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# A NOTE ON A CLASS OF RINGS FOUND AS $G_a$-INVARIANTS FOR LOCALLY TRIVIAL ACTIONS ON NORMAL AFFINE VARIETIES

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ABSTRACT. This paper concerns the type of ring that can be realized as a ring of invariants for a locally trivial $G_a$-action on a normal, affine variety.

Results involving ideal-transforms and a counterexample to the problem of Zariski are utilized to achieve an example of a locally trivial action on a normal, affine variety of dimension 4 that has a nonfinitely generated ring of invariants. This would also yield yet another example of a $G_a$-action on an affine variety that can be written locally as a translation but does not admit an equivariant trivialization.

**1. Introduction.** The main result of this paper is to show that a class of rings can be realized as rings of invariants for additive group actions. The background is Hilbert's fourteenth problem, which asks the following: "Let $k$ be an algebraically closed field and $x_1, \dots, x_n$ algebraically independent elements over $k$. Let $L$ be a subfield of $k(x_1, \dots, x_n)$ containing $k$. Is the ring $L \cap k[x_1, \dots, x_n]$ finitely generated over $k$?" [**13**, p. 1]. Of particular interest is the case in which this intersection is the ring of invariants for a group action.

We first introduce some notation that will be used throughout the paper. Let $k$ be an algebraically closed field of characteristic 0. We say that a $k$-algebra is affine if it is finitely generated as a $k$-algebra and that it is a normal domain if it is an integral domain that is integrally closed in its quotient field. Let $G_a = (k, +)$ denote the additive group on $k$. By an affine variety we will mean an irreducible, closed subset of $k^n$ with respect to the Zariski topology. If $X \subseteq k^n$ is an affine variety, then when $G_a$ act as automorphisms of the affine $k$-domain $k[X]$, it is well known that the associated $k$-homomorphism $k[X] \to k[X, t]$ is equivalent to a locally nilpotent $k$-derivation $D : k[X] \to k[X]$. That is, for a $G_a$-action $\sigma : G_a \times X \to X$, where for each $t \in G_a$, $\sigma_t \in \operatorname{Aut} X$,

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the associated $\hat{\sigma} : k[X] \to k[X,t]$ defined by $\hat{\sigma}(P)(x) = P(\sigma_t(x))$ for $P \in k[X]$ and $x \in X$, can be realized in terms of a locally nilpotent derivation

$$\hat{\sigma}(P) = \sum_{i \geq 0} \frac{t^i}{i!} D^i(P),$$

where $D^i(P)$ represents $i$ compositions of $D$ so that $D^1 = D$ and $D^0$ is the identity. In turn, $D$ can be represented in terms of $\hat{\sigma}$

$$D = \frac{\hat{\sigma}(P) - P}{t}\Big|_{t=0}.$$

In short, $G_a$-actions on an affine variety $\sigma : G_a \times X \to X$ are in one-to-one correspondence with locally nilpotent derivations $D : k[X] \to k[X]$ on the corresponding coordinate ring. The kernel of $D$, $\ker D$, equals the ring of invariants, $k[X]^{G_a}$, for the $G_a$-action.

Progress towards solving the fourteenth problem of Hilbert includes the generalization known as the problem of Oscar Zariski: "Let $k$ be an algebraically closed field and $k[a_1, \dots, a_n]$ an affine normal domain. Let $L$ be a subfield of $k(a_1, \dots, a_n)$ that contains $k$. Is the ring $k[a_1, \dots, a_n] \cap L$ finitely generated over $k$?" Zariski answered in the affirmative when the transcendence degree of $L$ over $k$, $\operatorname{tr} d._k L$, is less than or equal to 2. This implies that any ring of invariants for a $G_a$-action on $X = \mathbf{C}^n$ with $n \leq 3$ ($\mathbf{C}$ the set of complex numbers) must be finitely generated.

Rees gave a counterexample to the problem of Zariski when $\operatorname{tr} d._k L = 3$ [**16**]. Nagata later provided the first counterexamples to the original fourteenth problem itself [**11, 12**].

More recent work towards finding examples of nonfinitely generated rings of $G_a$-invariants for actions on $\mathbf{C}^n$ began with that of Roberts [**17**]. This eventually led to the construction of a $G_a$-action on $k^5$, where $k$ is assumed only to be a field of characteristic zero, for which the associated ring of invariants in $k[x_1, x_2, x_3, x_4, x_5]$ is nonfinitely generated [**3**]. Another recent breakthrough under the same assumption, that $k$ is a field of characteristic zero only, shows that every triangular derivation of $k[x_1, x_2, x_3, x_4]$ has a kernel that is finitely generated [**4**].

An action $\sigma : G_a \times X \to X$ is said to be *fixed-point free* or to act *freely* if, for any $x \in X$, $\sigma_t(x) = x$ only when $t = 0$. Let $A_n$ denote the affine $k$-domain $k[a_1, \dots , a_n]$ where $n \geq 1$, and let $\sigma$ be defined by the locally nilpotent derivation $D : A_n \to A_n$. Then $\sigma$ acts freely if and only if the set $\{D(a_1), \dots , D(a_n)\}$ has no common zeros. Such an action $\sigma$ is called *triangular* if $D(a_j) \in A_{j-1}$, for all $j \in \{1, \dots , n\}$, where $A_0$ is taken to be the ground field $k$.

An action $\sigma : G_a \times X \to X$ is said to be *equivariantly trivial* if there exists an affine variety $Y$ so that $X$ is $G_a$-equivariantly isomorphic to $Y \times k$ with the action fixing the first coordinate and acting as a translation of the second. In this case there exists an $s \in k[X]$ for which $D(s) = 1$. Such an $s$ is called a *slice*, or *global slice*. Since for the ring of invariants $k[X]^{G_a}$, $k[X] = k[X]^{G_a}[s]$ and $k[X]^{G_a} = k[Y]$, it follows that the ring of invariants for a $G_a$-action that admits a global slice is necessarily finitely generated over $k$.

In more generality a $G_a$-action is termed *locally trivial* if there exists a set $\{U_i\}_{i\in\Gamma}$ of $G_a$-stable, open (in the Zariski topology) sets $U_i$ for which $X = \cup_{i\in\Gamma} U_i$ and where for each $i \in \Gamma$ there exists a slice $s_i \in k[U_i]$, the ring of regular functions globally defined on $U_i$. Although acting freely is a necessary condition for any locally trivial $G_a$-action (hence for those that are equivariantly trivial), the ring of invariants for a locally trivial $G_a$-action need not be affine. Indeed, in [**5**, Corollary 2.10] a class of locally trivial $G_a$-actions on factorial affine varieties when $k = \mathbf{C}$ are constructed each with a nonfinitely generated ring of invariants. The smallest integer for which there is known to be a triangular, locally trivial $G_a$-action on $\mathbf{C}^n$ which is not equivariantly trivial is $n = 5$. The first such counterexample was given in [**18**].

**2. Ideal-transforms and normal ring extensions.** All rings are assumed to be commutative with identity. As mentioned in the first section, by a *normal domain* we mean an integral domain that is integrally closed in its quotient field. The *derived normal ring* of an integral domain $A$ means the integral closure of $A$ in its quotient field.

In discovering counterexamples to Hilbert's original fourteenth problem, Nagata used a technique for constructing a ring defined by an ideal $I$ of an integral domain $R$.

**Definition 2.1.** If $K(R)$ represents the quotient ring of an integral domain $R$ and $I \subseteq R$ is an ideal, then the *ideal-transform* of $R$ with respect to $I$ (or $I$-transform of $R$) is

$$S(I;R) = \{f \in K(R) \mid fI^m \subseteq R \text{ for some positive integer } m\}.$$

*Remarks.* The $I$-transform of $R$ is an integral domain in $K(R)$ that contains $R$. If $R$ is a Noetherian normal domain, then the codimension or height of $I$, $\operatorname{ht}(I) \geq 2$ implies that $S(I;R) = R$. Since $\operatorname{ht}(I) = 0$ implies that $S(I;R) = K(R)$ for a Noetherian normal domain, the only $I$-transform of interest comes for the case when $I$ is of height 1. Note that the height of an ideal that is not prime is understood to be the smallest height of the primes which contain $I$ [**6**, p. 225].

Work by Nagata [**14**] and Ogoma [**15**] on the types of rings that may be realized as an $I$-transform of a normal affine domain, yielded the following result.

**Proposition 2.1.** *A ring $R$ over a Noetherian domain $B$ has the form $\tilde{A} \cap L$ with the derived normal ring $\tilde{A}$ of an affine domain $A$ over $B$ and with a quotient field $L$ over $B$ if and only if $R$ is the $I$-transform of the derived normal ring $\tilde{C}$ of an affine domain $C$ over $B$ for an ideal $I$ of $\tilde{C}$* [**15**, Corollary 2.4].

The following results will also be needed to prove the theorem in the next section.

**Proposition 2.2.** *If $R$ is a normal domain and $T$ is a multiplicatively closed subset of $R$ with $0 \notin T$, then $T^{-1}R$ is a normal domain.*

For proof, see [**7**, Lemma 5.63], [**9**, Theorem VIII.5.8] or the more general result of [**6**, Proposition 4.13].

**Proposition 2.3.** *If $N$ is a normal domain and $x$ is transcendental over $K(N)$, then $N[x]$ is a normal domain.*

*Proof.* We need to show that $N[x]$ is an integral domain integrally closed in its quotient field. For $a, b \in N[x] - \{0\}$, the leading terms of

$a$ and $b$, $a_d x^d$ and $b_f x^f$, respectively, are nonzero in $N[x]$. Since $N$ is a domain, $a_d b_f x^{d+f} \neq 0$ so $ab \neq 0$. Therefore, $N[x]$ is a domain.

Let $c \in K(N)(x)$ be a zero for $g(T) \in (N[x])[T]$ and assume the leading coefficient of $g(T)$ is 1. Since $g(T) \in (K(N)[x])[T]$, it follows that $c$ is in the integral closure of $K(N)[x]$ in the quotient field $K(N)(x)$ of $K(N)[x]$. Since $K(N)[x]$ is a principal ideal domain, hence a unique factorization domain, it is integrally closed and so $c \in K(N)[x]$. By [**1**, Chapter 5], $N[x]$ is integrally closed in $K(N)[x]$ and so $c \in N[x]$. □

**Lemma 2.4.** *If $N_1$ and $N_2$ are normal domains and $S$ is a ring for which $S = N_1 \cap N_2$, then $S$ is a normal domain.*

*Proof.* Clearly $S$ is an integral domain. For $i \in \{1, 2\}$, since $S \subseteq N_i$ implies that $K(S) \subseteq K(N_i)$, it follows that if $a$ is in the integral closure of $S$ in $K(S)$, then $a$ is in the integral closure of $N_i$ in $K(N_i)$. Since $N_i$ is integrally closed, $a \in N_1 \cap N_2 = S$. □

**3. A nonfinitely generated ring of invariants.** Assume now that $k$ is an algebraically closed field of characteristic 0. A nonzero element $h$ of a ring $R$ is said to be *regular* if it is not a zero divisor. That is, if $h \cdot x = 0$ for an $x \in R$, then $x = 0$. If $R$ is an integral domain over $k$, then the notation $\operatorname{tr} d._k R$ represents the transcendence degree of the quotient field for $R$, $K(R)$, over $k$.

**Theorem 3.1.** *Let $A$ be an affine normal domain over $k$. Let $L$ be a field such that $k \subseteq L \subseteq K(A)$. Set $R = L \cap A$. Then there exists a normal, affine variety $X \subseteq k^n$ such that $k[X]$ is a proper ring extension of $R$ and a locally trivial $G_a$-action $\sigma : G_a \times X \to X$ such that $R = k[X]^{G_a}$, the ring of invariants for the action.*

*Proof.* Since $A$ is an affine normal domain, it equals its derived normal ring $\tilde{A}$. Since $L$ is assumed to be a subfield of a field containing $A$, Proposition 2.1 can be applied to $R = A \cap L$. Thus there exists an affine domain $C$ for which $R$ is the ideal-transform of the derived normal domain $\tilde{C}$ with respect to an ideal $I \subset \tilde{C}$. Since $C$ is an affine domain, its integral closure in its quotient field, $\tilde{C}$ is also affine [**6**, Corollary 13.13]. So assume $N = \tilde{C}$ is the normal affine domain with ideal $I \subset N$

such that $R = S(I; N)$.

We can choose $L$ to be $K(R)$, the quotient field of $R$. The reason is since $R \subset L$, and $L$ is a field, $K(R) \subseteq L$, which implies that $A \cap K(R) \subseteq A \cap L = R$. Since $R \subset A \cap K(R)$, it follows that $R = A \cap K(R)$. Additionally, $k \subseteq K(R) \subseteq K(A)$, so Proposition 2.1 can be applied and we have $R = S(I; N)$, where $N$ and $I$ are defined as before.

In [**10**, Theorem 2.6] it is shown that an ideal-transform such as $S(I; N)$ equals $S(J; N)$ where $J = (h_1, h_2)N$ for regular elements $h_1, h_2 \in N$. Additionally, it is shown that

$$R = N_{h_1} \cap N_{h_2},$$

where $N_{h_i}$ is the localization of $N$ at the multiplicatively closed set $\{1, h_i, h_i^2, \dots h_i^m, \dots\}$ for $i = 1, 2$. Since $N$ is a normal domain, by Proposition 2.2, each localization $N_{h_i}$ is a normal domain. Therefore, by Lemma 2.4, $R$ is a normal domain as well. By the definition of ideal-transform, $N \subseteq R \subseteq K(N)$, so $K(N) = K(R)$.

Let $y_1$ and $y_2$ be transcendental elements over $K(N)$. Define

$$S = \frac{N[y_1, y_2]}{y_1 h_1 + y_2 h_2 - 1}.$$

For $i = 1, 2$, let $g_i$ represent the residue class of $y_i$. Then $S = N[g_1, g_2]$ and $g_1 h_1 + g_2 h_2 = 1$ in $S$.

Let $i, j \in \{1, 2\}$ be unequal. For any $x \in (N[(1/h_i), g_j] \cap K(N))$, $x = g_j^m n_m + \cdots + g_j n_1 + n_0$, where each $n_l \in N[1/h_i]$ and $n_m \neq 0$. Since $x \in K(N)$ and $g_j$ is transcendental over $K(N)$, this forces $m$ to be 0, so that $x \in N[1/h_i]$. Therefore,

$$N\left[\frac{1}{h_i}\right] = N\left[\frac{1}{h_i}, g_j\right] \cap K(N).$$

Since $g_i$ equals $(1 - g_j h_j)/h_i$ in $S[1/h_i]$, it follows that $g_i \in N[(1/h_i), g_j]$. Therefore, $S[1/h_i] \subseteq N[(1/h_i), g_j]$, which implies that

$$(*) \qquad S\left[\frac{1}{h_i}\right] = N\left[\frac{1}{h_i}, g_j\right].$$

**Claim.** $S = S[1/h_1] \cap S[1/h_2]$.

The proof of this claim can be accomplished by showing that $N[(1/h_1), g_2] \cap N[(1/h_2), g_1] \subseteq N[g_1, g_2]$. If $a \in N[(1/h_1), g_2] \cap N[(1/h_2), g_1]$, then $ah_1^p \in N[g_2]$ and $ah_2^q \in N[g_1]$ for some positive integers $p$ and $q$. Since $g_1h_1 + g_2h_2 = 1$, in the expansion of $a = a(g_1h_1 + g_2h_2)^{p+q}$, every term will contain either the factor $ah_1^p$ or $ah_2^q$, so every term in this expansion for $a$ is in $N[g_1, g_2]$.

Therefore,

$$\begin{aligned} R &= N\left[\frac{1}{h_1}\right] \cap N\left[\frac{1}{h_2}\right] \\ &= \left(N\left[\frac{1}{h_1}, g_2\right] \cap K(N)\right) \cap \left(N\left[\frac{1}{h_2}, g_1\right] \cap K(N)\right) \\ &= \left(S\left[\frac{1}{h_1}\right] \cap K(N)\right) \cap \left(S\left[\frac{1}{h_2}\right] \cap K(N)\right) \\ &= S \cap K(N) = S \cap K(R), \end{aligned}$$

and so

$$R = S \cap K(R) \subseteq S.$$

Since $K(S)$ contains two elements $g_1$ and $g_2$ transcendental over $K(N)$, it follows that $K(S) \neq K(N) = K(R)$. Therefore, $R \neq S$, and so $S$ is a proper ring extension of $R$. Since, for $i \in \{1, 2\}$, $N_{h_i} = N[1/h_i]$ is a normal domain, and for $j = 3 - i$, $g_j$ is transcendental over $K(N) = K(N[1/h_i])$, by Proposition 2.3, $N[(1/h_i), g_j]$ is a normal domain. So, since $S[1/h_i] = N[(1/h_i), g_j]$ is a normal domain, it follows that $S = S[1/h_1] \cap S[1/h_2]$ is a normal domain that is a normal affine domain over $k$, hence a coordinate ring $k[X]$ of a normal affine variety $X$. Note that, since $S = N[g_1, g_2]$, it follows that $\operatorname{tr} d._k S = \operatorname{tr} d._k N + 1$.

Define a $k$-derivation $D : S = k[X] \to k[X]$, by the rule $D(N) = 0$, $D(g_1) = -h_2$ and $D(g_2) = h_1$. Clearly, $D$ is locally nilpotent, so it defines the $G_a$-action $\sigma : G_a \times X \to X$. Since $(h_1, h_2) \subseteq (\operatorname{im} D \cap \ker D)$ generates the unit ideal in $k[X]$, the zero-set $Z(h_1, h_2)$ is empty. Therefore, $U_1 = X - Z(h_1)$ and $U_2 = X - Z(h_2)$ are quasi-affine varieties that cover $X$ and are $G_a$-stable. Since $s_1 = g_2/h_1$ and $s_2 = -g_1/h_2$ in $K(S)$ are local slices defined on $U_1$ and $U_2$, respectively, $\sigma$ is locally trivial.

To finish the proof of the theorem, we need to show that $R = \ker D$. Denote $\ker D$ by $C_0$. Note that, since $N \subseteq C_0$ and $K(N) = K(R)$, it follows that $K(R) \subseteq K(C_0)$. Also, by [**2**], $C_0 = K(C_0) \cap S$. Since $R = K(R) \cap S$, we will be done if it can be shown that $K(R) = K(C_0)$. It will be enough to show that $K(C_0) \subset K(N)$.

We know that if $D^e$ is the extension of $D$ to $S[1/h_1]$ by the quotient rule for derivations, then $D^e$ is locally nilpotent and $\ker D^e = C_0[1/h_1]$ by [**2**]. Therefore, by [**19**, Proposition 2.1] $S[1/h_1] = C_0[(1/h_1), s_1]$. Since $C_0[(1/h_1), s_1] = C_0[(1/h_1), g_2]$ and by $(*)$ $S[1/h_1] = N[(1/h_1), g_2]$, it follows that $K(S) = K(N)(g_2) = K(C_0)(g_2)$. If $x \in K(C_0)$, then $x = p/q$, where $D(p) = D(q) = 0$. In other words, $p, q \in C_0[g_2]$ and $\deg_{g_2} p = \deg_{g_2} q = 0$. Since $x = p/q \in K(N)(g_2)$ and $\deg_{g_2} p = \deg_{g_2} q = 0$, it follows that $x \in K(N)$. $\square$

**Corollary 3.2.** *There is a dimension four normal affine variety $X \subseteq \mathbf{C}^n$ and a locally trivial $G_a$-action $\sigma : G_a \times X \to X$ for which the ring of invariants $C_0 = \mathbf{C}[X]^{G_a} \subset \mathbf{C}[X]$ is nonfinitely generated.*

*Proof.* In [**13**, pp. 57–60], the counterexample due to Rees of a nonfinitely generated $k$-domain $R$ where $k$ is of arbitrary characteristic is shown to be of the form $A \cap L$ where $A$ is a normal $k$-domain and $L = K(R)$. If we assume $k = \mathbf{C}$, then Theorem 3.1 can be applied and there is a proper ring extension $S$ of $R$ that is a normal affine domain and $S = \mathbf{C}[X]$ is the coordinate ring for a normal affine variety $X \subseteq \mathbf{C}^n$ for some $n$. Also, there exists a locally trivial $G_a$-action $G_a \times X \to X$ for which $R$ is the ring of invariants.

From the proof of Theorem 3.1, we know that there exists a normal affine $\mathbf{C}$-domain $N$ for which $S = N[g_1, g_2]$ and an ideal $I \subset N$ such that $R$ equals the $I$-transform of $N$. By the remarks made after Definition 2.1, since $N$ is a Noetherian normal domain, $R \neq N$, since $N$ is affine over $\mathbf{C}$ and $R$ is not, and $R \neq K(R)$ (since this would lead to the false statement $K(N) \subset N[g_1, g_2]$). Consequently, it must be that $\operatorname{ht} I = 1$. Additionally, from the proof of Theorem 3.1, $\operatorname{tr} d._{\mathbf{C}} S = \operatorname{tr} d._{\mathbf{C}} N + 1$. From [**13**], we know that $\operatorname{tr} d._{\mathbf{C}} R = 3$. Since $K(R) = K(N)$, it follows that $\operatorname{tr} d._{\mathbf{C}} S = 4$, and so by [**8**, Propositions I.17, I.18A], $S$ is the coordinate ring for a normal affine variety $X$ of dimension 4. $\square$

The $G_a$-action defined in Corollary 3.2, as with any locally trivial $G_a$-action with a nonfinitely generated ring of invariants, is another example of a locally trivial $G_a$-action that is not equivariantly trivial. Results from [**3**] and [**18**] leave some unanswered questions.

**Question 1.** Is there a $G_a$-action on $\mathbf{C}^4$ that has a nonfinitely generated ring of invariants?

**Question 2.** Is there a triangular $G_a$-action on $\mathbf{C}^4$ that is locally trivial but not equivariantly trivial?

Also,

**Question 3.** What is the smallest value of $n$ for which $X \subseteq \mathbf{C}^n$ where $X$ is the normal, affine variety defined in Corollary 3.2?

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