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# INTERVAL OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. New oscillation criteria established in this paper for the second order nonlinear equations

$$(r(t)\psi(x(t))x'(t))' + F(t, x(t),\ x'(t),\ x(\tau(t)),\ x'(\tau(t))) = 0$$

are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of $[t_0, \infty)$ rather than on the whole half-line. Our results are more general and sharper than some previous results and handle the cases which are not covered by known results. Several examples that show the generality of our results are also included.

**1. Introduction.** We are concerned here with the oscillatory behavior of solutions of the second order nonlinear differential equation

(1)
$$(r(t)\psi(x(t))x'(t))' + F(t, x(t),\ x'(t),\ x(\tau(t)),\ x'(\tau(t))) = 0, \quad t \geq t_0$$

where $F : [t_0, \infty) \times R^4 \to R$ is a continuous function. In what follows, we always assume without mention that

$(A_1)$ $r : I = [t_0, \infty) \to (0, \infty)$ is continuously differentiable;

$(A_2)$ $\psi : R \to R$ is continuously differentiable and $\psi(x) > 0$ for $x \neq 0$;

$(A_3)$ $\tau : I \to R$ is continuously differentiable with $\tau'(t) > 0$ for all $t \in I$, $\tau(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$;

$(A_4)$ there exist functions $q, f_0, f$ and $g$ such that

$$\begin{aligned} &F(t, x(t), x'(t),\ x(\tau(t)),\ x'(\tau(t)))\operatorname{sgn} x \\ &\qquad\qquad \geq q(t) f_0(x(t)) f(x(\tau(t))) g(x'(t),\ x'(\tau(t)))\operatorname{sgn} x \end{aligned}$$

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where the functions $q, f_0, f$ and $g$ satisfy the following assumptions:

$(B_1)$ $q : I = [t_0, \infty) \to (0, \infty)$ is continuous and $q(t) \not\equiv 0$; that means that there exists a sequence $\{t_k\}$ of real numbers $t_k \to \infty$ as $k \to \infty$ such that $q(t_k) \neq 0$;

$(B_2)$ $f_0, f : R \to R$ are continuous, $xf(x) > 0$ for $x \neq 0$ and $f_0(x) \geq K_0 > 0$ where $K_0$ is a constant;

$(B_3)$ $g : R \times R \to R$ is continuous and $g(x, y) \geq C$ for some constant $C > 0$.

Let $\chi : [\tau(t_0), t_0] \to R$. By a solution of equation (1), we mean a continuously differentiable function $x(t) : [\tau(t_0), \infty) \to R$ such that $x(t) = \chi(t)$ for $\tau(t_0) \leq t \leq t_0$, $r(t)\psi(x(t))x'(t)$ is continuously differentiable on $[t_0, \infty)$ and $x(t)$ satisfies equation (1) for $t \in [t_0, \infty)$.

We restrict our attention to proper solutions of equation (1), i.e., nonconstant solutions which exist on some ray $[T, \infty)$, where $T \geq t_0$, and satisfy $\sup_{t \geq T}\{|x(t)|\} > 0$. A proper solution $x(t)$ of equation (1) is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Finally, equation (1) is called oscillatory if all its proper solutions are oscillatory.

Oscillation for equation (1) and the nonlinear delay equation

$$(r(t)x'(t))' + q(t)f(x(\tau(t)))g(x'(t)) = 0, \quad t \geq t_0, \tag{2}$$

as well as for the nonlinear ordinary differential equation

$$(r(t)x'(t))' + q(t)f(x(t))g(x'(t)) = 0, \quad t \geq t_0, \tag{3}$$

and the linear ordinary differential equation

$$(r(t)x'(t))' + q(t)x(t) = 0, \quad t \geq t_0, \tag{4}$$

have been discussed by numerous authors. Some results can be found in [**1**–**16**] and the references therein. On the one hand, the recent paper by Rogovchenko [**12**] contains various conditions for nonlinear delay equations obtained by use of an integral averaging technique similar to that exploited in [**10**]. On the other hand, the results in [**1, 2, 4, 6, 10**–**12, 14**] and the references therein as well as most known oscillation criteria involve integrals of $r$ and $q$ and hence require the information of $r$ and $q$ on the entire half-line $[t_0, \infty)$. However, from the Sturm separation theorem, we see that oscillation for equation (4) is the only interval property, i.e., if there exists a sequence of subintervals $[a_i, b_i]$ of $[t_0, \infty)$, as $a_i \to \infty$, such that for each $i$ there exists a solution of equation (4) that has at least two zeros in $[a_i, b_i]$, then every solution of equation (4) is oscillatory, no matter how "bad" equation (4) is (or $r$ and $q$ are) on the remaining part of $[t_0, \infty)$.

In 1993, El-Sayed [**3**] established an interval criteria for oscillation of a forced second order equation. But the result is not very sharp because a comparison to equations with constant coefficient is used in the proof.

In 1997, Huang [**5**] presented the following results for interval oscillation of the linear ODE (4) with $r(t) = 1$.

**Theorem 1.1** [**5**]. *If there exist $t^0 > t_0$ such that for every $n \in N$*

$$\int_{2^n t^0}^{2^{n+1} t^0} q(s)\, ds \geq \frac{3 - 2\sqrt{3}}{2^{n+1} t^0},$$

*then every solution of equation* (4) *with $r(t) = 1$ is oscillatory.*

But Huang's oscillation criterion fails to apply to Euler's equation

$$x''(t) + \frac{\beta}{t^2}\, x(t) = 0,$$

see Li and Agarwal [**8**]. Of course, we know that Euler's equation is oscillatory if $\beta > 1/4$ and is nonoscillatory if $\beta \leq 1/4$.

We remark that Kong [**7**] employed the technique from the work of Philos [**10**] to obtain several interval oscillation results for the second order linear ODE (4). However, these results do not apply to the nonlinear ODE (3). In [**8**] Li and Agarwal further studied interval oscillation criteria for nonlinear ODEs. We note that the results of El-Sayed [**3**], Huang [**5**], Kong [**7**], Li and Agarwal [**8, 9**] cannot be applied to the nonlinear delay differential equation (1).

Motivated by the ideas of Rogovchenko [**12**] and Yang, et al. [**15**] in this paper, by using the generalized Riccati technique and an averaging technique and by considering the function $H(t, s)k(s)$ which may not have a nonpositive partial derivative on $D_0$ with respect to the second variable, we relax the usual assumption $(\partial H(t,s)/\partial s) \leq 0$ on $D_0 = \{(t,s) : t > s \geq t_0\}$ in [**10**, etc.], and we extend to delay differential equations an idea of Li and Agarwal [**8**] on interval criteria for oscillation of solutions of second order nonlinear ODEs.

This paper is organized as follows. In Section 2 we obtain some oscillation theorems for equation (1) when the function $f(x)$ is smooth. New interval oscillation criteria of equation (1) are obtained by making use of the technique similar to that exploited by Philos [**10**] and Kong [**7**] for second order linear ordinary differential equations. The interval oscillation criteria established in this section for second order nonlinear delay differential equations (1) are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of $[t_0, \infty)$ rather than on the entire half-line.

Further, in Section 3 we obtain interval oscillation criteria for equation (1) when the function $f(x)$ is not smooth. Our results (see Theorems 2.1–2.12 and Theorems 3.1–3.6) complement a number of existing results and handle the cases that are not covered by known criteria in [**1**–**16**] and others. Further, several examples that show the sharpness of our results are also included.

By choosing appropriate functions $H$, $k$ and $\rho$, we derive a series of explicit oscillation criteria which extend, improve and unify a number of existing results.

**2. Oscillation results for smooth $f(x)$.** In this section we consider oscillation of equation (1) when the function $f(x)$ is smooth. Throughout this section we use the notation

$$D_0 = \{(t,s) : t > s \geq t_0\}; \quad D = \{(t,s) : t \geq s \geq t_0\}.$$

**Theorem 2.1.** *Suppose that for $x \neq 0$*

$(H_0^1)$ *there exist constants $K$ and $L^{-1}$ such that*

$$f'(x) \geq K > 0; \quad 0 < \psi(x) \leq L^{-1}.$$

*Let functions $H \in C(D; R)$, $h_1, h_2 \in C(D_0; R)$, $k, \rho \in C^1([t_0, \infty); (0, \infty))$ satisfy the following conditions*:

($H1$) *$H(t,t) = 0$ for $t \ge t_0$, $H(t,s) > 0$ on $D_0$;*

($H2$) *$\partial(H(t,s)k(t))/\partial t + H(t,s)k(t)\rho'(t)/\rho(t) = h_1(t,s)$, for all $(t,s) \in D_0$;*

($H3$) *$\partial(H(t,s)k(s))/\partial s + H(t,s)k(s)(\rho'(s)/\rho(s)) = -h_2(t,s)$, for all $(t,s) \in D_0$.*

*Assume also that for each sufficiently large $T_0 \ge t_0$, there exist increasing divergent sequences of positive numbers $\{a_n\}, \{b_n\}, \{c_n\}$ with $T_0 \le a_n < c_n < b_n$ such that*

$$\begin{aligned}
&\frac{CK_0}{H(c_n,a_n)} \int_{a_n}^{c_n} H(s,a_n)k(s)\rho(s)q(s)\,ds \\
&\qquad + \frac{CK_0}{H(b_n,c_n)} \int_{c_n}^{b_n} H(b_n,s)k(s)\rho(s)q(s)\,ds \\
&\qquad > \frac{1}{4KL}\frac{1}{H(c_n,a_n)} \int_{a_n}^{c_n} \frac{\rho(s)r(\tau(s))}{\tau'(s)H(s,a_n)k(s)}\, h_1^2(s,a_n)\,ds \\
&\qquad + \frac{1}{4KL}\frac{1}{H(b_n,c_n)} \int_{c_n}^{b_n} \frac{\rho(s)r(\tau(s))}{\tau'(s)H(b_n,s)k(s)}\, h_2^2(b_n,s)\,ds.
\end{aligned} \tag{5}$$

*Then equation* (1) *is oscillatory.*

*Proof*. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \ge T_1 \ge t_0$ because a similar analysis holds for $x(t) < 0$ and $x(\tau(t)) < 0$. Then, by ($A_4$) and (1), we obtain that

$$(r(t)\psi(x(t))x'(t))' \le 0 \quad \text{for} \quad t \ge T = \max\{T_0, T_1\}. \tag{6}$$

Define

$$v(t) = \rho(t)\,\frac{r(t)\psi(x(t))x'(t)}{f(x(\tau(t)))}. \tag{7}$$

Differentiating (7) and making use of (1) and the assumptions of the theorem, it follows that for all $t \ge T_0$,

$$\begin{aligned}
v'(t) = \frac{\rho'(t)}{\rho(t)}\,v(t) - \rho(t)\,\frac{F(t,x(t),x'(t),\, x(\tau(t)),\, x'(\tau(t)))}{f(x(\tau(t)))} \\
- \frac{f'(x(\tau(t)))x'(\tau(t))\tau'(t)}{f(x(\tau(t)))}\,v(t).
\end{aligned} \tag{8}$$

By (6) and $(A_3)$, we conclude that

$$r(\tau(t))\psi(x(\tau(t)))x'(\tau(t)) \geq r(t)\psi(x(t))x'(t).$$

Consequently, by $(H_0^1)$, $(A_3)$ and (8) for $t \geq T_1$, we obtain

$$v'(t) \leq \frac{\rho'(t)}{\rho(t)}\, v(t) - CK_0\rho(t)q(t) - \frac{KL\tau'(t)}{r(\tau(t))\rho(t)}\, v^2(t). \tag{9}$$

Next we multiply (9) with $t$ replaced by $s$, by $H(t,s)k(s)$ and integrate from $t_1$ to $t$, $b_n \geq t_2 \geq t > t_1 \geq c_n \geq T$. After some simple computations, we get

(10)

$$\begin{aligned}
&\int_{t_1}^{t} H(t,s)k(s)CK_0\rho(s)q(s)\,ds \\
&\leq H(t,t_1)k(t_1)v(t_1) + \int_{t_1}^{t} \left[\frac{\partial}{\partial s}(H(t,s)k(s)) + H(t,s)k(s)\frac{\rho'(s)}{\rho(s)}\right] v(s)\,ds \\
&\quad - \int_{t_1}^{t} H(t,s)k(s)\,\frac{KL\tau'(s)}{r(\tau(s))\rho(s)}\, v^2(s)\,ds \\
&= H(t,t_1)k(t_1)v(t_1) - \int_{t_1}^{t} h_2(t,s)v(s)\,ds \\
&\quad - \int_{t_1}^{t} H(t,s)k(s)\,\frac{KL\tau'(s)}{r(\tau(s))\rho(s)}\, v^2(s)\,ds \\
&= H(t,t_1)k(t_1)v(t_1) + \int_{t_1}^{t} \frac{\rho(s)r(\tau(s))}{4KL\tau'(s)H(t,s)k(s)}\, h_2^2(t,s)\,ds \\
&\quad - \int_{t_1}^{t} \Bigg[\sqrt{\frac{KLH(t,s)k(s)\tau'(s)}{r(\tau(s))\rho(s)}}\, v(s) \\
&\qquad + \frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{KLH(t,s)k(s)\tau'(s)}}\, h_2(t,s)\Bigg]^2 ds.
\end{aligned}$$

From (10), we conclude that

$$\begin{aligned}
&\int_{t_1}^{t} CK_0H(t,s)k(s)\rho(s)q(s)\,ds \\
&\qquad \leq H(t,t_1)k(t_1)v(t_1) + \frac{1}{4KL}\int_{t_1}^{t_2} \frac{\rho(s)r(\tau(s))}{\tau'(s)H(t,s)k(s)} h_2^2(t,s)\,ds.
\end{aligned} \tag{11}$$

Now put $t_1 = c_n$ and let $t = t_2 - b_n^-$ in (11). Dividing both sides by $H(b_n, c_n)$ gives

$$(12)\quad \frac{1}{H(b_n,c_n)}\int_{c_n}^{b_n} CK_0 H(b_n,s)k(s)\rho(s)q(s)\,ds \\ \le k(c_n)v(c_n) + \frac{1}{4KLH(b_n,c_n)}\int_{c_n}^{b_n}\frac{\rho(s)r(\tau(s))}{\tau'(s)H(b_n,s)k(s)}\,h_2^2(b_n,s)\,ds.$$

Next go back to (9) and repeat the calculations multiplying first by $H(s,t)k(s)$ instead of by $H(t,s)k(s)$ and then integrating from $t$ to $t_2$, $b_n \ge t_2 > t \ge t_1 \ge c_n \ge T$. The result is

$$(13)\quad \int_t^{t_2} CK_0 H(s,t)k(s)\rho(s)q(s)\,ds \\ \le -H(t_2,t)k(t_2)v(t_2) + \frac{1}{4KL}\int_t^{t_2}\frac{\rho(s)r(\tau(s))}{\tau'(s)H(s,t)k(s)}\,h_1^2(s,t)\,ds.$$

Let $t = t_1 \to a_n^+$ and put $t_2 = c_n$. Then divide both sides in (13) by $H(c_n, a_n)$ to get

$$(14)\quad \frac{1}{H(c_n,a_n)}\int_{a_n}^{c_n} CK_0 H(s,a_n)k(s)\rho(s)q(s)\,ds \\ \le -k(c_n)v(c_n) + \frac{1}{4H(c_n,a_n)}\int_{a_n}^{c_n}\frac{\rho(s)r(\tau'(s))}{\tau'(s)H(s,a_n)k(s)}\,h_1^2(s,a_n)\,ds.$$

Now we claim that every nontrivial solution of equation (1) has at least one zero $t_n > a_n$ for $a_n > T_1$. This follows since $x(t) > 0$ and $x(\tau(t)) > 0$ for $t > T_1$. Adding (12) and (14), we get the inequality

$$\frac{CK_0}{H(c_n,a_n)}\int_{a_n}^{c_n} H(s,a_n)k(s)\rho(s)q(s)\,ds \\ + \frac{CK_0}{H(b_n,c_n)}\int_{c_n}^{b_n} H(b_n,s)k(s)\rho(s)q(s)\,ds \\ \le \frac{1}{4KL}\,\frac{1}{H(c_n,a_n)}\int_{a_n}^{c_n}\frac{\rho(s)r(\tau(s))}{\tau'(s)H(s,a_n)k(s)}\,h_1^2(s,a_n)\,ds \\ + \frac{1}{4KL}\,\frac{1}{H(b_n,c_n)}\int_{c_n}^{b_n}\frac{\rho(s)r(\tau(s))}{\tau'(s)H(b_n,s)k(s)}\,h_2^2(b_n,s)\,ds,$$

which contradicts the assumption (5). Thus the claim holds, i.e., no nontrivial solution of equation (1) can be eventually positive. Hence, equation (1) is oscillatory. □

The following result gives the possibility of considering new classes of equations in which $\psi(x)$ is unbounded.

**Theorem 2.2.** *Suppose that for* $x \neq 0$

$(H_0^2)$ *there exists a constant* $\gamma$ *such that* $f'(x)/\psi(x) \leq \gamma$.

*Let functions* $H \in C(D;R)$, $h_1, h_2 \in C(D_0;R)$, $k, \rho \in C^1([t_0,\infty);(0,\infty))$ *satisfy the conditions* (H1)–(H3) *in Theorem* 2.1. *Assume also that for each sufficiently large* $T_0 \geq t_0$, *there exist increasing divergent sequences of positive numbers* $\{a_n\}, \{b_n\}, \{c_n\}$ *with* $T_0 \leq a_n < c_n < b_n$ *such that*

$$\begin{aligned} &\frac{CK_0}{H(c_n,a_n)} \int_{a_n}^{c_n} H(s,a_n)k(s)\rho(s)q(s)\,ds \\ &\qquad + \frac{CK_0}{H(b_n,c_n)} \int_{c_n}^{b_n} H(b_n,s)k(s)\rho(s)q(s)\,ds \\ &\qquad > \frac{1}{4\gamma}\frac{1}{H(c_n,a_n)} \int_{a_n}^{c_n} \frac{\rho(s)r(\tau(s))}{\tau'(s)H(s,a_n)k(s)}\, h_1^2(s,a_n)\,ds \\ &\qquad + \frac{1}{4\gamma}\frac{1}{H(b_n,c_n)} \int_{c_n}^{b_n} \frac{\rho(s)r(\tau(s))}{\tau'(s)H(b_n,s)k(s)}\, h_2^2(b_n,s)\,ds. \end{aligned} \tag{15}$$

*Then equation* (1) *is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of equation (1). As in Theorem 2.1, without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq T_0 \geq t_0$, so that (6) holds. Again define the function $v(t)$ by (7). Then differentiate to obtain (8). By (6) and $(H_0^2)$, for $t \geq T_1$, we conclude from (8) that

$$v'(t) \leq \frac{\rho'(t)}{\rho(t)}\, v(t) - CK_0\rho(t)q(t) - \frac{\gamma\tau'(t)}{r(\tau(t))\rho(t)}\, v^2(t). \tag{16}$$

Therefore, starting with the inequality (16), by (1) and (16), we can proceed as in the proof of Theorem 2.1. □

As immediate consequences of Theorems 2.1 and 2.2 we get the following theorems.

**Theorem 2.3.** *Let condition* (5) *in Theorem* 2.1 *be replaced by*

(17)
$$\limsup_{t\to\infty}\int_l^t\Big[CK_0H(s,l)k(s)\rho(s)q(s)-\tfrac{\rho(s)r(\tau(s))}{4KL\tau'(s)H(t,l)k(s)}\,h_1^2(s,l)\Big]\,ds>0$$

*and*

(18)
$$\limsup_{t\to\infty}\int_l^t\Big[CK_0H(t,s)k(s)\rho(s)q(s)-\tfrac{\rho(s)r(\tau(s))}{4KL\tau'(s)H(t,s)k(s)}\,h_2^2(t,s)\Big]\,ds>0$$

*for each sufficient large* $l\ge T_0\ge t_0$ *with the other conditions unchanged. Then equation* (1) *is oscillatory.*

*Proof*. For any $T\ge T_0\ge t_0$, let $a_n=T$. In (17) we choose $l=a_n$. Then there exists $c_n>a_n$ such that

(19)
$$\int_{a_n}^{c_n}\Big[CK_0H(s,a_n)k(s)\rho(s)q(s)-\tfrac{\rho(s)r(\tau(s))}{4KL\tau'(s)H(t,a_n)k(s)}\,h_1^2(s,a_n)\Big]\,ds>0.$$

In (18) we choose $l=c_n$. Then there exists $b_n>c_n$ such that

(20)
$$\int_{c_n}^{b_n}\Big[CK_0H(b_n,s)k(s)\rho(s)q(s)-\tfrac{\rho(s)r(\tau(s))}{4KL\tau'(s)H(b_n,s)k(s)}\,h_2^2(b_n,s)\Big]\,ds>0.$$

Combining (19) and (20) we obtain (5). The conclusion thus comes from Theorem 2.1. $\square$

**Theorem 2.4.** *Let condition* (15) *in Theorem* 2.2 *be replaced by*

(21)
$$\limsup_{t\to\infty}\int_l^t\Big[CK_0H(s,l)k(s)\rho(s)q(s)-\tfrac{\rho(s)r(\tau(s))}{4\gamma\tau'(s)H(t,l)k(s)}\,h_1^2(s,l)\Big]\,ds>0$$

*and*

(22)
$$\limsup_{t\to\infty}\int_l^t\Big[CK_0H(t,s)k(s)\rho(s)q(s)-\tfrac{\rho(s)r(\tau(s))}{4\gamma\tau'(s)H(t,s)k(s)}\,h_2^2(t,s)\Big]\,ds>0$$

*for each sufficient large* $l \geq T_0 \geq t_0$ *with the other conditions unchanged. Then equation* (1) *is oscillatory.*

*Proof.* Similar to the proof of Theorem 2.3. □

If $h_1(t,s)$ and $h_2(t,s)$ are replaced by $h_1(t,s)\sqrt{H(t,s)k(s)}$ and $h_2(t,s)\sqrt{H(t,s)k(s)}$, respectively, in Theorems 2.1–2.4, we have the following theorems. The proofs are quite similar, so we omit the details.

**Theorem 2.5.** *Suppose that condition* $(H_0^1)$ *in Theorem* 2.1 *holds. Let functions* $H \in C(D;R)$, $h_1, h_2 \in C(D_0;R)$, $k, \rho \in C^1([t_0,\infty);(0,\infty))$ *satisfy the following conditions*:

(H1) $H(t,t) = 0$ *for* $t \geq t_0$, $H(t,s) > 0$ *on* $D_0$;

(H2) $\dfrac{\partial}{\partial t}(H(t,s)k(t)) + H(t,s)k(t)\dfrac{\rho'(t)}{\rho(t)} = h_1(t,s)\sqrt{H(t,s)k(t)}$, *for all* $(t,s) \in D_0$;

(H3) $\dfrac{\partial}{\partial s}(H(t,s)k(s)) + H(t,s)k(s)\dfrac{\rho'(s)}{\rho(s)} = -h_2(t,s)\sqrt{H(t,s)k(s)}$, *for all* $(t,s) \in D_0$.

*Assume also that for each sufficiently large* $T_0 \geq t_0$, *there exist increasing divergent sequences of positive numbers* $\{a_n\}, \{b_n\}, \{c_n\}$ *with* $T_0 \leq a_n < c_n < b_n$ *such that*

$$
\begin{aligned}
&\frac{CK_0}{H(c_n,a_n)} \int_{a_n}^{c_n} H(s,a_n)k(s)\rho(s)q(s)\,ds \\
&\qquad + \frac{CK_0}{H(b_n,c_n)} \int_{c_n}^{b_n} H(b_n,s)k(s)\rho(s)q(s)\,ds \\
&\qquad > \frac{1}{4KL}\frac{1}{H(c_n,a_n)} \int_{a_n}^{c_n} \frac{\rho(s)r(\tau(s))}{\tau'(s)}\, h_1^2(s,a_n)\,ds \\
&\qquad + \frac{1}{4KL}\frac{1}{H(b_n,c_n)} \int_{c_n}^{b_n} \frac{\rho(s)r(\tau(s))}{\tau'(s)}\, h_2^2(b_n,s)\,ds.
\end{aligned} \tag{23}
$$

*Then equation* (1) *is oscillatory.*

**Theorem 2.6.** *Suppose that condition* $(H_0^2)$ *in Theorem* 2.2 *holds. Let functions* $H\in C(D;R)$, $h_1, h_2\in C(D_0;R)$, $k, \rho\in C^1([t_0,\infty);(0,\infty))$ *satisfy the conditions* (H1)–(H3) *in Theorem* 2.5. *Assume also that for each sufficiently large* $T_0 \geq t_0$, *there exist increasing divergent sequences of positive numbers* $\{a_n\}, \{b_n\}, \{c_n\}$ *with* $T_0 \leq a_n < c_n < b_n$ *such that*

$$\begin{aligned}
&\frac{CK_0}{H(c_n,a_n)}\int_{a_n}^{c_n} H(s,a_n)k(s)\rho(s)q(s)\,ds \\
&\qquad + \frac{CK_0}{H(b_n,c_n)}\int_{c_n}^{b_n} CK_0H(b_n,s)k(s)\rho(s)q(s)\,ds \\
&\qquad > \frac{1}{4\gamma}\frac{1}{H(c_n,a_n)}\int_{a_n}^{c_n}\frac{\rho(s)r(\tau(s))}{\tau'(s)}\,h_1^2(s,a_n)\,ds \\
&\qquad + \frac{1}{4\gamma}\frac{1}{H(b_n,c_n)}\int_{c_n}^{b_n}\frac{\rho(s)r(\tau(s))}{\tau'(s)}\,h_2^2(b_n,s)\,ds.
\end{aligned} \tag{24}$$

*Then equation* (1) *is oscillatory.*

**Theorem 2.7.** *Let condition* (23) *in Theorem* 2.5 *be replaced by*

$$\limsup_{t\to\infty}\int_l^t\left[CK_0H(s,l)k(s)\rho(s)q(s) - \frac{\rho(s)r(\tau(s))}{4KL\tau'(s)}\,h_1^2(s,l)\right]ds > 0 \tag{25}$$

*and*

$$\limsup_{t\to\infty}\int_l^t\left[CK_0H(t,s)k(s)\rho(s)q(s) - \frac{\rho(s)r(\tau(s))}{4KL\tau'(s)}\,h_2^2(t,s)\right]ds > 0 \tag{26}$$

*for each sufficient large* $l \geq T_0 \geq t_0$ *with the other conditions unchanged. Then equation* (1) *is oscillatory.*

**Theorem 2.8.** *Let condition* (24) *in Theorem* 2.6 *be replaced by*

$$\limsup_{t\to\infty}\int_l^t\left[CK_0H(s,l)k(s)\rho(s)q(s) - \frac{\rho(s)r(\tau(s))}{4\gamma\tau'(s)}\,h_1^2(s,l)\right]ds > 0 \tag{27}$$

*and*

(28)
$$\limsup_{t\to\infty}\int_l^t\left[CK_0H(t,s)k(s)\rho(s)q(s)-\frac{\rho(s)r(\tau(s))}{4\gamma\tau'(s)}\,h_2^2(t,s)\right]ds>0$$

*for each sufficient large $l \ge T_0 \ge t_0$ with the other conditions unchanged. Then equation* (1) *is oscillatory.*

Next define

$$R(t)=\int_l^{\tau(t)}\frac{1}{r(s)}\,ds,\quad \tau(t)\ge l\ge t_0, \tag{29}$$

and let

$$H(t,s)=[R(t)-R(s)]^\lambda,\quad t\ge t_0, \tag{30}$$

where $\lambda>1$ is a constant.

**Theorem 2.9.** *Let $\lim_{t\to\infty}R(t)=\infty$ hold. Then equation* (1) *is oscillatory provided that for each $l\ge t_0$ and there exists $\lambda>1$ such that the following inequalities are satisfied*:

(31)
$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t CK_0[R(s)-R(l)]^\lambda q(s)\,ds>\frac{\lambda^2}{4KL(\lambda-1)}$$

*and*

(32)
$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t CK_0[R(t)-R(s)]^\lambda q(s)\,ds>\frac{\lambda^2}{4KL(\lambda-1)}.$$

*Proof.* Pick $\rho(t)\equiv k(t)\equiv 1$. It is easy to see that

$$h_1(t,s)=\lambda[R(t)-R(s)]^{\lambda/2-1}\,\frac{\tau'(t)}{r(\tau(t))}$$

and

$$h_2(t,s) = \lambda[R(t) - R(s)]^{\lambda/2-1}\,\frac{\tau'(s)}{r(\tau(s))}$$

in $(H_1)$ and $(H_2)$ of Theorem 2.5. Note that

$$\begin{aligned}
\int_l^t \frac{r(\tau(s))}{4KL\tau'(s)}\,h_1^2(s,l)\,ds
&= \int_l^t \frac{r(\tau(s))}{4KL\tau'(s)}\lambda^2[R(s) - R(l)]^{\lambda-2}\left[\frac{\tau'(t)}{r(\tau(t))}\right]^2 ds \\
&= \frac{\lambda^2}{4KL(\lambda-1)}[R(t) - R(l)]^{\lambda-1}
\end{aligned}$$

and

$$\begin{aligned}
\int_l^t \frac{r(\tau(s))}{4KL\tau'(s)}\,h_2^2(t,s)\,ds
&= \int_l^t \frac{r(\tau(s))}{4KL\tau'(s)}\lambda^2[R(t) - R(s)]^{\lambda-2}\left[\frac{\tau'(s)}{r(\tau(s))}\right]^2 ds \\
&= \frac{\lambda^2}{4KL(\lambda-2)}[R(t) - R(l)]^{\lambda-1}.
\end{aligned}$$

Since $\lim_{t\to\infty} R(t) = \infty$, it follows that

$$\lim_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \frac{r(\tau(s))}{4KL\tau'(s)}\,h_1^2(s,l)\,ds = \frac{\lambda^2}{4KL(\lambda-1)} \tag{33}$$

and

$$\lim_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \frac{r(\tau(s))}{4KL\tau'(s)}\,h_2^2(t,s)\,ds = \frac{\lambda^2}{4KL(\lambda-1)}. \tag{34}$$

From (31) and (33), we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{CK_0H(s,l)q(s) - \frac{r(\tau(s))}{4KL\tau'(s)}\,h_1^2(s,l)\right\} ds \\
&= \limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t CK_0[R(s)-R(l)]^{\lambda} q(s)\,ds - \frac{\lambda^2}{4KL(\lambda-1)} > 0.
\end{aligned}$$

It follows that

$$\limsup_{t\to\infty} \int_l^t \left\{ CK_0[R(s)-R(l)]^\lambda q(s) - \frac{r(\tau(s))}{4KL\tau'(s)}\, h_1^2(s,l) \right\} ds > 0;$$

i.e., (25) holds. Similarly, (32) implies that (26) holds. From Theorem 2.7, equation (1) is oscillatory. □

**Theorem 2.10.** *Let* $\lim_{t\to\infty} R(t) = \infty$ *hold. Then equation* (1) *is oscillatory provided that for each* $l \geq t_0$ *there exists* $\lambda > 1$ *such that the following inequalities are satisfied*:

(35)
$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t CK_0[R(s)-R(l)]^\lambda q(s)\, ds > \frac{\lambda^2}{4\gamma(\lambda-1)}$$

*and*

(36)
$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t CK_0[R(t)-R(s)]^\lambda q(s)\, ds > \frac{\lambda^2}{4\gamma(\lambda-1)}.$$

*Proof.* This proof follows as in the proof of Theorem 2.9. □

Now let $k(t) = 1$ and $H(t,s) = H(t-s)$ in Theorem 2.5. We have that $\partial(H(t-s))/\partial t = -\partial(H(t-s))/\partial s$. Denote this common value by $h(t-s)$. Then

$$h_1(t,s) = \frac{h(t-s)}{\sqrt{H(t-s)}} + \frac{\rho'(t)}{\rho(t)}\sqrt{H(t-s)},$$

and

$$h_2(t,s) = \frac{h(t-s)}{\sqrt{H(t-s)}} - \frac{\rho'(t)}{\rho(t)}\sqrt{H(t-s)}.$$

Applying Theorem 2.5 gives

**Theorem 2.11.** *Assume that for any* $T \geq t_0$, *there exists* $T \leq a_n < c_n$ *such that*

(37)

$$\int_{a_n}^{c_n} CK_0 H(s-a_n)\{\rho(s)(q(s)+\rho(2c_n-s)q(2c_n-s)\}\,ds$$
$$> \frac{1}{4KL}\int_{a_n}^{c_n}\left[\rho(s)\frac{r(\tau(s))}{\tau'(s)}+\rho(2c_n-s)\,\frac{r(\tau(2c_n-s))}{\tau'(2c_n-s)}\right]\frac{h^2(s-a_n)}{H(s-a_n)}\,ds$$
$$+\frac{1}{2KL}\int_{a_n}^{c_n}\left[\rho'(s)\frac{r(\tau(s))}{\tau'(s)}-\rho'(2c_n-s)\,\frac{r(\tau(2c_n-s))}{\tau'(2c_n-s)}\right]h(s-a_n)\,ds$$
$$+\frac{1}{4KL}\int_{a_n}^{c_n}\left[\frac{[\rho'(s)]^2}{\rho(s)}\,\frac{r(\tau(s))}{\tau'(s)}+\frac{[\rho'(2c_n-s)]^2}{\rho(2c_n-s)}\,\frac{r(\tau(2c_n-s))}{\tau'(2c_n-s)}\right]$$
$$\times H(s-a_n)\,ds.$$

*Then equation* (1) *is oscillatory.*

*Proof.* Let $b_n = 2c_n - a_n$. Then $H(b_n-c_n) = H(c_n-a_n) = H(b_n-a_n)/2$ and, for any $w \in L[a,b]$, we have

$$\int_{c_n}^{b_n} w(s)\,ds = \int_{a_n}^{c_n} w(2c-s)\,ds.$$

Hence

$$\int_{c_n}^{b_n} H(b_n-s)\rho(s)q(s)\,ds = \int_{a_n}^{c_n} H(s-a_n)\rho(2c_n-s)q(2c_n-s)\,ds$$

and

$$\int_{c_n}^{b_n}\frac{\rho(s)r(\tau(s))}{\tau'(s)}\,h_2^2(b_n-s)\,ds$$
$$=\int_{c_n}^{b_n}\frac{\rho(s)r(\tau(s))}{\tau'(s)}\left[\frac{h(b_n-s)}{\sqrt{H(b_n-s)}}-\frac{\rho'(s)}{\rho(s)}\sqrt{H(b_n-s)}\right]^2 ds$$
$$=\int_{a_n}^{c_n}\frac{\rho(2c_n-s)r(\tau(2c_n-s))}{\tau'(2c_n-s)}$$
$$\times\left[\frac{h(s-a_n)}{\sqrt{H(s-a_n)}}-\frac{\rho'(2c_n-s)}{\rho(2c_n-s)}\sqrt{H(s-a_n)}\right]^2 ds.$$

Thus (37) implies (23) and therefore equation (1) is oscillatory by Theorem 2.5. □

**Theorem 2.12.** *Assume that for any $T \geq t_0$, there exist $T \leq a_n < c_n$ such that*

(38)
$$\begin{aligned}
&\int_{a_n}^{c_n} CK_0 H(s-a_n)\{\rho(s)q(s)+\rho(2c_n-s)q(2c_n-s)\}\,ds \\
&> \frac{1}{4\gamma}\int_{a_n}^{c_n}\left[\rho(s)\,\frac{r(\tau(s))}{\tau'(s)}+\rho(2c_n-s)\,\frac{r(\tau(2c_n-s))}{\tau'(2c_n-s)}\right]\frac{h^2(s-a_n)}{H(s-a_n)}\,ds \\
&\quad+\frac{1}{2\gamma}\int_{a_n}^{c_n}\left[\rho'(s)\,\frac{r(\tau(s))}{\tau'(s)}-\rho'(2c_n-s)\,\frac{r(\tau(2c_n-s))}{\tau'(2c_n-s)}\right]h(s-a_n)\,ds \\
&\quad+\frac{1}{4\gamma}\int_{a_n}^{c_n}\left[\frac{[\rho'(s)]^2}{\rho(s)}\,\frac{r(\tau(s))}{\tau'(s)}+\frac{[\rho'(2c_n-s)]^2}{\rho(2c_n-s)}\,\frac{r(\tau(2c_n-s))}{\tau'(2c_n-s)}\right] \\
&\qquad\qquad\times H(s-a_n)\,ds.
\end{aligned}$$

*Then equation* (1) *is oscillatory.*

*Proof.* The proof is similar to the proof of Theorem 2.11 but uses Theorem 2.6 instead of Theorem 2.5. □

**3. Oscillation results for nonsmooth $f(x)$.** In this section we consider the oscillation of equation (1) when the function $f(x)$ does not have a continuous derivative.

**Theorem 3.1.** *Suppose that for $x \neq 0$,*

$(H_0^3)$ *there exist constants $K$ and $L$ such that*

$$\frac{f(x)}{x} \geq K > 0; \quad 0 < \psi(x) \leq L^{-1}.$$

*Let functions $H \in C(D; R)$, $h_1, h_2 \in C(D_0; R)$, $k, \rho \in C^1([t_0, \infty); (0, \infty))$ satisfy the conditions* (H1)–(H3) *in Theorem* 2.1. *Assume also that for each sufficiently large $T_0 \geq t_0$, there exist increasing divergent sequences* *of positive numbers* $\{a_n\}, \{b_n\}, \{c_n\}$ *with* $T_0 \leq a_n < c_n < b_n$ *such that*

$$\begin{aligned}\frac{CKK_0}{H(c_n, a_n)} &\int_{a_n}^{c_n} H(s, a_n)k(s)\rho(s)q(s)\,ds \\ &+ \frac{CKK_0}{H(b_n, c_n)} \int_{c_n}^{b_n} H(b_n, s)k(s)\rho(s)q(s)\,ds \\ &> \frac{1}{4L}\,\frac{1}{H(c_n, a_n)} \int_{a_n}^{c_n} \frac{\rho(s)r(\tau(s))}{\tau'(s)H(s, a_n)k(s)}\,h_1^2(s, a_n)\,ds \\ &+ \frac{1}{4L}\,\frac{1}{H(b_n, c_n)} \int_{c_n}^{b_n} \frac{\rho(s)r(\tau(s))}{\tau'(s)H(b_n, s)k(s)}\,h_2^2(b_n, s)\,ds\end{aligned} \tag{39}$$

*holds. Then equation* (1) *is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq T_1 \geq t_0$. Thus, by $(A_4)$ and (1), (6) holds.

Define

$$v(t) = \rho(t)\,\frac{r(t)\psi(x(t))x'(t)}{x(\tau(t))}. \tag{40}$$

Differentiating (40) and making use of (1) and the assumptions of the theorem, it follows that for all $t \geq T_0$,

$$\begin{aligned}v'(t) = \frac{\rho'(t)}{\rho(t)}\,v(t) - \rho(t)\,&\frac{F(t, x(t), x'(t), x(\tau(t)), x'(\tau(t)))}{x(\tau(t))} \\ &- \frac{x'(\tau(t))\tau'(t)}{x(\tau(t))}v(t).\end{aligned} \tag{41}$$

By (6) $(H_0)$, $(A_4)$ and (8), for $t \geq T_1$, we obtain from (41) that

$$v'(t) \leq \frac{\rho'(t)}{\rho(t)}\,v(t) - CKK_0\rho(t)q(t) - \frac{L\tau'(t)}{r(\tau(t))\rho(t)}\,v^2(t). \tag{42}$$

Therefore, by (1) and (42), the rest of the proof is similar to that of Theorem 2.1. $\square$

Theorems 3.2–3.5 that follow have proofs similar to those of Theorems 2.3, 2.5, 2.7 and 2.9, respectively. The details are omitted.

**Theorem 3.2.** *Let condition* (39) *in Theorem* 3.1 *be replaced by*

(43)
$$\limsup_{t\to\infty}\int_l^t\Big[CKK_0H(s,l)k(s)\rho(s)q(s)-\tfrac{\rho(s)r(\tau(s))}{4L\tau'(s)H(t,l)k(s)}\,h_1^2(s,l)\Big]\,ds>0$$

*and*

(44)
$$\limsup_{t\to\infty}\int_l^t\Big[CKK_0H(t,s)k(s)\rho(s)q(s)-\tfrac{\rho(s)r(\tau(s))}{4L\tau'(s)H(t,s)k(s)}\,h_2^2(t,s)\Big]\,ds>0$$

*for each sufficient large* $l\ge T_0\ge t_0$ *with the other conditions unchanged. Then equation* (1) *is oscillatory.*

If $h_1(t,s)$ and $h_2(t,s)$ are replaced by $h_1(t,s)\sqrt{H(t,s)k(s)}$ and $h_2(t,s)\sqrt{H(t,s)k(s)}$ in Theorems 3.1 and 3.2, respectively, we have the following theorems. The proofs are similar, so we omit the details.

**Theorem 3.3.** *Suppose that condition* $(H_0^3)$ *in Theorem* 3.1 *holds. Let functions* $H\in C(D;R)$, $h_1,h_2\in C(D_0;R)$, $k,\rho\in C^1([t_0,\infty);(0,\infty))$ *satisfy the conditions* (H1)–(H3) *in Theorem* 2.5. *Assume also that for each sufficiently large* $T_0\ge t_0$, *there exist increasing divergent sequences of positive numbers* $\{a_n\},\{b_n\},\{c_n\}$ *with* $T_0\le a_n<c_n<b_n$ *such that*

(45)
$$\begin{aligned}&\frac{CKK_0}{H(c_n,a_n)}\int_{a_n}^{c_n}H(s,a_n)k(s)\rho(s)q(s)\,ds\\&\qquad+\frac{CKK_0}{H(b_n,c_n)}\int_{c_n}^{b_n}H(b_n,s)k(s)\rho(s)q(s)\,ds\\&\qquad>\frac{1}{4L}\,\frac{1}{H(c_n,a_n)}\int_{a_n}^{c_n}\frac{\rho(s)r(\tau(s))}{\tau'(s)}\,h_1^2(s,a_n)\,ds\\&\qquad+\frac{1}{4L}\,\frac{1}{H(b_n,c_n)}\int_{c_n}^{b_n}\frac{\rho(s)r(\tau(s))}{\tau'(s)}\,h_2^2(b_n,s)\,ds,\end{aligned}$$

*holds. Then equation* (1) *is oscillatory.*

**Theorem 3.4.** *Let condition* (45) *in Theorem* 3.3 *be replaced by*

(46)
$$\limsup_{t\to\infty}\int_l^t\left[CKK_0H(s,l)k(s)\rho(s)q(s)-\frac{\rho(s)r(\tau(s))}{4L\tau'(s)}\,h_1^2(s,l)\right]ds>0$$

*and*

(47)
$$\limsup_{t\to\infty}\int_l^t\left[CKK_0H(t,s)k(s)\rho(s)q(s)-\frac{\rho(s)r(\tau(s))}{4L\tau'(s)}\,h_2^2(t,s)\right]ds>0$$

*for each sufficient large* $l \geq T_0 \geq t_0$ *with the other conditions unchanged. Then equation* (1) *is oscillatory.*

**Theorem 3.5.** *Let* $\lim_{t\to\infty} R(t) = \infty$ *hold. Then equation* (1) *is oscillatory provided that for each* $l \geq t_0$ *there exists* $\lambda > 1$ *such that*

(48)
$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t CKK_0[R(s)-R(l)]^\lambda q(s)\,ds>\frac{\lambda^2}{4L(\lambda-1)}$$

*and*

(49)
$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t CKK_0[R(t)-R(s)]^\lambda q(s)\,ds>\frac{\lambda^2}{4L(\lambda-1)}.$$

*Proof.* This theorem can be proved in a manner quite similar to the proof of Theorem 2.9. The details are omitted here. □

Modifying the proof of Theorem 2.11 by using Theorem 3.5 and Theorem 2.5, we obtain

**Theorem 3.6.** *Assume that for any* $T \geq t_0$, *there exists* $T \leq a_n < c_n$

*such that*

(50)

$$\begin{aligned}
&\int_{a_n}^{c_n} CKK_0 H(s-a_n)\{\rho(s)q(s)+\rho(2c_n-s)q(2c_n-s)\}\,ds\\
&> \frac{1}{4L}\int_{a_n}^{c_n}\left[\rho(s)\,\frac{r(\tau(s))}{\tau'(s)}+\rho(2c_n-s)\,\frac{r(\tau(2c_n-s))}{\tau'(2c_n-s)}\right]\frac{h^2(s-a_n)}{H(s-a_n)}\,ds\\
&\quad+\frac{1}{2L}\int_{a_n}^{c_n}\left[\rho'(s)\,\frac{r(\tau(s))}{\tau'(s)}-\rho'(2c_n-s)\,\frac{r(\tau(2c_n-s))}{\tau'(2c_n-s)}\right]h(s-a_n)\,ds\\
&\quad+\frac{1}{4L}\int_{a_n}^{c_n}\left[\frac{[\rho'(s)]^2}{\rho(s)}\,\frac{r(\tau(s))}{\tau'(s)}+\frac{[\rho'(2c_n-s)]^2}{\rho(2c_n-s)}\,\frac{r(\tau(2c_n-s))}{\tau'(2c_n-s)}\right]\\
&\qquad\qquad\times H(s-a_n)\,ds.
\end{aligned}$$

*Then equation* (1) *is oscillatory.*

**4. Remarks and examples.** The results in this paper involve Kamenev's type conditions and improve and extend the results of Rogovchenko [**12**], Li and Agarwal [**8**], Huang [**5**], Kamenev [**6**] and Philos [**10**].

*Remark* 4.1. From Theorems 2.1–2.12 and Theorems 3.1–3.6, we can derive different explicit sufficient conditions for the oscillation of equation (1) by appropriate choice of functions $H(t,s)$, $k(s)$ and $\rho(s)$. For instance, if we choose $H(t,s)=(t-s)^\alpha$, $H(t,s)=[R(t)-R(s)]^\alpha$, $H(t,s) = [\log U(t)/U(s)]^\alpha$ or $H(t,s) = [\int_s^t (1/w(z))\,dz]^\alpha$, etc., for $t\ge s\ge t_0$; then $k(s)$ and $\rho(s)$ may be chosen 1 and $s$, respectively, etc., and $\alpha>1$ is a constant, $R(t)=\int_{t_0}^t ds/u(s)$ and $U(t)=\int_t^\infty ds/u(s)<\infty$ for $t\ge t_0$. Also $w\in C([t_0,\infty),(0,\infty))$ satisfies $\int_{t_0}^\infty (1/w(z))\,dz=\infty$.

The conditions in this paper are sharper than conditions in [**1–16**]. We will see that the oscillations cannot be demonstrated by most other known criteria in the following examples.

**Example 4.2.** Assume $\alpha, a, b, c\ge 0$, $(A_3)$ and $[\tau'(t)/\tau^2(t)]'\le 0$.

Consider the delay equation

$$\begin{aligned}\Big[\frac{1}{1+a\cos^2 x(t)}\,x'(t)\Big]' + \frac{\alpha\tau'(t)}{\tau^2(t)}\,\big[x(\tau(t))+bx^3(\tau(t))\big] \\ \times\big[1+c(\sin x'(t))^2+c(\cos x(\tau(t))^2\big]=0\end{aligned} \tag{51}$$

where $t \geq 1$.

Here $L = C = K = K_0 = 1$ and

$$R(t)=\int_l^{\tau(t)}\frac{1}{r(s)}\,ds=\tau(t)-l,\quad R'(t)=\tau'(t),\quad \lim_{t\to\infty}R(t)=\infty.$$

Then for $\lambda > 1$,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{CK_0}{R^{\lambda-1}(t)}\int_l^t[R(s)-R(l)]^\lambda q(s)\,ds\\ &\quad=\limsup_{t\to\infty}\frac{\alpha}{(\tau(t)-l)^{\lambda-1}}\int_l^t[\tau(s)-\tau(l)]^\lambda\,\frac{\tau'(s)}{\tau^2(s)}\,ds\\ &\quad=\limsup_{t\to\infty}\frac{\alpha}{(\tau(t)-l)^{\lambda-1}}\int_l^t\left(\frac{\tau(s)-\tau(l)}{\tau(s)}\right)^\lambda\tau'(s)\tau^{\lambda-2}(s)\,ds\\ &\quad\geq\frac{\alpha(1-\varepsilon)}{\lambda-1}\end{aligned} \tag{52}$$

for any $\varepsilon \in (0,1)$, since $\lim_{t\to\infty}\big((\tau(s)-\tau(l))/\tau(s)\big)^\lambda = 1$. Next we will prove that

$$\int_l^t CK_0[R(t)–R(s)]^\lambda\,\frac{\alpha\tau'(s)}{\tau^2(s)}\,ds \geq \int_l^t CK_0[R(s)–R(l)]^\lambda\,\frac{\alpha\tau'(s)}{\tau^2(s)}\,ds. \tag{53}$$

Let

$$F(t)=\int_l^t CK_0\big\{[R(t)-R(s)]^\lambda-[R(s)-R(l)]^\lambda\big\}\,\frac{\alpha\tau'(s)}{\tau^2(s)}\,ds.$$

Then $F(l) = 0$ and, for $t \geq l$,

$$
\begin{aligned}
F'(t) &\\
&= CK_0 \int_l^t \lambda[R(t)-R(s)]^{\lambda-1} R'(t)\, \frac{\alpha\tau'(t)}{\tau^2(t)}\, ds - CK_0[R(t)-R(l)]^{\lambda}\, \frac{\alpha\tau'(t)}{\tau^2(t)}\\
&\geq CK_0 \int_l^t \lambda[R(t)-R(s)]^{\lambda-1} R'(s)\, \frac{\alpha\tau'(t)}{\tau^2(t)}\, ds - CK_0[R(t)-R(l)]^{\lambda}\, \frac{\alpha\tau'(t)}{\tau^2(t)}\\
&\geq CK_0\, \frac{\alpha\tau'(t)}{\tau^2(t)} \int_l^t \lambda[R(t)-R(s)]^{\lambda-1} R'(s)\, ds\\
&\quad - CK_0[R(t)-R(l)]^{\lambda}\, \frac{\alpha\tau'(t)}{\tau^2(t)} = 0.
\end{aligned}
$$

Hence $F(t) \geq F(l) = 0$ for $t \geq l$, i.e., (53) holds. By (52) and (53), for any $\alpha > 1/4$, there exists $\lambda > 1$ such that $\alpha/(\lambda-1) > \lambda^2/(4KL(\lambda-1))$. This means that conditions in Theorem 2.9 hold for some $\lambda$. Therefore, equation (51) is oscillatory for $\alpha > 1/4$.

However, the oscillations cannot be demonstrated by other known criteria in [**1**–**16**]. Further, we note that Euler's equation $x''(t) + (\alpha/t^2)x(t) = 0$, i.e., (51) with $a = b = c = 0$ and $\tau(t) = t$, is oscillatory if $\alpha > 1/4$. This implies that our results are sharp.

**Example 4.3.** Let $\alpha \geq 0$, $(A_3)$ and $[\tau'(t)/\tau^2(t)]' \leq 0$. Consider the delay equation

$$
\begin{aligned}
&\left[(1+5x^2(t))x'(t)\right]' + \frac{\alpha\tau'(t)}{\tau^2(t)} \left[x(\tau(t)) + x^3(\tau(t))\right] \\
&\qquad \times \left[1 + (\sin x'(t))^2\right] = 0 \quad \text{for } t \geq 1.
\end{aligned} \tag{54}
$$

Similar to the use of Theorem 2.9 in Example 4.2, now by Theorem 2.10, equation (1) is oscillatory for $\alpha > 1/4$. However, $\psi(x) = 1 + 5x^2$ is an unbounded function.

In Examples 4.2 and 4.3, we can obtain some interesting results. For example, $\tau(t)$ may be chosen to be $t$, $t-\delta$, $t-e^{-t}$, etc.

**Example 4.4.** Let $\alpha \geq 0$ and $b \geq 0$. Consider the delay equation

$$
\begin{aligned}
&\left[\frac{1-e^{-x^2(t)}}{2(t+1)}\, x'(t)\right]' + \frac{2\alpha t}{(t^2-1)^2}\, x(t-1)\, e^{b(1+\sin x(t-1))} \\
&\qquad \times \left[1 + (\sin x(t-1))^2\right] = 0 \quad \text{for } t \geq 1.
\end{aligned} \tag{55}
$$

It is easy to see $L = C = K = K_0 = 1$ and

$$R(t) = \int_l^{t-1} \frac{1}{r(s)}\, ds = \int_l^{t-1} 2(s+1)\, ds = t^2 - (l+1)^2,$$
$$R'(t) = 2t, \qquad \lim_{t\to\infty} R(t) = \infty$$

and

$$\frac{f(x)}{x} = [1 + \sin x]e^{b(1+\sin x^2)} \geq 1 = C > 0.$$

Then for $\lambda > 1$ we obtain

$$\begin{aligned}(56)\quad & \limsup_{t\to\infty} \frac{CKK_0}{R^{\lambda-1}(t)} \int_l^t [R(s) - R(l)]^\lambda q(s)\, ds \\ & \quad = \limsup_{t\to\infty} \frac{\alpha}{(t^2 - l^2)^{\lambda-1}} \int_l^t (s^2 - l^2)^\lambda \frac{2s}{(s^2 - l^2)^2}\, ds \geq \frac{\alpha}{\lambda - 1}\end{aligned}$$

as in Example 4.2. Also, as in Example 4.2, we can prove that

$$\begin{aligned}(57)\quad & \int_l^t CKK_0[R(t) - R(s)]^\lambda \frac{2\alpha s}{(s^2-1)^2}\, ds \\ & \quad \geq \int_l^t CKK_0[R(s) - R(l)]^\lambda \frac{2\alpha s}{(s^2-1)^2}\, ds.\end{aligned}$$

By (56) and (57) for any $\alpha > 1/4$, there exists $\lambda > 1$ such that $\alpha/(\lambda - 1) > \lambda^2/(4L(\lambda - 1))$. This means that all conditions of Theorem 3.4 hold for some $\lambda$. Thus, equation (55) is oscillatory for $\alpha > 1/4$. However, $f'(y) \geq 0$ is not satisfied, the results in [**1**–**16**] fail to apply equation (55).

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