ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 3, Fall 2004

EXISTENCE OF POSITIVE SOLUTIONS OF A SINGULAR INITIAL PROBLEM FOR A NONLINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

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ABSTRACT. A singular Cauchy problem for a system of nonlinear differential equations is considered. It is shown that, under certain assumptions, there exists its positive solution with asymptotic behavior similar, in a certain sense, to asymptotic behavior of the solution of the identical initial problem for implicit system of auxiliary nonlinear (and nondifferential) equations. Solution of the mentioned implicit system serves as the base for the construction of a "funnel," having the vertex coinciding with the initial point, in which the graph of a solution of a given singular problem is located. The main result gives sufficient conditions for the existence of a parametric family of such solutions having positive coordinates. In the proof of the main result, we apply the topological retract method. With this connection the character of every boundary point of this "funnel" is tested and as a result, we conclude that each of its boundary points is either the point of strict egress or the point of strict ingress with respect to the system considered. Corresponding computations use the properties of implicitly defined functions. As a special case, the linear system is considered, too. Illustrative examples show that the assumptions of the main result can easily be verified without the construction of any implicit function.

1. Introduction. Let us consider the system of nonlinear differential equations

$$g(x)y' = A(x)\alpha(y) - \omega(x) \tag{1}$$

together with the initial condition

$$y(0^+) = 0. \tag{2}$$

Here $y = (y_1, \dots, y_n)^T$ is the vector of unknown functions, $\alpha(y) = (\alpha_1(y_1), \dots, \alpha_n(y_n))^T$, $A(x)$ is an $n \times n$ matrix with elements $a_{ij}(x)$, $i, j = 1, \dots, n$, $\omega(x) = (\omega_1(x), \dots, \omega_n(x))^T$ and $g(x) = \operatorname{diag}(g_1(x), \dots, g_n(x))$ is a diagonal matrix with the indicated diagonal entries.

With $\mathbf{R}^n_{>0}$ we denote the set of all component-wise positive vectors v in $\mathbf{R}^n$, i.e., $v = (v_1, \dots , v_n)$ with $v_i > 0$ for $i = 1, \dots , n$. For $u, v \in \mathbf{R}^n$, we say $u \ll v$ if $v - u \in \mathbf{R}^n_{>0}$.

The symbol I_s, used throughout this paper, indicates an interval of the form $(0, s]$ with a fixed $s > 0$. The system (1) is considered under the following main assumptions: Let there be intervals I_{x_0}, I_{y_0} such that

(C$_1$) $g_i \in C(I_{x_0}, \mathbf{R}^+)$, $i = 1, \dots , n$, with $\mathbf{R}^+ = (0, \infty)$;

(C$_2$) $\alpha \in C^1(I_{y_0}, \mathbf{R}^n)$, $\alpha(y) \gg 0$ on $I_{y_0}, \alpha'(y) \gg 0$ on I_{y_0} and $\alpha(0^+) = 0$;

(C$_3$) $\omega \in C^1(I_{x_0}, \mathbf{R}^n)$;

(C$_4$) $a_{ij} \in C^1(I_{x_0}, \mathbf{R})$, $a_{ii}(x) \neq 0$, $i, j = 1, \dots , n$ and $\det A(x) \neq 0$ on I_{x_0};

(C$_5$) $\alpha_i(y) \leq M\alpha_i'(y)$, $i = 1, \dots , n$, on I_{y_0} with a constant $M \in \mathbf{R}^+$;

(C$_6$) $\Omega(x) \equiv A^{-1}(x)\omega(x) \gg 0$, $\Omega'(x) \gg 0$ on I_{x_0} and $\Omega(0^+) = 0$.

Let us define the notion of a positive solution of the problem (1), (2):

Definition 1. A function $y = y(x) \in C^1(I_{x^*}, \mathbf{R}^n)$ with $0 < x^* \leq x_0$ is said to be a *solution* of the singular problem (1), (2) on interval I_{x^*} if y satisfies (1) on I_{x^*} and $y(0^+) = 0$. If, except this, $y(x) \gg 0$ on I_{x^*}, we say that the solution of (1), (2) is *positive*.

The system (1) can be written in its normal form as

$$y' = f(x, y) \equiv g^{-1}(x) \cdot [A(x)\alpha(y) - \omega(x)].$$

The problem (1), (2) will be a singular problem if the vector $f(0^+, 0^+)$ is undefined. This condition is not involved in the group of conditions above and will hold (under the supposition that the ith equation of the system cannot be canceled by $g_i(x)$ for $x \in I_{x_0}$) if, in addition, $g_i(0^+) = 0$ for an $i \in \{1, \dots , n\}$. The latter condition is implicitly contained in the conditions of results formulated below. The advantage of using the form (1) consists of an easy application of the results concerning implicit functions.

With this one exception, the condition $g(0^+) = 0$ (null matrix) will help us to explain the motivation of our investigation. Let us put

$g(x) \equiv 0$ in (1). Then

$$0 = A(x)\alpha(y) - \omega(x) \tag{3}$$

or

$$y = \varphi(x) \equiv \alpha^{-1}[\Omega(x)] \tag{4}$$

where α^{-1} is the inverse function with respect to α. If

$$\lim_{x \to 0^+} \alpha^{-1}[\Omega(x)] = 0,$$

the solution (4) of system (3) satisfies the initial condition (2). We can expect that, in the case when $g(x) \to 0$ sufficiently quickly if $x \to 0^+$, the curve (4) can serve, in a certain sense, as an *approximation of solution* of the problem (1), (2).

Many books and articles are devoted to the investigation of various singular initial problems. Let us cite at least the pioneering work by Chechyk [**6**], the monograph of Kiguradze [**8**] and the papers by Balla [**1**], Diblík [**3, 4**], Konyukhova [**9**], Nowak [**10**] and O'Regan [**11**].

Problem (1), (2) under the conditions formulated has not been considered yet. The main progress in the investigation of this problem consists of employing the function $y = \varphi$, defined by (3), (4), in building an $(n+1)$-dimensional *funnel*, i.e., the domain Ω^0 in the proof of the main result, which contains the graph of the solution of considered problem (1), (2). An additional advantage of our approach lies in the fact that the conditions of the main result can be verified without the concrete construction of the implicitly given function φ itself. Only its existence and some of its properties (which are a consequence of the theory of implicit functions) are necessary. Except the implicitly defined functions, known qualitative properties for the solutions of differential equations and the topological method of Ważewski are also used in the proof of the main result.

The main result (Theorem 2 in Section 3 below) indicates sufficient conditions for the existence of a family of positive solutions of the problem (1), (2). This result is then generalized to a larger class of systems (Theorem 3 in Section 4). Except the system (1) the corresponding linear system is considered in the paper under weaker conditions (Theorem 4, Section 5). Illustrative examples are considered in Section 6.

2. Preliminaries.

2.1 Implicit functions. In this section we give a summary of properties of implicitly defined functions that are used in the proof of the main result. Consider an implicit equation

$$\tilde{\alpha}(y) = \tilde{\omega}(x). \tag{5}$$

Lemma 1. *Let the following assumptions be valid*:

1. $\tilde{\alpha} \in C^1(I_{y_0}, \mathbf{R})$, $\tilde{\alpha} > 0$ *on* I_{y_0}, $\tilde{\alpha}' > 0$ *on* I_{y_0} *and* $\tilde{\alpha}(0^+) = 0$;
2. $\tilde{\omega} \in C^1(I_{x_0}, \mathbf{R})$, $\tilde{\omega} > 0$ *on* I_{x_0}, $\tilde{\omega}' > 0$ *on* I_{x_0} *and* $\tilde{\omega}(0^+) = 0$.

Then there exists the unique solution

$$y = \tilde{\varphi}(x) \equiv \tilde{\alpha}^{-1}[\tilde{\omega}(x)]$$

of equation (5), *defined on an interval* $I_{\delta_0} \subset I_{x_0}$, *with properties* $\tilde{\varphi} \in C^1(I_{\delta_0}, \mathbf{R}), \tilde{\varphi}(0^+) = 0$, $\tilde{\varphi} \in I_{y_0}$ *and* $\tilde{\varphi}' > 0$ *on* I_{δ_0}.

Proof. The proof is elementary and is therefore omitted. □

Remark 1. From Lemma 1 the next property immediately follows which will be used in the sequel: let $\varepsilon_1, \varepsilon_2$ be two positive constants and $\varepsilon_1 < \varepsilon_2$. Then there exists an interval $I_{\delta_1} \subset I_{\delta_0}$ such that the inequality $\tilde{\varphi}(\varepsilon_1 x) < \tilde{\varphi}(\varepsilon_2 x)$ holds on I_{δ_1}.

Lemma 2. *Let all assumptions of Lemma* 1 *be valid and, moreover, let there exist a constant* $M \in \mathbf{R}^+$ *such that*

$$\tilde{\alpha}(y) \le M\tilde{\alpha}'(y), \quad y \in I_{y_0}.$$

Then the unique solution $y = \tilde{\varphi}(x)$ *of equation* (5), *defined on an interval* $I_{\delta_0} \subset I_{x_0}$, *satisfies the inequality*

$$\tilde{\varphi}'(x) \le M \cdot \frac{\tilde{\omega}'(x)}{\tilde{\omega}(x)}, \quad x \in I_{\delta_0}.$$

Proof. In view of (5) and affirmation of Lemma 1,

$$\tilde{\varphi}'(x) = \frac{\tilde{\omega}'(x)}{\tilde{\alpha}'[\tilde{\varphi}(x)]} = \frac{\tilde{\omega}'(x)}{\tilde{\omega}(x)} \cdot \frac{\tilde{\alpha}[\tilde{\varphi}(x)]}{\tilde{\alpha}'[\tilde{\varphi}(x)]} \le M \cdot \frac{\tilde{\omega}'(x)}{\tilde{\omega}(x)}, \quad x \in I_{\delta_0}. \qquad \square$$

Lemma 3. *Let the assumptions* (C_2)–(C_6) *be valid. Then the system of implicit equations*

$$\alpha(z) = \Omega(x) \tag{6}$$

defines the implicit vector function

$$z = \varphi(x) \equiv \alpha^{-1}[\Omega(x)] \tag{7}$$

uniquely on an interval I_{δ_2} *satisfying there the properties:*

$$\varphi \in C^1(I_{\delta_2}, \mathbf{R}^n),\ \varphi(0^+) = 0,\ \varphi_i \in I_{y_0},\ i = 1, \dots, n,$$

and

$$0 < \varphi_i'(x) \le M \cdot \frac{\Omega_i'(x)}{\Omega_i(x)}. \tag{8}$$

Proof. The proof follows from Lemmas 1 and 2 if we put $\tilde{\alpha} \equiv \alpha_i$ and $\tilde{\omega} \equiv \Omega_i$, $i = 1, \dots, n$, in their formulations. $\square$

2.2 Topological principle. In the proof of the main result the topological method of Ważewski [**5, 16**] is used. Therefore we give a short summary of it. Let us consider the system of differential equations

$$y' = f(x, y), \tag{9}$$

with $y \in \mathbf{R}^n$. Below it will be assumed that the righthand side of the system (9) is a continuous function defined on the open (x, y)-set Ω.

Definition 2 [**5**, p. 281]. An open subset Ω^0 of the set Ω is called a (u, v)-subset of Ω with respect to the system (9) if the following conditions are satisfied:

(1) There exist functions $v_i(x,y) \in C^1(\Omega)$, $i = 1, \dots, l$ and $u_j(x,y) \in C^1(\Omega)$, $j = 1, \dots, m$, such that

$$\Omega^0 = \{(x,y) : v_i(x,y) < 0, u_j(x,y) < 0 \text{ for all } i, j\}.$$

(2) $\dot{v}_\alpha(x,y) < 0$ holds for the derivatives of the functions $v_\alpha(x,y)$, $\alpha = 1, \dots, l$, along the trajectories of (9) on the set

$$V_\alpha = \{(x,y) : v_\alpha(x,y) = 0, v_i(x,y) \le 0, u_j(x,y) \le 0 \text{ for all } i, j \text{ and } \alpha, i \ne \alpha\}.$$

(3) $\dot{u}_\beta(x,y) > 0$ holds for the derivatives of the functions $u_\beta(x,y)$, $\beta = 1, \dots, m$, along the trajectories of (9) on the set

$$U_\beta = \{(x,y) : u_\beta(x,y) = 0, v_i(x,y) \le 0, u_j(x,y) \le 0 \text{ for all } i, j \text{ and } \beta, j \ne \beta\}.$$

The number l or the number m in this definition can be zero.

Definition 3. The point $(x_0, y_0) \in \Omega \cap \delta\Omega^0$ is called an *egress point* (or *ingress point*) of Ω^0 with respect to the system (9) if, for every solution of the problem $y(x_0) = y_0$, there exists an $\varepsilon > 0$ such that $(x, y(x)) \in \Omega^0$ for $x_0 - \varepsilon \le x < x_0 (x_0 < x \le x_0 + \varepsilon)$. An egress point (ingress point) (x_0, y_0) of Ω^0 is called a *strict egress point* (*strict ingress point*) of Ω^0 if $(x, y(x)) \notin \overline{\Omega}^0$ on the interval $x_0 < x \le x_0 + \varepsilon_1$ $(x_0 - \varepsilon_1 \le x < x_0)$ for a small $\varepsilon_1 > 0$. The set of all points of egress (strict egress) is denoted by Ω^0_e (Ω^0_{se}).

Lemma 4 [**5**, p. 281]. *Let Ω^0 be a (u,v)-subset of Ω with respect to the system* (9). *Then*

$$\Omega^0_{se} = \Omega^0_e = \left(\bigcup_{\beta=1}^{m} U_\beta \right) \Big\backslash \left(\bigcup_{\alpha=1}^{l} V_\alpha \right).$$

The following theorem formulates sufficient conditions for the existence of at least one solution, having its graph in Ω^0, see [**5**, p. 282].

Theorem 1. *Let* Ω^0 *be some* (u, v)*-subset of* Ω *with respect to the system* (9). *Let* S *be a nonempty compact subset of* $\Omega^0 \cup \Omega^0_e$ *such that the set* $S \cap \Omega^0_e$ *is not a retract of* S *but is a retract of* Ω^0_e. *Then there is at least one point* $(x_0, y_0) \in S \cap \Omega^0$ *such that the graph of a solution* $y(x)$ *of the Cauchy problem* $y(x_0) = y_0$ *lies in* Ω^0 *on its righthand maximal interval of existence.*

3. Main result.

Theorem 2 (Main result). *Suppose that the conditions* (C_1)–(C_6) *are satisfied. Let, moreover, there exist two constants* $\delta \in (0, 1)$ *and* $K > 1$ *such that on an interval* $I_{x^{**}} \subset I_{x_0}$:

A) *for* $i = 1, \dots, p \leq n$:

$$a_{ij}(x) \geq 0 \tag{10}$$

with $j \neq i$, $j = 1, \dots, n$,

$$\frac{a_{ii}(x)}{a_{ii}(\delta x)}\, a_{ij}(\delta x) \leq a_{ij}(x) \leq \frac{a_{ii}(x)}{a_{ii}(Kx)}\, a_{ij}(Kx) \tag{11}$$

with $j \neq i$, $j = 1, \dots, n$,

$$\frac{a_{ii}(x)}{a_{ii}(\delta x)}\, \omega_i(\delta x) > \omega_i(x) + \delta M g_i(x)\, \frac{\Omega_i'(\delta x)}{\Omega_i(\delta x)}, \tag{12}$$

and

$$\frac{a_{ii}(x)}{a_{ii}(Kx)}\, \omega_i(Kx) - \omega_i(x) \leq 0. \tag{13}$$

B) *For* $i = p + 1, \dots, n$:

$$a_{ij}(x) \leq 0 \tag{14}$$

with $j \neq i$, $j = 1, \dots, n$,

$$\frac{a_{ii}(x)}{a_{ii}(Kx)}\, a_{ij}(Kx) \leq a_{ij}(x) \leq \frac{a_{ij}(x)}{a_{ii}(\delta x)}\, a_{ij}(\delta x) \tag{15}$$

with $j \neq i$, $j = 1, \dots, n$,

$$\frac{a_{ii}(x)}{a_{ii}(Kx)}\,\omega_i(Kx) > \omega_i(x) + KMg_i(x)\,\frac{\Omega_i'(Kx)}{\Omega_i(Kx)} \tag{16}$$

and

$$\frac{a_{ii}(x)}{a_{ii}(\delta x)}\,\omega_i(\delta x) - \omega_i(x) \le 0. \tag{17}$$

Then there exists an $(n-p)$*-parametric family of solutions of the problem* (1), (2) *having positive coordinates on an interval* $I_{x^*} \subset I_{x^{**}}$.

Remark 2. It is easy to see that, in the special case of constant matrix, i.e., in the case $A(x) \equiv A =$ const, the conditions of Theorem 2 are significantly reduced. Namely, the conditions (11) and (15) are omitted and the rest of the conditions are simplified.

Remark 3. In the proof of Theorem 2 the topological principle mentioned in Part 2.2 is used. Successful application of it needs the construction of an appropriate (u, v)-subset. This construction is very technical since it is necessary to verify that every part of the boundary of such a (u, v)-subset is transversal to integral curves of the system (1). This leads, except others, to the conclusion that $\Omega^0_{se} = \Omega^0_e$ for the below-defined set Ω^0. Note in this connection that the notion of a (u, v)-subset in the (x, y)-space is similar to the notion of an isolating block (which is often used, e.g., for computation of the Conley index) and that in this way there is an analogy with constructions of (u, v)-subsets and isolating blocks by Liapunov functions. Let us refer, e.g., to the works [**2, 7, 12–15, 17**] (and to the references therein).

Proof of Theorem 2.

3.1 The case $p \in \{1, \dots, n-1\}$**.** Suppose at first $p \in \{1, \dots, n-1\}$. Let φ be the implicit function defined on the interval I_{δ_2} by means of relation (7). Define a domain Ω^0 of the form

$$\Omega^0 = \{(x, y) \in \mathbf{R} \times \mathbf{R}^n : x \in (0, \delta_3), \varphi(\delta x) \ll y \ll \varphi(Kx)\},$$

supposing, without loss of generality, that $\delta_3 \le \min\{\delta_2, x^{**}\}$ is sufficiently small. (Obviously, in accordance with Remark 1, $\varphi(\delta x) \ll \varphi(x) \ll \varphi(Kx)$, $x \in I_{\delta_3}$.)

3.1.1 *Construction of a* (u,v)*-subset of* $\mathbf{R} \times \mathbf{R}^n$*.* Let us construct a (u,v)-subset of $\mathbf{R} \times \mathbf{R}^n$. Define auxiliary functions

$$\begin{gathered} v_j(x,y) \equiv v_j(x,y_j) \equiv (y_j - \varphi_j(\delta x))(y_j - \varphi_j(Kx)), \\ j = 1, \dots, p, \\ u_k(x,y) \equiv u_k(x,y_k) \equiv (y_k - \varphi_k(\delta x))(y_k - \varphi_k(Kx)), \\ k = p+1, \dots, n, \end{gathered}$$

and

$$u_{n+1}(x,y) \equiv u_{n+1}(x) \equiv x - \delta_3.$$

Then

$$\begin{aligned} \Omega^0 = \{(x,y) \in \mathbf{R} \times \mathbf{R}^n : v_j(x,y) < 0, \quad u_k(x,y) < 0, \\ j = 1, \dots, p; \quad k = p+1, \dots, n+1\}. \end{aligned}$$

In the next we will show that all points of the sets

$$\begin{aligned} V_\beta = \{(x,y) \in \mathbf{R} \times \mathbf{R}^n : v_\beta(x,y) = 0,\ v_j(x,y) \le 0,\ u_k(x,y) \le 0, \\ j \ne \beta;\ j = 1, \dots, p;\ k = p+1, \dots, n+1\},\ \beta = 1, \dots, p, \end{aligned}$$

are the points of strict ingress of the set Ω^0 with respect to the system (1) and that all points of the sets

$$\begin{aligned} U_\gamma = \{(x,y) \in \mathbf{R} \times \mathbf{R}^n : u_\gamma(x,y) = 0,\ u_k(x,y) \le 0,\ v_j(x,y) \le 0, \\ k \ne \gamma;\ k = p+1, \dots, n+1;\ j = 1, \dots, p\}, \\ \gamma = p+1, \dots, n+1, \end{aligned}$$

are the points of strict egress of the set Ω^0 with respect to the system (1). (For the corresponding definitions of notions *ingress* and *egress* point, etc., see Section 2, Definition 3. For more details, the reader is referred to the book [**5**]. This technique is also explained in detail, e.g., in the papers [**3, 4**].)

For verifying this, compute the full derivatives of the functions $v_\beta(x,y)$, $\beta = 1,\dots,p$, along the trajectories of the system (1) on corresponding sets V_β at first. We get

$$\begin{aligned}\frac{dv_\beta(x,y)}{dx} &= (y'_\beta - \delta\varphi'_\beta(\delta x))(y_\beta - \varphi_\beta(Kx))\\ &\quad + (y_\beta - \varphi_\beta(\delta x))(y'_\beta - K\varphi'_\beta(Kx))\\ &= \left[\frac{\sum_{j=1}^n a_{\beta j}(x)\alpha_j(y_j) - \omega_\beta(x)}{g_\beta(x)} - \delta\varphi'_\beta(\delta x)\right](y_\beta - \varphi_\beta(Kx))\\ &\quad + (y_\beta - \varphi_\beta(\delta x))\left[\frac{\sum_{j=1}^n a_{\beta j}(x)\alpha_j(y_j) - \omega_\beta(x)}{g_\beta(x)} - K\varphi'_\beta(Kx)\right].\end{aligned}$$

If $(x,y) \in V_\beta$ for a fixed β, then either $y_\beta = \varphi_\beta(\delta x)$ and $\varphi_j(\delta x) \le y_j \le \varphi_j(Kx)$, $j = 1,\dots,n$, $j \ne \beta$ or $y_\beta = \varphi_\beta(Kx)$ and $\varphi_j(\delta x) \le y_j \le \varphi_j(Kx)$, $j = 1,\dots,n$, $j \ne \beta$.

In the first case, i.e., if

$$(x,y) \in V_\beta, y_\beta = \varphi_\beta(\delta x), \varphi_j(\delta x) \le y_j \le \varphi_j(Kx), \quad j = 1,\dots,n \text{ and } j \ne \beta, \tag{18}$$

we have

$$\begin{aligned}&\left.\frac{dv_\beta(x,y)}{dx}\right|_{(x,y)\in V_\beta, y_\beta = \varphi_\beta(\delta x)}\\ &= \left[\frac{\alpha_{\beta\beta}(x)\alpha_\beta(\varphi_\beta(\delta x)) + \sum_{j=1, j\ne\beta}^n a_{\beta j}(x)\alpha_j(y_j) - \omega_\beta(x)}{g_\beta(x)} - \delta\varphi'_\beta(\delta x)\right]\\ &\quad \times (\varphi_\beta(\delta x) - \varphi_\beta(Kx)) \qquad \text{in view of (3), (4) and (C}_4\text{)}\\ &= \left[\frac{\frac{a_{\beta\beta}(x)}{\alpha_{\beta\beta}(\delta x)}\left(\omega_\beta(\delta x) - \sum_{j=1, j\ne\beta}^n a_{\beta j}(\delta x)\alpha_j(\varphi_j(\delta x))\right)}{g_\beta(x)}\right.\\ &\quad \left. + \frac{\sum_{j=1, j\ne\beta}^n a_{\beta j}(x)\alpha_j(y_j) - \omega_\beta(x)}{g_\beta(x)} - \delta\varphi'_\beta(\delta x)\right](\varphi_\beta(\delta x) - \varphi_\beta(Kx))\\ &\qquad \text{in view of (10), (C}_2\text{) and (18)}\end{aligned}$$

$$\leq \left[\frac{\frac{a_{\beta\beta}(x)}{a_{\beta\beta}(\delta x)}\omega_\beta(\delta x) - \omega_\beta(x) + \sum_{j=1, j\neq\beta}^{n}[a_{\beta j}(x) - \frac{a_{\beta\beta}(x)}{a_{\beta\beta}(\delta x)}a_{\beta j}(\delta x)]\alpha_j(\varphi_j(\delta x))}{g_\beta(x)} \right.$$
$$\left. - \delta\varphi_\beta'(\delta x)\right](\varphi_\beta(\delta x) - \varphi_\beta(Kx)) \qquad \text{in view of (8), (C}_2\text{) and (11)}$$
$$\leq \left(\frac{\frac{a_{\beta\beta}(x)}{a_{\beta\beta}(\delta x)}\omega_\beta(\delta x) - \omega_\beta(x)}{g_\beta(x)} - \delta M \cdot \frac{\Omega_\beta'(\delta x)}{\Omega_\beta(\delta x)} \right)(\varphi_\beta(\delta x) - \varphi_\beta(Kx))$$
$$\text{in view of (12)}$$
$$< 0.$$

Thus, all the points $(x, y) \in V_\beta$ if $y_\beta = \varphi_\beta(\delta x)$ are points of strict ingress.

In the second case, i.e., if

$$(x, y) \in V_\beta, y_\beta = \varphi_\beta(Kx), \varphi_j(\delta x) \leq y_j \leq \varphi_j(Kx), \quad j = 1, \ldots, n \text{ and } j \neq \beta, \tag{19}$$

we get

$$\left.\frac{dv_\beta(x,y)}{dx}\right|_{(x,y)\in V_\beta, y_\beta = \varphi_\beta(Kx)}$$
$$= (\varphi_\beta(Kx) - \varphi_\beta(\delta x))$$
$$\times \left[\frac{a_{\beta\beta}(x)\alpha_\beta(\varphi_\beta(Kx)) + \sum_{j=1, j\neq\beta}^{n} a_{\beta j}(x)\alpha_j(y_j) - \omega_\beta(x)}{g_\beta(x)} - K\varphi_\beta'(Kx) \right]$$
$$\text{in view of (3), (4) and (C}_4\text{)}$$
$$= \left[\frac{\frac{a_{\beta\beta}(x)}{a_{\beta\beta}(Kx)}(\omega_\beta(Kx) - \sum_{j=1, j\neq\beta}^{n} a_{\beta j}(Kx)\alpha_j(\varphi_j(Kx)))}{g_\beta(x)} \right.$$
$$\left. + \frac{\sum_{j=1, j\neq\beta}^{n} a_{\beta j}(x)\alpha_j(y_j) - \omega_\beta(x)}{g_\beta(x)} - K\varphi_\beta'(Kx) \right](\varphi_\beta(Kx) - \varphi_\beta(\delta x))$$
$$\text{in view of (10), (C}_2\text{) and (19)}$$

$$\le (\varphi_\beta(Kx)-\varphi_\beta(\delta x))\Bigg[\frac{\frac{a_{\beta\beta}(x)}{a_{\beta\beta}(Kx)}\omega_\beta(Kx)-\omega_\beta(x)}{g_\beta(x)}$$
$$+\frac{\sum_{j=1,j\neq\beta}^{n}[a_{\beta j}(x)-\frac{a_{\beta\beta}(x)}{a_{\beta\beta}(Kx)}a_{\beta j}(Kx)]\alpha_j(\varphi_j(Kx))}{g_\beta(x)}-K\varphi'_\beta(Kx)\Bigg]$$

in view of (8), (11) and (13)

$$< 0.$$

This means that all the points $(x,y)\in V_\beta$ if $y_\beta=\varphi_\beta(Kx)$ are also points of strict ingress and, in both of the cases considered,

$$\left.\frac{dv_\beta(x,y)}{dx}\right|_{(x,y)\in V_\beta}<0,\quad \beta=1,\dots,p. \tag{20}$$

Now let us compute the full derivative of the functions $u_\gamma(x,y)$, $\gamma=p+1,\dots,n$, along the trajectories of the system (1) on corresponding sets U_γ. As above,

$$\begin{aligned}\frac{du_\gamma(x,y)}{dx}&=(y'_\gamma-\delta\varphi'_\gamma(\delta x))(y_\gamma-\varphi_\gamma(Kx))\\&\quad+(y_\gamma-\varphi_\gamma(\delta x))(y'_\gamma-K\varphi'_\gamma(Kx))\\&=\Bigg[\frac{\sum_{j=1}^{n}a_{\gamma j}(x)\alpha_j(y_j)-\omega_\gamma(x)}{g_\gamma(x)}-\delta\varphi'_\gamma(\delta x)\Bigg](y_\gamma-\varphi_\gamma(Kx))\\&\quad+(y_\gamma-\varphi_\gamma(\delta x))\Bigg[\frac{\sum_{j=1}^{n}a_{\gamma j}(x)\alpha_j(y_j)-\omega_\gamma(x)}{g_\gamma(x)}-K\varphi'_\gamma(Kx)\Bigg].\end{aligned}$$

If $(x,y)\in U_\gamma$ for a fixed γ, then either $y_\gamma=\varphi_\gamma(\delta x)$ and $\varphi_j(\delta x)\le y_j\le\varphi_j(Kx)$, $j=1,\dots,n$, $j\neq\gamma$ or $y_\gamma=\varphi_\gamma(Kx)$ and $\varphi_j(\delta x)\le y_j\le\varphi_j(Kx)$, $j=1,\dots,n$, $j\neq\gamma$.

In the first case, i.e., if

$$(x,y)\in U_\gamma, y_\gamma=\varphi_\gamma(\delta x),\varphi_j(\delta x)\le y_j\le\varphi_j(Kx),\quad j=1,\dots,n \text{ and } j\neq\gamma, \tag{21}$$

we have

$$\begin{aligned}
&\frac{du_\gamma(x,y)}{dx}\bigg|_{(x,y)\in U_\gamma, y_\gamma=\varphi_\gamma(\delta x)} \\
&= \left[\frac{a_{\gamma\gamma}(x)\alpha_\gamma(\varphi_\gamma(\delta x)) + \sum_{j=1,j\neq\gamma}^n a_{\gamma j}(x)\alpha_j(y_j) - \omega_\gamma(x)}{g_\gamma(x)} - \delta\varphi_\gamma'(\delta x)\right] \\
&\quad \times (\varphi_\gamma(\delta x) - \varphi_\gamma(Kx)) \qquad \text{in view of (3), (4) and (C}_4\text{)} \\
&= \left[\frac{\frac{a_{\gamma\gamma}(x)}{a_{\gamma\gamma}(\delta x)}\Big(\omega_\gamma(\delta x) - \sum_{j=1,j\neq\gamma}^n a_{\gamma j}(\delta x)\alpha_j(\varphi_j(\delta x))\Big)}{g_\gamma(x)}\right. \\
&\quad \left. + \frac{\sum_{j=1,j\neq\gamma}^n a_{\gamma j}(x)\alpha_j(y_j) - \omega_\gamma(x)}{g_\gamma(x)} - \delta\varphi_\gamma'(\delta x)\right](\varphi_\gamma(\delta x) - \varphi_\gamma(Kx)) \\
&\qquad \text{in view of (14), (C}_2\text{) and (21)} \\
&\geq \left[\frac{\frac{a_{\gamma\gamma}(x)}{a_{\gamma\gamma}(\delta x)}\omega_\gamma(\delta x) - \omega_\gamma(x) + \sum_{j=1,j\neq\gamma}^n [a_{\gamma j}(x) - \frac{a_{\gamma\gamma}(x)}{a_{\gamma\gamma}(\delta x)}a_{\gamma j}(\delta x)]\alpha_j(\varphi_j(\delta x))}{g_\gamma(x)}\right] \\
&\quad \times (\varphi_\gamma(\delta x) - \varphi_\gamma(Kx)) - \delta\varphi_\gamma'(\delta x)(\varphi_\gamma(\delta x) - \varphi_\gamma(Kx)) \\
&\qquad \text{in view of (8), (15) and (17)} \\
&> 0.
\end{aligned}$$

Thus, all the points $(x,y) \in U_\gamma$ if $y_\gamma = \varphi_\gamma(\delta x)$ are points of strict egress.

In the second case, i.e., if

$$(x,y) \in U_\gamma, y_\gamma = \varphi_\gamma(Kx), \varphi_j(\delta x) \leq y_j \leq \varphi_j(Kx), \quad j = 1, \dots, n \text{ and } j \neq \gamma, \tag{22}$$

we get

$$\frac{du_\gamma(x,y)}{dx}\bigg|_{(x,y)\in U_\gamma, y_\gamma=\varphi_\gamma(Kx)}$$

$$= \big(\varphi_\gamma(Kx)-\varphi_\gamma(\delta x)\big)$$
$$\times\left[\frac{a_{\gamma\gamma}(x)\alpha_\gamma(\varphi_\gamma(Kx))+\sum_{j=1,j\neq\gamma}^n a_{\gamma j}(x)\alpha_j(y_j)-\omega_\gamma(x)}{g_\gamma(x)}-K\varphi'_\gamma(Kx)\right]$$

in view of (3), (4) and (C_4)

$$=\big(\varphi_\gamma(Kx)-\varphi_\gamma(\delta x)\big)\left[\frac{\frac{a_{\gamma\gamma}(x)}{a_{\gamma\gamma}(Kx)}\big(\omega_\gamma(Kx)-\sum_{j=1,j\neq\gamma}^n a_{\gamma j}(Kx)\alpha_j(\varphi_j(Kx))\big)}{g_\gamma(x)}\right.$$
$$\left.+\frac{\sum_{j=1,j\neq\gamma}^n a_{\gamma j}(x)\alpha_j(y_j)-\omega_\gamma(x)}{g_\gamma(x)}-K\varphi'_\gamma(Kx)\right]$$

in view of (14), (C_2) and (22)

$$\geq\big(\varphi_\gamma(Kx)-\varphi_\gamma(\delta x)\big)\left[\frac{\frac{a_{\gamma\gamma}(x)}{a_{\gamma\gamma}(Kx)}\omega_\gamma(Kx)-\omega_\gamma(x)}{g_\gamma(x)}\right.$$
$$\left.+\frac{\sum_{j=1,j\neq\gamma}^n[a_{\gamma j}(x)-\frac{a_{\gamma\gamma}(x)}{\alpha_{\gamma\gamma}(Kx)}a_{\gamma j}(Kx)]\alpha_j(\varphi_j(Kx))}{g_\gamma(x)}-K\varphi'_\gamma(Kx)\right]$$

in view of (8), (C_2) and (15)

$$\geq\big(\varphi_\gamma(Kx)-\varphi_\gamma(\delta x)\big)\left[\frac{\frac{a_{\gamma\gamma}(x)}{a_{\gamma\gamma}(Kx)}\omega_\gamma(Kx)-\omega_\gamma(x)}{g_\gamma(x)}-KM\frac{\Omega'_\gamma(Kx)}{\Omega_\gamma(Kx)}\right]$$

in view of (16)

$$>0.$$

This means that the points $(x,y)\in U_\gamma$ if $y_\gamma=\varphi_\gamma(Kx)$ are also points of strict egress and, in both of the cases considered,

$$\frac{du_\gamma(x,y)}{dx}\bigg|_{(x,y)\in U_\gamma}>0,\quad \gamma=p+1,\dots,n. \tag{23}$$

Finally, the relation

$$\frac{du_{n+1}}{dx}>0, \tag{24}$$

obviously holds. So the set Ω^0 is the (u,v)-set.

3.1.2 *Application of Theorem* 1. The goal of this part of the proof is to show that, in the closure of the (u,v)-subset Ω^0 it is possible to find a compact and connected set S such that its boundary ∂S is a retract of all strict egress points Ω^0_{se} of Ω^0, but ∂S is not a retract of the set S itself (let us refer to Remark 3 and to Part 2.2). This no-retraction statement will be proved explicitly, with the aid of an appropriate analytical construction.

The above inequalities (20), (23) and (24) simultaneously say that, if orientation of the x-axis is changed into reverse orientation, points $(x,y) \in V_\beta$, $\beta = 1,\dots,p$ will be the points of strict egress of the set Ω^0 with respect to the system (1) and points $(x,y) \in U_\gamma$, $\gamma = p+1,\dots,n+1$, will be the points of strict ingress of the set Ω^0 with respect to the system (1). For all points of egress Ω^0_e and all points of strict egress Ω^0_{se} of the set Ω^0 with respect to the system (1), the relation

$$\Omega^0_e = \Omega^0_{se} = \Big(\bigcup_{\beta=1}^{p} V_\beta\Big) \Big\backslash \Big(\bigcup_{\gamma=p+1}^{n+1} U_\gamma\Big)$$

holds, see Lemma 4.

Let us recall that δ_3 is a fixed positive number. For every fixed $y^0_{p+1},\dots,y^0_n$ satisfying the inequalities $u_k(\delta_3,y^0_k) < 0$, $k = p+1,\dots,n$, we define a compact set

$$S = S_{y^0_{p+1},\dots,y^0_n} \subset \Omega^0 \cup \Omega^0_{se}$$

as

$$S = \{(\delta_3, y_1,\dots,y_p, y^0_{p+1},\dots,y^0_n) : v_j(\delta_3,y_j) \le 0, j = 1,\dots,p\}.$$

Its boundary can be written as $\partial S = \cup_{\beta=1}^p \mathcal{V}_\beta$ with

$$\begin{aligned}\mathcal{V}_\beta\{(\delta_3, y_1,\dots,y_p, y^0_{p+1},\dots,y^0_n) : v_\beta(\delta_3,y_\beta) = 0,& \\ v_j(\delta_3,y_j) \le 0, \quad j \ne \beta,\ j = 1,\dots,p\}.&\end{aligned}$$

It is easy to verify that $S \cap \Omega^0_{se} = \cup_{\beta=1}^p \mathcal{V}_\beta$, i.e., $S \cap \Omega^0_{se} = \partial S$. Since δ_3 and $y^0_{p+1},\dots,y^0_n$ are fixed, we can in the space of variables $(y_1,\dots,y_p)$

rewrite the set S in a simpler way. It has the form of the product of p closed intervals

$$S = \{(y_1, \dots, y_p) : a_j \le y_j \le A_j,\ j = 1, \dots, p\}$$

with constants $a_j = \varphi_j(\delta\delta_3)$, $A_j = \varphi_j(K\delta_3)$, $a_j < A_j$, $j = 1, \dots, p$. It becomes clear that S is homeomorphic to a p-dimensional closed ball and that ∂S is not a retract of S (since the boundary of a p-dimensional ball is not its retract). Let us show that $S \cap \Omega^0_{se}$ is a retract of Ω^0_{se}. Define, for $(x, y) \in \Omega^0_{se}$, a map

$$\pi : (x, y) \longrightarrow (\delta_3, \tilde{y}_1, \dots, \tilde{y}_p, y^0_{p+1}, \dots, y^0_n) \in S \cap \Omega^0_{se}$$

with

$$\tilde{y}_i = \varphi_i(\delta\delta_3) + (y_i - \varphi_i(\delta x)) \cdot \frac{\varphi_i(K\delta_3) - \varphi_i(\delta\delta_3)}{\varphi_i(Kx) - \varphi_i(\delta x)}$$

and $i = 1, \dots, p$. The map π is continuous and maps the set Ω^0_{se} into $S \cap \Omega^0_{se}$. The points of the set $S \cap \Omega^0_{se}$ are stationary points and, consequently, $S \cap \Omega^0_{se}$ is a retract of Ω^0_{se}. Now we are in a position to apply Theorem 1. So there exists a solution $y = y(x)$ of (1) having its initial data in $S \cap \Omega^0$ such that $(x, y(x)) \in \Omega^0$ on interval $(0, \delta_3)$. This solution, due to the form of Ω^0, clearly satisfies the initial condition (2). Since we can vary the constants $y^0_{p+1}, \dots, y^0_n$ within the intervals indicated in the definition of S, we conclude that the set of initial data for solutions having the same property generate an $(n-p)$-dimensional set. Put $x^* \in (0, \min\{x^{**}, \delta_3\})$. The theorem is proved for the case $p \in \{1, \dots, n-1\}$.

3.2 The case $p = n$. For $p = n$ the proof remains the same as above. In this case we have no possibility to vary any constant. So there exists (at least) one solution with the properties indicated.

3.3 The case $p = 0$. If $p = 0$, then $\Omega^0_{se} = \varnothing$, and we can vary all constants. This case needs no application of topological principle since every point $(x, y) \in \partial\Omega^0$ with $x \ne 0$ is the strict ingress point of the set Ω^0 with respect to the system (1) (if the orientation of x-axis is reversed). In this case the set of initial data for solutions, having the properties indicated, is n-dimensional. This completes the proof. $\square$

From the proof of Theorem 2, we get the following:

Corollary 1. *If Theorem* 2 *holds, then there exists an* $(n-p)$*-parametric family of solutions* $y = y^*(x)$ *of the problem* (1), (2), *each of which satisfies on interval* I_{x^*} *the inequalities*

$$\varphi(\delta x) \ll y^*(x) \ll \varphi(Kx).$$

Remark 4. Let us consider the initial problem (25), (2) where

$$g_i(x)y_i' = (-1)^{s_i}\left[\sum_{j=1}^{n} a_{ij}(x)\alpha_j(y_j) - \omega_i(x)\right], \quad i = 1, \dots, n, \tag{25}$$

and $s_i \in \{0, 1\}$. Suppose that conditions (C_1)–(C_6) are satisfied. Let, except this, in the case when $s_i = 0$, conditions (14)–(17) hold, and in the case when $s_i = 1$, conditions (10)–(13) hold (with a_{ij} and ω_i changed to $-a_{ij}$ and $-\omega_i$, $j = 1, \dots, n$, consequently). It is easy to see that, after an appropriate renumbering of dependent variables, Theorem 2 (with a suitable p) can be applied to the problem (25), (2) and therefore this problem is equivalent with the problem (1), (2).

4. Generalization. Let us point out that the condition (C_5) permits us to get the inequalities (8) for coordinates of implicitly given, by relation (6) (or by (3), (4)), function φ. Condition (C_5) is easily verifiable. Nevertheless, the change of this condition by a general one:

(C_5^*) There exist functions $f_i(y) \in C^1(I_{y_{00}}, \mathbf{R})$ with

$$y_{00} = \max_{j=1,\dots,n}\{\alpha_j(y_0)\}$$

and $f_i > 0$ on $I_{y_{00}}$, $i = 1, \dots, n$, such that inequalities

$$\alpha_i(y) \le f_i(\alpha_i(y))\alpha_i'(y)$$

hold on interval I_{y_0},

leads to an improvement of inequalities (8) by the inequalities

$$0 < \varphi_i'(x) \le f_i(\Omega_i(x))\frac{\Omega_i'(x)}{\Omega_i(x)} \tag{26}$$

with $i = 1, \dots, n$ on interval I_{δ_2} (see the proof of Lemma 2) and, consequently, leads to an improvement of affirmation of the main result. Namely, the following holds.

Theorem 3. *Suppose that conditions of Theorem* 2 *are satisfied if*:

a) *Condition* (C_5) *is exchanged for condition* (C_5^*);

b) *Inequality* (12) *is exchanged for inequality*

$$\frac{a_{ii}(x)}{a_{ii}(\delta x)}\,\omega_i(\delta x) > \omega_i(x) + \delta g_i(x) f_i(\Omega_i(\delta x))\,\frac{\Omega_i'(\delta x)}{\Omega_i(\delta x)} \tag{27}$$

with a constant $\delta \in (0,1)$ *on interval* $I_{x^{**}}$;

c) *Inequality* (16) *is exchanged for the inequality*

$$\frac{a_{ii}(x)}{a_{ii}(Kx)}\,\omega_i(Kx) > \omega_i(x) + K g_i(x) f_i(\Omega_i(Kx))\,\frac{\Omega_i'(Kx)}{\Omega_i(Kx)} \tag{28}$$

with a constant $K > 1$ *on interval* $I_{x^{**}}$. *Then the conclusion of Theorem* 2 *remains valid.*

The proof is omitted since it is a variant of the proof of Theorem 2.

5. The linear case. The corresponding result for the linear case can be obtained as a straightforward consequence of Theorem 3 and Corollary 1 if

$$f_i(\alpha_i(y_i)) \equiv \alpha_i(y_i) \equiv y_i, \quad i = 1, \dots, n.$$

Then, as it follows from (3), (4), $\varphi(x) \equiv \Omega(x)$; (26) obviously holds and the inequalities (27), (28) turn into the inequalities (30), (31) indicated below. Let us put $\alpha(y) \equiv y$ in (1) and consider the linear system

$$g(x)y' = A(x)y - \omega(x). \tag{29}$$

Theorem 4. *Suppose that conditions* (C_1), (C_3), (C_4), (C_6), (10), (11), (13)–(15) *and* (17) *are satisfied. Let, moreover, for* $x \in I_{x^{**}}$,

$$\frac{a_{ii}(x)}{a_{ii}(\delta x)}\,\omega_i(\delta x) > \omega_i(x) + \delta g_i(x)\Omega_i'(\delta x) \tag{30}$$

with $i = 1, \ldots, p \leq n$ *and a constant* $\delta \in (0,1)$, *and*

$$\frac{a_{ii}(x)}{a_{ii}(Kx)}\omega_i(Kx) > \omega_i(x) + Kg_i(x)\Omega_i'(Kx) \tag{31}$$

with $i = p+1, \ldots, n$, *and a constant* $K > 1$. *Then there exists an* $(n-p)$*-parametric family of solutions of* $y = y^*(x)$ *of the problem* (29), (2), *having positive coordinates on an interval* I_{x^*}, *each of which satisfies the inequalities*

$$\Omega(\delta x) \ll y^*(x) \ll \Omega(Kx).$$

6. Examples.

Example 1. Let us consider a linear singular problem of the type (29), (2):

$$\begin{cases} x^2y_1' = -5y_1 + y_2 + y_3 + x + x^2, \\ x^2y_2' = y_1 - 5y_2 + y_3 + x + x^2, \\ x^2y_3' = -2y_1 - 3y_2 + 2y_3 - x + 3x^2, \end{cases} \tag{32}$$

$$y_1(0^+) = y_2(0^+) = y_3(0^+) = 0. \tag{33}$$

Here $g_1(x) = g_2(x) = g_3(x) = x^2$, $a_{11} = a_{22} = -5$, $a_{12} = a_{13} = a_{21} = a_{23} = 1, a_{31} = -2$, $a_{32} = -3$, $a_{33} = 2$, $\alpha_i(y_i) = y_i$, $i = 1,2,3$, $\omega_1(x) = \omega_2(x) = -x - x^2$, $\omega_3(x) = x - 3x^2$, $\Omega(x) = (x - x^2/3, x - x^2/3, 3x - 7x^2/3)^T$, $p = 2$ and $n = 3$. This problem has, by Theorem 4, a one-parametric family of positive solutions. Indeed, the general solution of the system considered is expressed by means of relations

$$\begin{cases} y_1 = x + 11C_1\exp(6/x) + C_2\exp(3/x) + C_3\exp(-1/x), \\ y_2 = x - 10C_1\exp(6/x) + C_2\exp(3/x) + C_3\exp(-1/x), \\ y_3 = 3x - C_1\exp(6/x) + C_2\exp(3/x) + 5C_3\exp(-1/x) \end{cases} \tag{34}$$

with arbitrary constants C_1, C_2 and C_3. We get this one-parametric family by setting $C_1 = C_2 = 0$ in (34).

By the same theorem there exists a one-parametric family of positive solutions of linear singular problem (35), (33) (the general solution of which cannot be expressed analytically in a closed form), where

$$
\begin{cases} x^3 y_1' = -5y_1 + y_2 + y_3 + x + x^2, \\ x^4 y_2' = y_1 - 5y_2 + y_3 + x + x^2, \\ x^5 y_3' = -2y_1 - 3y_2 + 2y_3 - x + 3x^2 \end{cases} \tag{35}
$$

(here, unlike in the previous system (32), $g_1(x) = x^3$, $g_2(x) = x^4$ and $g_3(x) = x^5$ is put). Moreover, by Theorem 2, there exists a one-parametric family of positive solutions of nonlinear problem (36), (33), where

$$
\begin{cases} x^3 y_1' = -5y_1^2 + y_2^5 + y_3^3 + x + x^2, \\ x^4 y_2' = y_1^2 - 5y_2^5 + y_3^3 + x + x^2, \\ x^5 y_3' = -2y_1^2 - 3y_2^5 + 2y_3^3 - x + 3x^2, \end{cases} \tag{36}
$$

(here, unlike in the previous system (35), $\alpha_1(y_1) = y_1^2$, $\alpha_2(y_2) = y_2^5$ and $\alpha_3(y_3) = y_3^3$ is put).

Example 2. The following example shows that conditions of Theorem 2 are in some sense sharp. Let us consider the problem

$$
x^4 y' = \frac{y}{y+\varepsilon} - x, \quad y(0^+) = 0,
$$

where $\varepsilon \in (0, 1]$ is a parameter. Here $n = 1$, $g(x) = x^4$, $\alpha(y) = y/(y+\varepsilon)$, $a_{11} = 1$ and $\omega(x) = \Omega(x) = x$. By Theorem 2, with $p = 0$, this problem has a one-parametric family of positive solutions for every fixed constant ε on a corresponding interval. Nevertheless, for $\varepsilon = 0$, the problem has no solution since the general solution of the equation obtained (define $y/y := 1$):

$$
x^4 y' = 1 - x
$$

has the form $y(x) = -1/(3x^3) + 1/(2x^2) + C$, $C = \text{const}$.

Acknowledgments. The authors would like to thank the referee for his/her helpful suggestions incorporated into this paper. This work was supported by grants 1/8055/01 and 1/0026/03 of the Grant Agency of Slovak Republic (VEGA).

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