\{b_n,\, n \geq 1\}$ *be a sequence of positive nondecreasing real numbers and* $\{X_n, n \geq 1\}$ *a sequence of mean zero, and square integrable* AANA *random variables such that* (1) *holds for* $1 \leq m < k+m \leq n$ *and for all coordinatewise continuous functions* $f$ *and* $g$ *whenever the righthand side of* (1) *is finite. Let* $A^2 = \sum_{k=1}^{n-1} q^2(k)$ *and* $\sigma_k^2 = EX_k^2$, $k \geq 1$. *Then for* $\varepsilon > 0$ *and for any positive integer* $m < n$, *we have*

$$(7)\quad P\left\{\max_{m\leq k\leq n}\left|\frac{\sum_{j=1}^{k} X_j}{b_k}\right| \geq \varepsilon\right\} \leq 32\varepsilon^{-2}(A+(1+A^2)^{1/2})^2\left(\sum_{j=m+1}^{n}\frac{\sigma_j^2}{b_j^2}+\sum_{j=1}^{m}\frac{\sigma_j^2}{b_m^2}\right).$$

*Proof.* By Theorem 2.3 we have

$$\begin{aligned}
&P\left\{\max_{m\leq k\leq n}\left|\frac{\sum_{j=1}^{k} X_j}{b_k}\right| \geq \varepsilon\right\}\\
&\leq P\left\{\left|\frac{\sum_{j=1}^{m} X_j}{b_m}\right| \geq \frac{\varepsilon}{2}\right\} + P\left\{\max_{m+1\leq k\leq n}\left|\frac{\sum_{j=m+1}^{k} X_j}{b_k}\right| \geq \frac{\varepsilon}{2}\right\}\\
&\leq P\left\{\frac{1}{b_m}\max_{1\leq k\leq m}\left|\sum_{j=1}^{k} X_j\right| \geq \frac{\varepsilon}{2}\right\} + P\left\{\max_{m+1\leq k\leq n}\left|\frac{\sum_{j=m+1}^{k} X_j}{b_k}\right| \geq \frac{\varepsilon}{2}\right\}\\
&\leq 32\varepsilon^{-2}(A+(1+A^2)^{1/2})^2\left(\sum_{j=1}^{m}\frac{\sigma_j^2}{b_m^2}+\sum_{j=m+1}^{n}\frac{\sigma_j^2}{b_j^2}\right).
\end{aligned}$$

Hence the proof is complete.

## 3. Strong laws of large numbers for AANA random variables.

**Theorem 3.1.** *Let* $\{b_n,\, n \geq 1\}$ *be a sequence of positive nondecreasing real numbers and* $\{X_n, n \geq 1\}$ *a sequence of mean zero, square integrable random variables such that* (1) *holds for* $1 \leq m < k+m \leq n$ *and* *for all coordinatewise increasing continuous functions $f$ and $g$ whenever the righthand side of* (1) *is finite. Let $\sigma_k^2 = EX_k^2$, $k \geq 1$, and $S_n = \sum_{i=1}^n X_i$. Assume*

$$\sum_{k=1}^{\infty} q^2(k) < \infty, \tag{8}$$

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{b_k^2} < \infty. \tag{9}$$

*Then, for any* $0 < r < 2$,

(A) $E(\sup_n(|S_n|/b_n)^r < \infty$,

(B) $0 < b_n \uparrow \infty$ *implies* $S_n/b_n \to 0$ *a.s., as* $n \to \infty$.

*Proof.* Put $B^2 = \sum_{k=1}^{\infty} q^2(k) < \infty$.

(A) Note that

$$E\left(\sup_n \frac{|S_n|}{b_n}\right)^r < \infty \Longleftrightarrow \int_1^{\infty} P\left(\sup_n \frac{|S_n|}{b_n} > t^{1/r}\right) < \infty.$$

By Theorem 2.3, it follows from (8) and (9) that

$$\begin{aligned}\int_1^{\infty} P\left(\sup_n \frac{|S_n|}{b_n} > t^{1/r}\right) dt \\ &\leq 8 \int_1^{\infty} t^{-2/r}\, dt (B + (1+B^2)^{1/2})^2 \sum_{k=1}^{\infty} \frac{\sigma_k^2}{b_k^2} < \infty \\ &\leq 8(B + (1+B^2)^{1/2})^2 \sum_{k=1}^{\infty} \frac{\sigma_k^2}{b_k^2} \int_1^{\infty} t^{-2/r}\, dt < \infty.\end{aligned}$$

Hence the proof of (A) is complete.

(B) By Theorem 2.4 we get

$$P\left\{\max_{m \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon\right\} \leq 32\varepsilon^{-2}(B+(1+B^2)^{1/2})^2 \left(\sum_{j=m+1}^{n} \frac{\sigma_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{\sigma_j^2}{b_m^2}\right).$$

But, by assumptions (8) and (9), we have

(10)
$$P\left\{\sup_{k\geq m}\frac{|S_k|}{b_k}\geq\varepsilon\right\}=\lim_{n\to\infty}P\left\{\max_{m\leq k\leq n}\frac{|S_k|}{b_k}\geq\varepsilon\right\}$$
$$\leq 32\varepsilon^{-2}(B+(1+B^2)^{1/2})^2\left(\sum_{j=m+1}^{\infty}\frac{\sigma_j^2}{b_j^2}+\sum_{j=1}^{m}\frac{\sigma_j^2}{b_m^2}\right).$$

By the Kronecker lemma and (9) we get

$$\sum_{j=1}^{m}\frac{\sigma_j^2}{b_m^2}\longrightarrow 0\quad\text{as } m\to\infty. \tag{11}$$

Hence, by combining (9), (10) and (11) we have

$$\lim_{n\to\infty}P\left\{\sup_{k\geq n}\frac{|S_k|}{b_k}\geq\varepsilon\right\}=0,$$

i.e., $S_n/b_n\to 0$ a.s. as $n\to\infty$.

**Corollary 3.2.** *Let* $\{X_n,\, n\geq 1\}$ *be a sequence of mean zero, square integrable random variables such that* (1) *holds for* $1\leq m<k+m\leq n$ *and for all coordinatewise increasing continuous functions* $f$ *and* $g$ *where the righthand side of* (1) *is finite. Assume* $B^2=\sum_{k=1}^{\infty}q(k)^2<\infty$. *Then, for* $0<t<2$,

$$P\left(\sup_{k\geq m}\frac{|S_k|}{k^{1/t}}\geq\varepsilon\right)\leq 32\varepsilon^{-2}(B+(1+B^2)^{1/2})^2\frac{2}{2-t}\sup_k\sigma_k^2 m^{(t-2)/t},$$

*for all* $\varepsilon\geq 0$, $m\geq 1$, *where*

$$S_n=\sum_{j=1}^{n}X_j\quad\textit{and}\quad\sigma_n^2=\operatorname{Var}(X_n),\quad n\geq 1.$$

**Corollary 3.3.** *Let* $\{X_n,\, n\geq 1\}$ *be a sequence of mean zero, square integrable* AANA *random variables such that* (1) *holds for* $1\leq m<k+m\leq n$ *and for all coordinatewise increasing continuous*

*functions $f$ and $g$ whenever the righthand side of* (1) *is finite. Assume that*

$$\sum_{k=1}^{\infty} q^2(k) < \infty \quad \textit{and} \quad \sup_n \sigma_n^2 < \infty,$$

*where* $\sigma_n^2 = \mathrm{Var}\,(X_n)$, $n \ge 1$. *Then, for* $0 < t < 2$,

(A) $S_n/n^{1/t} \to 0$ *a.s. as* $n \to \infty$,

(B) $E\sup_n(|S_n|/n^{1/t})^r < \infty$ *for any* $0 < r < 2$, *where* $S_n = \sum_{j=1}^n X_j$.

**4. Strong laws of large numbers for weighted sums of AANA random variables.** Finally, we consider an almost convergence of weighted sums of AANA random variables as applications of Theorem 3.1.

**Theorem 4.1.** *Let* $\{a_{ni},\, 1 \le i \le n,\, n \ge 1\}$ *be a sequence of real numbers with* $a_{ni} = 0$, $i > n$, $\sup_{n\ge1}\sum_{i=1}^n |a_{ni}| < \infty$ *and* $\{b_n, n \ge 1\}$ *a sequence of positive nondecreasing real numbers such that* $0 < b_n \uparrow \infty$ *and let* $\{X_n, n \ge 1\}$ *be a sequence of mean zero, square integrable* AANA *random variables satisfying* (8) *and* (9). *Then, as* $n \to \infty$,

$$\sum_{i=1}^{n} \frac{a_{ni}X_i}{b_n} \longrightarrow 0 \quad a.s.$$

*Proof.* Define

$$S_k = \sum_{j=1}^{k} \frac{X_j}{b_k},$$

$$c_{nj} = \frac{b_j}{b_n}(a_{nj} - a_{nj+1}) \quad \text{for } 1 \le j \le n-1,$$

and

$$c_{nn} = a_{nn}.$$

Then

$$\sum_{j=1}^{n} \frac{a_{nj}X_j}{b_n} = \sum_{j=1}^{n} c_{nj}S_j, \tag{12}$$

$$\sum_{j=1}^{n} |c_{nj}| \le 2 \sup_{n\ge 1} \sum_{j=1}^{n} |a_{nj}|, \tag{13}$$

$$\lim_{n\to\infty} |c_{nj}| = 0 \quad \text{for every fixed } j. \tag{14}$$

It follows from (13) and (14) that we have, for every sequence of real numbers $d_n$ with $d_n \to 0$, $n \to \infty$,

$$\sum_{j=1}^{n} c_{nj} d_j \to 0 \quad \text{as } n \to \infty. \tag{15}$$

Hence from Theorem 3.1(B), (12) and (15), the desired result follows.

**Theorem 4.2.** *Let* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *be a sequence of real numbers with* $a_{ni} = 0$, $i > n \sup_{n\ge 1} \sum_{i=1}^{n} |a_{ni}| < \infty$, *and let* $\{X_n, n \ge 1\}$ *be a sequence of mean zero, square integrable* AANA *random variables with* $\sup_n \sigma_n^2 < \infty$ *and* $\sum_{k=1}^{\infty} q^2(k) < \infty$ *where* $\sigma_n^2 = EX_k^2$, $n \ge 1$. *Then, for* $0 < t < 2$,

$$\sum_{i=1}^{n} \frac{a_{ni} X_i}{n^{1/t}} \longrightarrow 0 \quad a.s. \tag{16}$$

*Proof.* By putting $b_n = n^{1/t}$ from Corollary 3.3 and Theorem 4.1, the result follows and the proof is omitted.

From Theorem 2 of [**2**] we can obtain the following Marcinkiewicz strong law of large numbers for sums of AANA random variables.

**Theorem 4.3.** *Let* $\{X_n, n \ge 1\}$ *be a sequence of identically distributed* AANA *random variables with* $EX_1 = 0$, $E|X_1|^t < \infty$ *for* $0 < t < 2$ *and satisfying* (1). *Let* $\sum_{k=1}^{\infty} q^2(k) < \infty$. *Then*

$$\sum_{j=1}^{n} \frac{X_j}{n^{1/t}} \longrightarrow 0 \quad a.s.$$

**Theorem 4.4.** *Let* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be a sequence of real numbers with* $\sup_{n\geq 1}\sum_{i=1}^{n}|a_{ni}| < \infty$*, and let* $\{X_n, n \geq 1\}$ *be a sequence of identically distributed* AANA *random variables with* $EX_1 = 0$ *and* $E|X_1|^t < \infty$ *for* $0 < t < 2$ *and satisfying* (1). *Let* $B^2 = \sum_{k=1}^{\infty} q^2(k) < \infty$. *Then, for* $0 < t < 2$ *as* $n \to \infty$,

$$\sum_{i=1}^{n} \frac{a_{ni}X_i}{n^{1/t}} \longrightarrow 0 \quad a.s. \tag{17}$$

*Proof.* Basically, using the ideas in the proof of Theorem 4.1 and Theorem 4.3 we can obtain (17) and the proof is omitted.

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