ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 4, Winter 2004

A FIX-FINITE APPROXIMATION THEOREM

ABDELKADER STOUTI

ABSTRACT. We prove that if C_i is a nonempty convex compact subset of a metrizable locally convex vector space for $i = 1, \dots, m$ such that $\cap_{i=1}^{i=m} C_i \neq \varnothing$ or $C_i \cap C_j = \varnothing$ for $i \neq j$, then for every $\varepsilon > 0$ and for every n-valued continuous multi-function $F : \cup_{i=1}^{i=m} C_i \to \cup_{i=1}^{i=m} C_i$ there exists an n-valued continuous multi-function $G : \cup_{i=1}^{i=m} C_i \to \cup_{i=1}^{i=m} C_i$ which is ε-near to F and has only a finite number of fixed points.

1. Introduction and preliminaries. Let A be a nonempty subset of a metrizable locally convex vector space. We say that A satisfies the fix-finite approximation property, FFAP, for a family $\mathcal{F}$ of self multi-functions, or self-maps, of A if, for every $F \in \mathcal{F}$ and all $\varepsilon > 0$, there exists $G \in \mathcal{F}$ which is ε-near to F and has only a finite number of fixed points.

In [**3, 5**], Hopf proved by a special construction that any finite polyhedron which is connected and has dimension greater than one satisfies the FFAP for the family of continuous self-maps. In [**8**], Schirmer extended this result to any n-valued continuous self multi-function. In [**1**], Baillon and Rallis showed that any finite union of closed convex subsets of a Banach space satisfies the FFAP for the family of compact self-maps.

In this paper we consider the more general case of metrizable locally convex vector spaces. Our first key result, Theorem 2.2, is a generalization of a theorem of Baillon-Rallis [**1**]. The main result in this paper is Theorem 3.7: Let C_i be a nonempty convex compact subset of a metrizable locally convex vector space for $i = 1, \dots, m$ such that $\cap_{i=1}^{i=m} C_i \neq \varnothing$ or $C_i \cap C_j = \varnothing$ for $i \neq j$; then $\cup_{i=1}^{i=m} C_i$ satisfies the FFAP for any n-valued continuous multi-function $F : \cup_{i=1}^{i=m} C_i \to \cup_{i=1}^{i=m} C_i$.

In the sequel we recall some definitions and well-known results for subsequent use. Let $\varepsilon > 0$ and let X be a topological space and (Y, d)

2000 AMS *Mathematics Subject Classification.* 41A65, 46A03, 46A55, 47H10.
Key words and phrases. Metrizable locally convex vector space, convex set, approximation, fixed points.
Received by the editors on August 24, 2001, and in revised form on May 8, 2003.

a metric space. We say that $f : X \to Y$ is a map if it is a single-valued function.

1) Two continuous maps f and g from X to Y are said to be ε-near if

$$d(f(x), g(x)) < \varepsilon \quad \text{for all } x \in X.$$

2) A homotopy $h_t : X \to Y$, $0 \le t \le 1$, is said to be an ε-homotopy if

$$\sup\{d(h_t(x), h_{t'}(x)) : t, t' \in [0,1]\} < \varepsilon \quad \text{for all } x \in X.$$

3) Two continuous maps f and g from X to Y are said to be ε-homotopic if there exists an ε-homotopy $(h_t)_{t\in[0,1]}$ from X to Y such that $h_0 = f$ and $h_1 = g$.

Let X and Y be two Hausdorff topological spaces and $f : X \to Y$ be a map. The map f is said to be compact if it is continuous and the closure of its range $\overline{f(X)}$ is a compact subset of Y.

Let Y be a metric space. One says that Y is an absolute neighborhood retract (ANR) if for any nonempty closed subset A of an arbitrary metric space X and for any continuous map $f : A \to Y$, then there exists an open subset U of X containing A and a continuous map $g : U \to Y$ which is an extension of f, i.e., $g(x) = f(x)$ for all $x \in A$.

In [**4, 6**], Dugundji established the homotopy extension theorem for ANRs.

Theorem 1.1. *Let X be a metrizable space and Y an* ANR. *For $\varepsilon > 0$ there exists $\delta > 0$ such that for any two δ-near maps $f, g : X \to Y$ and δ-homotopy $j_t : A \to Y$, where A is a closed subspace of X and $j_0 = f|_A$, $j_1 = g|_A$, there exists an ε-homotopy $h_t : X \to Y$ such that $h_0 = f$, $h_1 = g$ and $h_{t_{|A}} = j_t$ for all $t \in [0,1]$.*

In what follows we denote by $\mathbf{N}^*$ the set of all positive entire numbers. Let E be a metrizable locally convex vector space, then its topology can always be given by a decreasing sequence $(U_n)_{n\in\mathbf{N}^*}$ of absolutely convex neighborhoods of 0 with $\cap_{n\in\mathbf{N}^*} U_n = \{0\}$. Let $(p_n)_{n\in\mathbf{N}^*}$ be the increasing family of semi-norms defined by setting

$$p_n(x) = \inf\left\{\lambda > 0 : \frac{1}{\lambda}\, x \in U_n\right\} \quad \text{for all } x \in E.$$

Then the topology of E is defined by a translation-invariant metric d given by

$$d(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\,\frac{p_n(x-y)}{1+p_n(x-y)}.$$

For $n,k\in\mathbf{N}^*$, we denote $A(n,k)=\{y\in E : p_n(y)<\frac{1}{2}^k\}$. For all $x\in E$ and $r>0$, we set $B(x,r)=\{y\in E : d(x,y)<r\}$. Let $C(E)$ be the set of nonempty compact subsets of E. Let A and B be two elements of $C(E)$. We define the Hausdorff distance between A and B by

$$d_H(A,B)=\max\{\rho(A,B),\rho(B,A)\}\qquad\text{where } \rho(A,B)=\sup\{d(x,B) : x\in A\}.$$

A multi-function $F:E\to E$ is a map from E to the set of nonempty subsets of E. The range of F is $F(E)=\cup_{x\in E}F(x)$.

The multi-function $F:E\to E$ is continuous at $x_0\in E$ if, for every $\varepsilon>0$, there exists $\beta>0$ such that if $d(x_0,x)<\beta$, then $d_H(F(x),F(x_0))<\varepsilon$. The multi-function F is called continuous on E if it is continuous at every point of E.

The multi-function $F:E\to E$ is called compact if it is continuous and the closure of its range $\overline{F(E)}$ is a compact subset of E.

An element x of E is said to be a fixed point of the multi-function $F:E\to E$ if $x\in F(x)$. We denote by $\mathrm{Fix}\,(F)$ the set of fixed points of F.

Let F and G be two compact multi-functions from E to E. We define the Hausdorff distance between F and G by

$$d_H(F,G)=\sup\{d_H(F(x),G(x)) : x\in E\}.$$

Let $\varepsilon>0$ and F and G be two compact multi-functions. We say that F and G are ε-near if $d_H(F,G)<\varepsilon$.

2. Fix-finite approximation property for self-compact maps.

First we define the Schauder mapping.

Lemma 2.1. *For $i=1,\dots,n$, let C_i be a closed nonempty convex subset of a metrizable locally vector space (E,d). Set $C=\cup_{i=1}^{i=n}C_i$. Let*

$\varepsilon > 0$ and let K be a nonempty compact subset of C. Then there exist a finite polyhedron P contained in C and a continuous map $\pi_\varepsilon : K \to P$ which is $\frac{1}{2}\,\varepsilon$-near to Id_K.

Proof. Let $\varepsilon > 0$ and let K be a nonempty compact subset of $C = \cup_{i=1}^{i=n} C_i$. Replacing the norm by the distance in the first part of the proof of Baillon-Rallis's theorem [**1**], then there exist $q \in \mathbf{N}^*$ such that ${\frac{1}{2}}^{q-1} < \varepsilon$ and a finite number of points $x_1, \ldots, x_m$ of C and real numbers $\eta_{x_i} \in \left]0, {\frac{1}{2}}^{2q+3}\right[$ with $i = 1, \ldots, m$ satisfying the following two properties: $(*)$

(i) $K \subset \cup_{i=1}^{i=m} B(x_i, \eta_{x_i})$,

(ii) for all subsets $\{x_{i_l}, \ldots, x_{i_k}\}$ of $\{x_1, \ldots, x_m\}$ satisfying $\cap_{j=l}^{j=k} B(x_{i_j}, \eta_{x_{i_j}}) \cap K \neq \varnothing$, then $\operatorname{conv}\{x_{i_l}, \ldots, x_{i_k}\} \subset C$.

For $i = 1, \ldots, m$, let $\phi_i : K \to \mathbf{R}$ be the real continuous function defined by setting

$$\phi_i(x) = \max(0, \eta_{x_i} - d(x_i, x)) \quad \text{for all } x \in K.$$

Since for every $x \in K$ there exists $i \in \{1, \ldots, m\}$ such that $x \in B(x_i, \eta_{x_i})$, then $d(x_i, x) < \eta_{x_i}$. Hence $\phi_i(x) > 0$. Therefore, $\sum_{i=1}^{i=m} \phi_i(x) > 0$ for all $x \in K$. Now for $i = 1, \ldots, m$, we can define a continuous function by setting

$$\psi_i(x) = \frac{\phi_i(x)}{\sum_{i=1}^{i=m} \phi_i(x)} \quad \text{for all } x \in K.$$

We define the Schauder mapping $\pi_\varepsilon : K \to E$ by

$$\pi_\varepsilon(x) = \sum_{i=1}^{i=m} \psi_i(x) x_i \quad \text{for all } x \in K.$$

Let

$$Q = \left\{ \{x_{i_l}, \ldots, x_{i_k}\} \subset \{x_1, \ldots, x_m\} : \bigcap_{j=l}^{j=k} B(x_{i_j}, \eta_{x_{i_j}}) \cap K \neq \varnothing \right\}$$

and

$$P = \bigcup_{\{x_{i_l}, \ldots, x_{i_k}\} \in Q} \operatorname{conv}\{x_{i_l}, \ldots, x_{i_k}\}.$$

For each $x \in K$ we set $I(x) = \{i \in \{1, \dots, m\} : \psi_i(x) > 0\}$. Then $\pi_\varepsilon(x) = \sum_{i \in I(x)} \psi_i(x) x_i$ for all $x \in K$. It follows from $(*)$ that $\pi_\varepsilon(K) \subset P \subset C$.

If $i \in I(x)$, then $d(x_i, x) < \eta_{x_i} < \frac{1}{2}^{2q+3}$. From [**7**, p. 206], we have $B(0, \frac{1}{2}^{2q+3}) \subset A(q+1, q+1)$, hence $p_{q+1}(x_i - x) < \frac{1}{2}^{q+1}$, for all $x \in K$ and $i \in I(x)$. Therefore,

$$p_{q+1}(\pi_\varepsilon(x) - x) \leq \sum_{i \in I(x)} \psi_i(x) p_{q+1}(x_i - x) < \frac{1}{2^{q+1}} \quad \text{for all } x \in K.$$

From [**7**, p. 206] we have $A(q+1, q+1) \subset B(0, \frac{1}{2}^q)$, then

$$d(\pi_\varepsilon(x), x) < \frac{1}{2^q} < \frac{1}{2}\varepsilon \quad \text{for all } x \in K.$$

Thus π_ε is $\frac{1}{2}\varepsilon$-near to Id_K. $\square$

Now we are in a position to state our generalization of Baillon-Rallis's theorem [**1**].

Theorem 2.2. *For $i = 1, \dots, n$, let C_i be a nonempty closed convex subset of a metrizable locally convex vector space (E, d). Set $C = \cup_{i=1}^{i=n} C_i$. Let $\varepsilon > 0$ and D be a compact subset of E containing C. Then for every continuous map $f : D \to C$ there exists a continuous map $g : D \to C$ which is ε-near to f and has only a finite number of fixed points. In particular, any finite union of nonempty closed convex sets in a metrizable locally convex vector space satisfies the* FFAP *for the family of compact self-maps.*

Proof. Let $f : D \to C$ be a continuous map and $\varepsilon > 0$ be given. Since $K = \overline{f(D)}$ is a compact set, then by Lemma 2.1 there exist a finite polyhedron P contained in C and a continuous map $\pi_\varepsilon : K \to P$ which is $\frac{1}{2}\varepsilon$-near to Id_K. Set $f_\varepsilon = \pi_\varepsilon \circ f : D \to C$. Then f_ε is a continuous map $\frac{1}{2}\varepsilon$-near to f. Since P is a finite polyhedron, then it is a compact ANR [**2**]. By Theorem 1.1, for $\frac{1}{2}\varepsilon > 0$, there exists $\varepsilon > 0$ such that for any two continuous maps $h, u : P \to P$ δ-near and a δ-homotopy $j_t : P \to P$ with $j_0 = h|_P$ and $j_1 = u|_P$, there exists a $\frac{1}{2}\varepsilon$-homotopy $g_t : D \to P$ such that $g_0 = h$ and $g_1 = u$ and $g_{t_{|P}} = j_t$ for all $t \in [0, 1]$.

From [**3**, p. 40] for $\frac{1}{2}\delta > 0$, there exists $\lambda > 0$ such that if $\phi, \psi : P \to P$ are two continuous maps λ-near, then ϕ and ψ are $\frac{1}{2}\delta$-homotopic.

Since $f_\varepsilon|_P$ is a continuous self-map, then by Hopf's construction [**3, 5**] there exists a continuous map $h : P \to P$ which is λ-near to $f_\varepsilon|_P$ and has only a finite number of fixed points. Hence, from [**3**, p. 40] h and $f_\varepsilon|_P$ are $\frac{1}{2}\delta$-homotopic. Let $(h_t)_{(t\in[0,1])}$ be this $\frac{1}{2}\delta$-homotopy between h and $f_{\varepsilon|P}$ and define a new homotopy $(j_t)_{t\in[0,1]}$ by setting

$$j_t = \begin{cases} h_{2t} & \text{if } 0 \le t < \dfrac{1}{2}, \\ h_{2t-2} & \text{if } \dfrac{1}{2} \le t \le 1. \end{cases}$$

Therefore, $(j_t)_{t\in[0,1]}$ is a δ-homotopy satisfying $j_0 = j_1 = f_{\varepsilon|P}$ and $j_{1/2} = h$. Hence, by Theorem 1.1 there exists a $\frac{1}{2}\varepsilon$-homotopy $g_t : D \to P$ such that $g_0 = g_1 = f_\varepsilon$ and $g_{t|_P} = j_t$ for all $t \in [0,1]$. We set $g_{1/2} = g$. Then $g : D \to P$ is a compact map and $\operatorname{Fix}(g) = \operatorname{Fix}(h)$. Indeed, if $g(x) = x$, hence $x \in P$ and $g(x) = h(x) = x$. So $\operatorname{Fix}(g) \subset \operatorname{Fix}(h)$. On the other hand, $h = g|_P$. Then $\operatorname{Fix}(h) \subset \operatorname{Fix}(g)$. Thus g is $\frac{1}{2}\varepsilon$-near to f_ε and has only a finite number of fixed points. We claim that g is ε-near to f because

$$d(f(x), g(x)) \le d(f(x), f_\varepsilon(x)) + d(f_\varepsilon(x), g(x)) < \varepsilon$$
$$\text{for all } x \in D.$$

□

As consequences of Theorem 2.1, we obtain the following corollaries.

Corollary 2.3. *Any nonempty convex subset C of a metrizable locally convex vector space (E, d) satisfies the* FFAP *for every compact map $f : C \to C$.*

Proof. Let $\varepsilon > 0$ and $f : C \to C$ be a compact map. Then $\overline{f(C)}$ is a compact subset of E. By an argument similar to the proof of Lemma 2.1 for $\varepsilon > 0$ there exist a finite number of points $x_1, \dots, x_n$

of C and a Schauder mapping $\pi_\varepsilon : f(C) \to \operatorname{conv}\{x_1, \dots, x_n\}$ such that $d(\pi_\varepsilon(y), y) < \frac{1}{2}\varepsilon$ for all $y \in f(C)$. Set $f_\varepsilon = \pi_\varepsilon \circ f$. So $f_\varepsilon : C \to \operatorname{conv}\{x_1, \dots, x_n\}$ is a compact map and $d(f_\varepsilon(x), f(x)) < \frac{1}{2}\varepsilon$, for all $x \in C$. By Theorem 2.2 there exists a continuous map $g : C \to \operatorname{conv}\{x_1, \dots, x_n\}$ which is $\frac{1}{2}\varepsilon$-near to f_ε and has only a finite number of fixed points. We claim that the map g is ε-near to f because

$$d(f(x), g(x)) \le d(f(x), f_\varepsilon(x)) + d(f_\varepsilon(x), g(x)) < \varepsilon$$
$$\text{for all } x \in C.$$

□

Corollary 2.4. *Any nonempty open subset U of a metrizable locally convex vector space (E, d) satisfies the* FFAP *for compact maps $f : U \to U$.*

Proof. Let $\varepsilon > 0$ and $f : U \to U$ a compact map. Since U is an open subset of E, then for all $x \in U$ there exists $r(x) \in \left]0, \frac{1}{2}\varepsilon\right[$ such that $B(x, r(x)) \subset U$.

Since $\overline{f(U)}$ is a compact subset of E then there exists a finite number of points $x_1, \dots, x_n$ of U such that $\overline{f(U)} \subset \cup_{i=1}^{i=n} \overline{B}(x_i, r(x_i)) \subset U$. By Theorem 2.2 there exists a continuous map $g : U \to \cup_{i=1}^{i=n} \overline{B}(x_i, r(x_i))$ which is ε-near to f and has only a finite number of fixed points. □

3. Fix-finite approximation property for continuous self multi-functions. First we give the definition of an n-function.

Definition 3.1. Let E be a metrizable locally convex vector space and $F : E \to E$ be a multi-function. The multi-function F is called an n-function if there exist n continuous maps $f_i : E \to E$ where $i = 1, \dots, n$ such that $F(x) = \{f_1(x), \dots, f_n(x)\}$ for all $x \in E$ and $f_i(x) \ne f_j(x)$ for all $x \in E$ and $i, j = 1, \dots, n$ with $i \ne j$.

In the following lemma we study the FFAP for the family of n-functions.

Proposition 3.2. *Let C_i be a nonempty convex compact subset of a metrizable locally convex vector space for $i = 1, \dots, m$, then $\cup_{i=1}^{i=m} C_i$ satisfies the* FFAP *for any n-function*

$$F : \bigcup_{i=1}^{i=m} C_i \longrightarrow \bigcup_{i=1}^{i=m} C_i.$$

Proof. Set $C = \cup_{i=1}^{i=m} C_i$. Let $\varepsilon > 0$ and $F : C \to C$ be an n-function. Then there exist n continuous maps $f_i : C \to C$ such that $F(x) = \{f_1(x), \dots, f_n(x)\}$ for all $x \in C$ and $f_i(x) \neq f_j(x)$ for all $x \in C$ and $i, j = 1, \dots, n$ with $i \neq j$.

For all $i, j = 1, \dots, n$ with $i \neq j$ we define

$$\delta_{(i,j)}(F) = \min\{d(f_i(x), f_j(x)) : x \in C\}.$$

As each f_i is continuous for all $i = 1, \dots, n$ and C is compact, then for each $i, j = 1, \dots, n$ with $i \neq j$, we have $\delta_{(i,j)}(F) > 0$. Therefore,

$$\delta(F) = \min\{\delta_{(i,j)}(F) : i, j = 1, \dots, n,\ i \neq j\} > 0.$$

For a given $\varepsilon > 0$, we set $\lambda = \min(\frac{1}{2}\,\delta(F), \frac{1}{2}\,\varepsilon)$. By Theorem 2.2, for each $i = 1, \dots, n$, there exists a map $g_i : C \to C$ which is λ-near to f_i and has only a finite number of fixed points. Let $G : C \to C$ be the multi-function defined by $G(x) = \{g_i(x), \dots, g_n(x)\}$ for all $x \in C$.

Claim 1. *The multi-function G is an n-function. Indeed, if there exist $x_0 \in C$ and $i, j = 1, \dots, n$ with $i \neq j$ such that $g_i(x_0) = g_j(x_0)$, then*

$$d(f_i(x_0), f_j(x_0)) \leq d(f_i(x_0), g_i(x_0)) + d(f_j(x_0), g_j(x_0)) < 2\lambda.$$

Therefore, $\delta_{(i,j)}(F) < \delta(F)$. This is a contradiction and our claim is proved.

Claim 2. *The multi-function G is ε-near to F. Indeed, for all $i = 1, \dots, n$ and for every $x \in C$, we have $d(f_i(x), g_i(x)) < \frac{1}{2}\,\varepsilon$. Then $d_H(F, G) < \varepsilon$.*

Claim 3. *The multi-function G has only a finite number of fixed points. Indeed,* $\mathrm{Fix}\,(G) = \cup_{i=1}^{i=n}\mathrm{Fix}\,(g_i)$ *and for all $i = 1, \dots, n$, the map g_i has only a finite number of fixed points.* $\square$

Now we recall some definitions concerning n-valued continuous multi-functions.

Definition 3.3. Let X and Y be two Hausdorff topological spaces. A multi-function $F : X \to Y$ is said to be n-valued if for all $x \in X$, the subset $F(x)$ of Y consists of n points.

Definition 3.4. Let X and Y be two Hausdorff topological spaces, and let $F : X \to Y$ be an n-valued continuous multi-function. Then we can write $F(x) = \{y_1, \dots, y_n\}$ for all $x \in X$. We define a real function γ on X by

$$\gamma(x) = \inf\{d(y_i, y_j) : y_i, y_j \in F(x),\ i, j = 1, \dots, n,\ i \neq j\} \quad \text{for all } x \in X,$$

and the gap of F by

$$\gamma(F) = \inf\{\gamma(x) : x \in X\}.$$

Remark 3.5. Since the multi-function F is continuous then the function γ is also continuous [**5**, p. 76]. If X is compact, then $\gamma(F) > 0$.

Before giving Theorem 3.7 we recall the following lemma due to Schirmer [**8**].

Lemma 3.6. *Let X and Y be compact Hausdorff topological spaces. If X is path and simply connected and $F : X \to Y$ is an n-valued continuous multi-function, then F is an n-function.*

Now we give the main result in this paper.

Theorem 3.7. *Let C_i be a nonempty convex compact subset of a metrizable locally convex vector space for $i = 1, \dots, m$ such that*

$\cap_{i=1}^{i=m} C_i \neq \varnothing$ or $C_i \cap C_j = \varnothing$ for $i \neq j$, then $\cup_{i=1}^{i=m} C_i$ satisfies the FFAP *for any n-valued continuous multi-function $F : \cup_{i=1}^{i=m} C_i \to \cup_{i=1}^{i=m} C_i$.*

Proof. Let $\varepsilon > 0$ and $F : \cup_{i=1}^{i=m} C_i \to \cup_{i=1}^{i=m} C_i$ be an n-valued continuous multi-function. For the proof we distinguish the following two cases.

First case. $C_i \cap C_j = \varnothing$ for $i, j = 1, \dots, m$ and $i \neq j$. We have $F|_{C_i} : C_i \to \cup_{i=1}^{i=m} C_i$ is an n-valued continuous multi-function for $i = 1, \dots, m$. From Lemma 3.6, the multi-function $F|_{C_i}$ is an n-function for $i = 1, \dots, m$. Therefore, for each $i \in \{1, \dots, m\}$, there exist n continuous maps $f_{i_j} : C_i \to \cup_{i=1}^{i=m} C_i$ such that $F(x) = \{f_{i_1}(x), \dots, f_{i_n}(x)\}$ for all $x \in C_i$. Now for each $j \in \{1, \dots, n\}$ we can define a continuous map $h_j : \cup_{i=1}^{i=m} C_i \to \cup_{i=1}^{i=m} C_i$ by $h_j(x) = f_{i_j}(x)$ if $x \in C_i$. It follows that for all $x \in \cup_{i=1}^{i=m} C_i$ we have $F(x) = \{h_1(x), \dots, h_n(x)\}$. Thus, the multi-function F is an n-function. By Proposition 3.2 there exists an n multi-function $G : \cup_{i=1}^{i=m} C_i \to \cup_{i=1}^{i=m} C_i$ which is ε-near to F and has only a finite number of fixed points.

Second case. $\cap_{i=1}^{i=m} C_i \neq \varnothing$. It follows from Proposition 3.2 that $\cup_{i=1}^{i=m} C_i$ satisfies the FFAP for any n-valued continuous multi-function. $\square$

As a particular case of Theorem 3.7, we obtain the following

Corollary 3.8. *If C_1 and C_2 are two nonempty convex compact subsets of a metrizable locally convex vector space, then $C_1 \cup C_2$ satisfies the* FFAP *for any n-valued continuous multi-function $F : C_1 \cup C_2 \to C_1 \cup C_2$.*

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University Cadi Ayyad, FST Beni-Mellal, Department of Mathematics, P.O. Box 523, Beni-Mellal, Morocco
E-mail address: stouti@yahoo.com