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ON POINT VALUES OF BOEHMIANS

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ABSTRACT. The notion of a value of a Boehmian at a point, its properties and the concept of regular delta sequences are available in the literature. Let $\mathcal{E}$ be a Banach space. Denote by $C(\mathbf{R}^N, \mathcal{E})$ the space of all continuous $\mathcal{E}$-valued functions on $\mathbf{R}^N$ and by $\mathcal{D}(\mathbf{R}^N)$ the space of all infinitely differentiable real-valued functions with compact support in $\mathbf{R}^N$. Using $C(\mathbf{R}^N, \mathcal{E})$ as the top space and the usual delta sequences from $\mathcal{D}(\mathbf{R}^N)$ we can construct in a canonical way a Boehmian space $\mathcal{B} = \mathcal{B}(\mathbf{R}^N, \mathcal{E})$. In 1994, Piotr Mikusiński and Mourad Tighiouart asserted that, if for every representation $[f_n/\phi_n]$ of $F \in \mathcal{B}$ where (ϕ_n) is regular delta sequence we have $\lim_{n\to\infty} f_n(x_0) = a$, then $F(x_0) = a$. In this paper we shall point out that the proof of this theorem contains an error, produce a counterexample to show that the theorem is not valid and obtain modified conditions for its validity. As a consequence we shall also show that if $F = [f_n/\phi_n]$ where (ϕ_n) is a delta sequence made of one function and if $\lim_{n\to\infty} f_n(x_0) = a$ for every such representation, then F need not have a value at x_0. Incidentally, this observation settles one of the questions raised by Piotr Mikusiński and Mourad Tighiouart.

1. Introduction. The concept of Boehmians was first introduced and studied in [**3**]. Various spaces of Boehmians and their properties were available in the literature, for example, see [**2**]. In [**4**] the notion of a value of a Boehmian at a point was defined and its properties were studied. Further related results can be found in [**1**]. An equivalent condition for a Boehmian to have a value at a point is claimed in Theorem 2.4 of [**4**]. We shall first point out that the proof of this theorem contains an error. In addition we shall produce a counterexample and establish that this theorem is not valid. We shall also explain how the hypothesis of this theorem must be modified for

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its validity. We shall also provide yet another characterization for a Boehmian to have a value at a point. Further, we shall answer one of the questions posed in the concluding remarks of [**4**].

In Section 2 we give the preliminary definitions and results. In Section 3 we shall point out the error in the proof of Theorem 2.4 of [**4**] and give a counterexample to show that the theorem is not valid. In Section 4 we shall provide a characterization for a Boehmian to have a value at a point and use this to modify the hypothesis of Theorem 2.4 of [**4**] and prove its validity under the new hypothesis. In the last section we shall prove that in the definition of the value of a Boehmian at a point it is not sufficient to assume that $\lim_{n\to\infty} f_n(x_0) = a$ exists for every representation $[f_n/\phi_n]$ such that (ϕ_n) is a *delta sequence made of one function.* We shall also obtain a few more results in the context of the existence of point values for Boehmians.

2. Preliminaries.

The space $\mathcal{B}$ of Boehmians is defined in a canonical way as in [**2**] having $C(\mathbf{R}^N)$, the space of all complex-valued continuous functions defined on $\mathbf{R}^N$, as the top space and delta sequences from $\mathcal{D}(\mathbf{R}^N)$, the Schwartz testing function space of all complex-valued infinitely differentiable functions defined on $\mathbf{R}^N$ with compact support. For simplicity we shall always be concerned with functions having complex values instead of functions having values in a Banach space as in [**4**].

Let $\operatorname{supp}\phi$ denote the support of ϕ, the closure of $\{x\in\mathbf{R}^N/\phi(x)\neq 0\}$. Let $\mathcal{D}'(\mathbf{R}^N)$ denote the Schwartz distribution space. The convolution of two functions f and g is defined as $(f*g)(x) = \int_{\mathbf{R}^N} f(t)g(x-t)\,dt$ whenever the integral exists. The convolution of a distribution $f \in \mathcal{D}'(\mathbf{R}^N)$ and a testing function $\phi \in \mathcal{D}(\mathbf{R}^N)$ is defined as $(f*\phi)(x) = \langle f(t), \phi(x-t)\rangle$.

Definition 2.1. A sequence (ϕ_n) in $\mathcal{D}(\mathbf{R}^N)$ is called a delta sequence if

(i) $\displaystyle\int_{\mathbf{R}^N} \phi_n(x)\,dx = 1$ for all n.

(ii) There is an M such that $\displaystyle\int_{\mathbf{R}^N} |\phi_n(x)|\,dx \le M$ for all n.

(iii) $\operatorname{supp}\phi_n \to \{0\}$ as $n\to\infty$.

Definition 2.2 [**4**]. Let F be a Boehmian, and let $x \in \mathbf{R}^N$. If, for every representation $F = [f_n/\phi_n]$, we have $\lim_{n\to\infty} f_n(x) = a$, then we say that F has a value a at x and denote this by $F(x) = a$.

Definition 2.3. A delta sequence (ϕ_n) is called ρ-regular if, for every multi-index $k = (k_1, k_2, \dots , k_N)$ where $k_1, k_2, \dots , k_N$ are nonnegative integers, there exists a positive constant M_k such that

$$(\rho(\phi_n))^{|k|} \int_{\mathbf{R}^N} \left| \frac{\partial^{|k|}}{\partial x^k}\, \phi_n(x) \right| dx \le M_k \quad \text{for all } n$$

where $\rho(\phi) = \inf\{s > 0 \;/ \phi(t) = 0 \text{ if } \|t\| > s\}$ and $|k\,| = k_1 + k_2 + \cdots + k_N$.

What we have defined as a ρ-regular delta sequence here is termed as a regular delta sequence in [**4**].

Definition 2.4. A delta sequence (ϕ_n) is called r-regular if, for every multi-index $k = (k_1, k_2, \dots , k_N)$, there exists a positive constant M_k such that

$$(r(\phi_n))^{|k|} \int_{\mathbf{R}^N} \left| \frac{\partial^{|k|}}{\partial x^k}\, \phi_n(x) \right| dx \le M_k \quad \text{for all } n$$

where $r(\phi) = \sup\{\|x - y\|/x, y \in \operatorname{supp} \phi\}/2$.

Note that, as $r(\phi) \le \rho(\phi)$, we have every ρ-regular delta sequence is r-regular but not conversely. We shall give an example of a delta sequence which is r-regular but not ρ-regular.

Example 2.5. On $\mathbf{R}$ define

$$f_1(t) = \begin{cases} e^{(1/(t^2-1))} & \text{if } |t\,| < 1 \\ 0 & \text{if } |t\,| \ge 1, \end{cases}$$

and let $f(t) = f_1(t)/\int_{\mathbf{R}} f_1(t)\, dt$. Let $\delta_n(t) = n^2 f(n^2 t - n)$. Then (δ_n) is a delta sequence with $\operatorname{supp} \delta_n = [(1/n) - (1/n^2), (1/n) + (1/n^2)]$. Also

$\rho(\delta_n) = (1/n) + (1/n^2)$ and $r(\delta_n) = (1/n^2)$. Now

$$\begin{aligned}(\rho(\delta_n))^k \int_{\mathbf{R}} |\delta_n^{(k)}(t)\,|\,dt &= \Big(\frac{1}{n} + \frac{1}{n^2}\Big)^k (n^2)^{k+1} \int_{\mathbf{R}} |(f^{(k)})(n^2t-n)\,|\,dt \\ &= \Big(\frac{1}{n} + \frac{1}{n^2}\Big)^k (n^2)^k \int_{\mathbf{R}} |f^{(k)}(s)\,|ds \\ &= (n+1)^k C \text{ where } C = \int_{\mathbf{R}} |f^{(k)}(s)\,|ds > 0.\end{aligned}$$

Thus $(\rho(\delta_n))^k \int_{\mathbf{R}} |\delta_n^{(k)}(t)\,|dt$ cannot be bounded by any M_k. So (δ_n) is not ρ-regular. But

$$(r(\delta_n))^k \int_{\mathbf{R}} |\delta_n^{(k)}(t)\,|\,dt = (1/n^2)^k (n^2)^k \int_{\mathbf{R}} |f^{(k)}(s)\,|\,ds = M_k,$$

say. Thus (δ_n) is r-regular. □

We shall now give an example of a delta sequence which is not r-regular. Thus the class of r-regular delta sequences is not the whole of Δ. Thus this example establishes the significance of our Theorem 4.4 later.

Example 2.6. Let f be as in Example 2.5. Let $\phi_n(t) = (n^2/2) \times (f(n^2t-n)+f(n^2t+n))$. Then (ϕ_n) is a delta sequence with $\operatorname{supp}\phi_n = [-(1/n)-(1/n^2), -(1/n)+(1/n^2)] \cup [(1/n)-(1/n^2), (1/n)+(1/n^2)]$ and $r(\phi_n) = (1/n) + (1/n^2)$. Now

$$\begin{aligned}&(r(\phi_n))^k \int_{\mathbf{R}} |\phi_n^{(k)}(t)\,|\,dt \\ &\qquad \geq \Big[\Big(\frac{1}{n}\Big) + \Big(\frac{1}{n^2}\Big)\Big]^k \int_0^\infty |\phi_n^{(k)}(t)\,|\,dt \\ &\qquad = \Big[\Big(\frac{1}{n}\Big) + \Big(\frac{1}{n^2}\Big)\Big]^k \int_0^\infty \frac{(n^2)^{k+1}}{2} |(f^{(k)})(n^2t-n)\,|\,dt\end{aligned}$$

(when $t > 0$, $n^2t + n > 1$ and hence $f^{(k)}(n^2 + n) = 0$ for all k and n)

$$= (n+1)^k/2 \int_0^\infty |f^{(k)}(s)\,|\,ds.$$

Thus, if $k \geq 1$, then $(r(\phi_n))^k \int_{\mathbf{R}} |\phi_n^{(k)}(t)|\,dt$ cannot be bounded (by any M_k). So (ϕ_n) is not r-regular. □

Lemma 2.7. *If (ϕ_n) and (ψ_n) are r-regular delta sequences, then $(\phi_n * \psi_n)$ is also an r-regular delta sequence.*

Proof. Since (ϕ_n) and (ψ_n) are r-regular delta sequences, for every multi-index k, there exist M_k and N_k such that, for all n,

$$(r(\phi_n))^{|k|} \int_{\mathbf{R}^N} |(\partial^{|k|}/\partial x^k)\phi_n(x)\,|\,dx \leq M_k$$

and

$$(r(\psi_n))^{|k|} \int_{\mathbf{R}^N} |(\partial^{|k|}/\partial x^k)\psi_n(x)\,|dx \leq N_k.$$

Now

$$\begin{aligned}
&(r(\phi_n * \psi_n))^{|k|} \int_{\mathbf{R}^N} \Big|\frac{\partial^{|k|}}{\partial x^k}(\phi_n * \psi_n)(x)\Big|\,dx \\
&\quad\leq (r(\phi_n) + r(\psi_n))^{|k|} \int_{\mathbf{R^N}} \Big|\frac{\partial^{|k|}}{\partial x^k}(\phi_n * \psi_n)(x)\Big|\,dx \\
&\quad\leq 2^{|k|-1}((r(\phi_n))^{|k|} + (r(\psi_n))^{|k|}) \int_{\mathbf{R}^N} \Big|\frac{\partial^{|k|}}{\partial x^k}(\phi_n * \psi_n)(x)\Big|\,dx \\
&\quad\leq 2^{|k|-1}(r(\phi_n))^{|k|} \int_{\mathbf{R}^N} \Big|\Big(\Big(\frac{\partial^{|k|}}{\partial x^k}\,\phi_n\Big) * \psi_n\Big)(x)\Big|\,dx \\
&\qquad + 2^{|k|-1}(r(\psi_n))^{|k|} \int_{\mathbf{R}^N} \Big|\Big(\phi_n * \Big(\frac{\partial^{|k|}}{\partial x^k}\,\psi_n\Big)\Big)(x)\Big|\,dx \\
&\quad\leq 2^{|k|-1}(r(\phi_n))^{|k|} \int_{\mathbf{R}^N} \Big|\frac{\partial^{|k|}}{\partial x^k}\,\phi_n(x)\Big|\,dx \int_{\mathbf{R}^N} |\psi_n(x)\,|\,dx \\
&\qquad + 2^{|k|-1}(r(\psi_n))^{|k|} \int_{\mathbf{R}^N} |\phi_n(x)\,|\,dx \int_{\mathbf{R}^N} \Big|\frac{\partial^{|k|}}{\partial x^k}\,\psi_n(x)\Big|\,dx \\
&\quad\leq 2^{|k|-1}(M_k N_0 + M_0 N_k).
\end{aligned}$$

This completes the proof as the other properties of a delta sequence can be easily verified. □

3. Counter Example. In Theorem 2.4 of [**4**] it has been claimed that a Boehmian F has a value at a point x_0 if and only if, for all representation $[f_n/\phi_n]$ of F with ρ-regular delta sequence (ϕ_n), $\lim_{n\to\infty} f_n(x_0)$ exists. There is an error in the proof. On page 1045 of [**4**], the existence of a sequence of positive numbers $\gamma_1, \gamma_2, \ldots$ satisfying

$$\|f_{p_n}(x) - f_{p_n}(0)\| < \varepsilon \quad \text{whenever } \|x\| < \gamma_n$$

is proved. To be more specific, as γ_n depends on f_{p_n}, we shall write γ_n as γ_{p_n}. However, the array of inequalities on page 1046 [**4**] assumes

$$\sup_{\|x\|\leq\gamma_{p_n}} \|f_{q_n}(x) - f_{q_n}(0)\| < \varepsilon$$

where $\{q_n\}$ is a subsequence of $\{p_n\}$. This assumption is not valid because we only know that

$$\sup \|f_{q_n}(x) - f_{q_n}(0)\| < \varepsilon$$

when the supremum is taken over $\{x \ / \ \|x\| \leq \gamma_{q_n}\}$ and not over $\{x \ / \ \|x\| \leq \gamma_{p_n}\}$.

We shall now give an example of a Boehmian F not admitting a value at $x_0 = 0$ even though for every representation $[f_n/\phi_n]$ of F with ρ-regular delta sequence (ϕ_n) we have $\lim_{n\to\infty} f_n(0) = 0$.

Example 3.1. Let $A_n = [-(3/n) - (1/n^2), -(3/n)]$, $n = 1, 2, \ldots$, and let $A = \cup_{n=1}^{\infty} A_n$. Define

$$G(t) = \begin{cases} -|t|\sqrt{|t|} & \text{if } t \in A \\ |t|\sqrt{|t|} & \text{if } t \notin A. \end{cases}$$

Then G is a locally integrable function and hence G can be identified in $\mathcal{D}'(\mathbf{R})$. Let G' be the distributional derivative of G. Let F be the Boehmian representing G'. Let (ϕ_n) be any ρ-regular delta sequence. Then there exists M_1 such that $\rho(\phi_n)\int_{\mathbf{R}} |\phi_n'(t)|\, dt \leq M_1$ for all n. We know $[(G' * \phi_n)/\phi_n]$ is a representation of F. Now

$$\begin{aligned}
|(G' * \phi_n)(0)\,| = \,|(G * \phi_n')(0)\,| &= \Big|\int_{-\rho(\phi_n)}^{\rho(\phi_n)} G(-t)\phi_n'(t)\,dt\Big| \\
&\le \int_{-\rho(\phi_n)}^{\rho(\phi_n)} |G(-t)\,|\,|\phi_n'(t)\,|\,dt \\
&= \int_{-\rho(\phi_n)}^{\rho(\phi_n)} |t\,|\sqrt{|t\,|}\,|\phi_n'(t)\,|\,dt \\
&\le \sqrt{\rho(\phi_n)}\ \rho(\phi_n)\int_{\mathbf{R}} |\phi_n'(t)\,|\,dt \\
&\le \sqrt{\rho(\phi_n)}\ M_1.
\end{aligned}$$

Since $\rho(\phi_n) \to 0$, we have $(G' * \phi_n)(0) \to 0$ as $n \to \infty$.

Let $[f_n/\psi_n]$ be any representation of F with ρ-regular delta sequence (ψ_n). Then $[f_n/\psi_n] = [(G' * \phi_n)/\phi_n]$. This implies $f_n * \phi_m = (G' * \phi_m) * \psi_n$. Taking limit as $m \to \infty$, we get $f_n(x) = (G' * \psi_n)(x)$ for all x. Since (ψ_n) is ρ-regular, as above, we can prove that $(G' * \psi_n)(0) \to 0$ as $n \to \infty$. Therefore, $f_n(0) \to 0$ as $n \to \infty$. Thus for every representation $[f_n/\psi_n]$ of F, with ρ-regular delta sequence (ψ_n), we have $\lim_{n\to\infty} f_n(0) = 0$.

Let f be as in Example 2.5. Then $f \in C^\infty$, $\int_{\mathbf{R}} f(t)\,dt = 1$, $\operatorname{supp} f = [-1,1]$, $f(t) \ge 0$ for all $t \in \mathbf{R}$ and $f(-x) = f(x)$. Moreover $f'(t) \ge 0$ if $t < 0$ and $f'(t) \le 0$ if $t \ge 0$. Let $\delta_n(t) = n^2 f(n^2 t - 3n)$. Then (δ_n) is a delta sequence with $\operatorname{supp}\delta_n = [(3/n) - (1/n^2), (3/n) + (1/n^2)]$.

Now

$$\begin{aligned}
(G' * \delta_n)(0) = (G * \delta_n')(0) &= \int_{(3/n)-(1/n^2)}^{(3/n)+(1/n^2)} G(-t)\delta_n'(t)\,dt \\
&= n^4 \int_{(3/n)-(1/n^2)}^{(3/n)+(1/n^2)} G(-t) f'(n^2 t - 3n)\,dt \\
&= n^2 \int_{-1}^{1} G\Big(-\Big(\frac{s+3n}{n^2}\Big)\Big) f'(s)\,ds \\
&= n^2 \int_{-1}^{0} \Big(\frac{s+3n}{n^2}\Big)^{3/2} f'(s)\,ds + n^2 \int_0^1 -\Big(\frac{s+3n}{n^2}\Big)^{3/2} f'(s)\,ds
\end{aligned}$$

$$= n^2 \int_{-1}^{1} \Big(\frac{s+3n}{n^2}\Big)^{3/2} |f'(s)|\, ds$$
$$\geq n^2 \Big(\frac{1}{n}\Big)^{3/2} \int_{-1}^{1} |f'(s)|\, ds$$
$$= \sqrt{n}\, C$$

where
$$C = \int_{-1}^{1} |f'(s)|\, ds.$$

Since $C > 0$, $(G' * \delta_n)(0) \to \infty$ as $n \to \infty$. But $[(G' * \delta_n)/\delta_n]$is a representation of F. Thus F cannot have a value at 0.

4. Characterizations for the existence of point values of Boehmians.

Lemma 4.1. *If a Boehmian F has a value at x_0 and if $[f_n/\phi_n]$ is any representation of F, then $\{f_n\}$ is equicontinuous at x_0.*

Proof. Let $F(x) = a$ and let $\varepsilon > 0$. Then, by Theorem 2.5 of [**4**], there exists $\delta_0 > 0$ and $n_0 \in \mathbf{N}$ such that $|f_n(x) - a| \leq \varepsilon/2$ whenever $\|x - x_0\| \leq \delta_0$ and for all $n \geq n_0$. Thus $|f_n(x) - f_n(x_0)| \leq |f_n(x) - a| + |f_n(x_0) - a| < \varepsilon$ for all $n \geq n_0$ and for all x such that $\|x - x_0\| \leq \delta_0$. As the functions $f_i(x)$, $1 \leq i < n_0$, are continuous at x_0, there exist $\delta_i > 0$, $1 \leq i < n_0$, such that $|f_i(x) - f_i(x_0)| \leq \varepsilon$ whenever $\|x - x_0\| \leq \delta_i$ for $1 \leq i < n_0$. Let $\delta = \min\{\delta_i / \, 0 \leq i < n_0\}$. Then $\delta > 0$ and $|f_n(x) - f_n(x_0)| \leq \varepsilon$ for all n and for all x with $\|x - x_0\| \leq \delta$. This implies that $\{f_n\}$ is equicontinuous at x_0. □

Lemma 4.2. *If F is a Boehmian and $x_0 \in \mathbf{R}^N$ and if there exists at least one representation $[f_n/\delta_n]$ of F satisfying*

i) $\lim_{n\to\infty} f_n(x_0)$ *exists*

ii) $\{f_n\}$ *is equicontinuous at x_0,*

then F has a value at x_0.

Proof. Let $\lim_{n\to\infty} f_n(x_0) = a$. Let $[g_n/\phi_n]$ be any representation of F. Since $[f_n/\delta_n] = [g_n/\phi_n]$ we have $g_n * \delta_m = f_m * \phi_n$. Let $a_{m,n} =$

$(g_n * \delta_m)(x_0)$. Then $a_{m,n} = (f_m * \phi_n)(x_0)$. Since $\lim_{m\to\infty}(g_n * \delta_m) = g_n$ pointwise for all n, we have

$$\lim_{m\to\infty} a_{m,n} = g_n(x_0) \quad \text{for all } n.$$

Since (ϕ_n) is a delta sequence, there is a positive number M such that $\int_{\mathbf{R}^N} |\phi_n(t)|\,dt \le M$ for all n. Let $\varepsilon > 0$ be given. Since $\{f_n\}$ is equicontinuous at x_0, there exists $\delta > 0$ such that $|f_m(x_0 - t) - f_m(x_0)| < \varepsilon/(2M)$ whenever $\|t\| < \delta$ for all m. By definition of delta sequence, there exists n_1 such that $\rho(\delta_n) < \delta$ for all $n \ge n_1$. Since $f_m(x_0) \to a$, as $m \to \infty$ there exists n_2 such that $|f_m(x_0) - a| < \varepsilon/2$ for all $m \ge n_2$.

Let $n_0 = \max\{n_1, n_2\}$. If $m, n \ge n_0$, then we have

$$\begin{aligned} |a_{m,n} - a| &= |(f_m * \phi_n)(x_0) - a| \\ &\le |(f_m * \phi_n)(x_0) - f_m(x_0)| + |f_m(x_0) - a| \\ &\le \int_{\mathbf{R}^N} |f_m(x_0 - t) - f_m(x_0)|\ |\phi_n(t)|\,dt + \frac{\varepsilon}{2} \\ &\le \frac{\varepsilon}{2M} M + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This implies that the double limit $\lim_{m,n\to\infty} a_{m,n} = a$. Hence we have $\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} = a$. Thus by (1) we get $\lim_{n\to\infty} g_n(x_0) = a$. □

Remark 4.3. If F is a Boehmian having at least one representation $[f_n/\delta_n]$ satisfying (i) and (ii) of Lemma 4.2, then by Lemma 4.1 every representation of F satisfies (i) and (ii) of Lemma 4.2. Thus the existence of one representation of F satisfying (i) and (ii) of Lemma 4.2 is a necessary and sufficient condition for F to have a value at x_0.

Another necessary and sufficient condition is given in the following theorem. This theorem shows how the hypothesis of Theorem 2.4 of [**4**] must be modified.

Theorem 4.4. *Let F be a Boehmian and $x_0 \in \mathbf{R}^N$. Then F has a value at x_0 if and only if, for all representation $[f_n/\phi_n]$ of F with r-regular delta sequence (ϕ_n), $\lim_{n\to\infty} f_n(x_0)$ exists.*

Proof. If F has a value at x_0, then, by definition, for all representation $[f_n/\phi_n]$ of F, $\lim_{n\to\infty} f_n(x_0)$ exists. To prove the converse, let $[f_n/\phi_n]$ be any representation of F with r-regular delta sequence (ϕ_n). Such delta sequences actually exist. Indeed we can always represent Boehmians with a corresponding delta sequence being ρ-regular, see Lemma 2.3 of [**4**]. Then $\lim_{n\to\infty} f_n(x_0)$ exists, say a. Suppose $\{f_n\}$ is not equicontinuous at x_0. Then there exists $\varepsilon > 0$ such that, for any $\eta > 0$, there are infinitely many n such that

$$\sup_{\|x-x_0\|\le\eta} |f_n(x) - f_n(x_0)\,| > 3\varepsilon. \tag{1}$$

We can further prove that, for the same ε, for any $\eta > 0$, there are infinitely many n such that

$$\sup_{\|x-x_0\|\le\eta} |f_n(x) - a\,| > 2\varepsilon. \tag{2}$$

Indeed, since $\lim_{n\to\infty} f_n(x_0) = a$, there exists n_0 such that

$$|f_n(x_0) - a\,| < \varepsilon \quad \text{for all } n \ge n_0. \tag{3}$$

As there are infinitely many n satisfying (1), using (3) and the fact $|f_n(x) - a\,| \ge |f_n(x) - f_n(x_0)\,| - |f_n(x_0) - a\,|$ we get (2). Thus, by (2), there exists a sequence $n_1 < n_2 < \dots$ of positive integers and a sequence $x_1, x_2, \dots$ of points of $\mathbf{R}^N$ such that

$$|f_{n_k}(x_k) - a\,| > 2\varepsilon \quad \text{and} \quad \|x_k - x_0\| < \frac{1}{k} \quad \text{for } k = 1, 2, \dots . \tag{4}$$

As each f_{n_k} is continuous at x_k, there exists $\gamma_k > 0$ such that

$$|f_{n_k}(x) - f_{n_k}(x_k)\,| < \varepsilon \quad \text{whenever } \|x - x_k\| < \gamma_k. \tag{5}$$

We may assume γ_k decreases to 0. Let

$$f_1(t) = \begin{cases} e^{(1/(\|t\|^2-1))} & \text{if } \|t\| < 1 \\ 0 & \text{if } \|t\| \ge 1, \end{cases}$$

and let $f(t) = (f_1(t))/\int_{\mathbf{R}^N} f_1(t)\,dt$. Let $\delta_k(t) = 1/(\gamma_k^N) f[(t/\gamma_k) - ((x_k - x_0)/\gamma_k)]$. Then $\operatorname{supp}\delta_k = B_{\gamma_k}(x_k - x_0)$, the ball of radius γ_k

and center $(x_k - x_0)$, and (δ_k) is a delta sequence with $r(\delta_k) = \gamma_k$. Also it is easy to see that (δ_k) is r-regular. Consider

$$\Big|\int_{\mathbf{R}^N} (f_{n_k}(x) - f_{n_k}(x_k))\delta_k(x - x_0)\,dx\Big| \\ \leq \int_{\mathbf{R}^N} |f_{n_k}(x) - f_{n_k}(x_k)|\,\delta_k(x - x_0)\,dx < \varepsilon,$$

because of (5) whenever $x - x_0 \in \operatorname{supp} \delta_k$. That is,

$$\Big|\int_{\mathbf{R}^N} (f_{n_k}(x) - f_{n_k}(x_k))\delta_k(x - x_0)\,dx\Big| < \varepsilon \quad \text{for all } k. \tag{6}$$

Let $\check{\delta_n}(t) = \delta_n(-t)$. Now using (4) and (6), we get

$$\begin{aligned} |(f_{n_k} * \check{\delta_k})(x_0) - a\,| &= \Big|\int_{\mathbf{R}^N} f_{n_k}(x)\delta_k(x - x_0)\,dx - a\Big| \\ &\geq \Big|\int_{\mathbf{R}^N} f_{n_k}(x_k)\delta_k(x - x_0)\,dx - a\Big| \\ &\quad - \Big|\int_{\mathbf{R}^N} (f_{n_k}(x) - f_{n_k}(x_k))\delta_k(x - x_0)\,dx\Big| \\ &\geq |f_{n_k}(x_k) - a\,| - \varepsilon \geq \varepsilon. \end{aligned}$$

That is $|(f_{n_k} * \check{\delta_k})(x_0) - a\,| \geq \varepsilon$ for all k. Thus, $(f_{n_k} * \check{\delta_k})(x_0)$ cannot converge to a. Since (δ_k) is an r-regular delta sequence, $(\check{\delta_k})$, and hence $(\phi_{n_k} * \check{\delta_k})$ are r-regular delta sequences. Now we have a representation $\big[(f_{n_k} * \check{\delta_k})/(\phi_{n_k} * \check{\delta_k})\big]$ of F, with r-regular delta sequence $(\phi_{n_k} * \check{\delta_k})$ such that $(f_{n_k} * \check{\delta_k})(x_0)$ do not converge to a. This is a contradiction. Thus $\{f_n\}$ is equicontinuous. The theorem follows from Lemma 2.7. $\square$

5. Miscellaneous results. In the sequel we shall give two more results on point value of Boehmians.

Theorem 5.1. *If a Boehmian F has a value at each point of a compact set $K \subseteq \mathbf{R}^N$, then for every representation $[f_n/\phi_n]$ of F we have*

i) $\lim_{n\to\infty} f_n(x)$ *exists for all* $x \in K$.

ii) $\{f_n\}$ *is equicontinuous on* K.

Proof. Since F has a value at all points of K, $\lim_{n\to\infty} f_n(x)$ exists for all $x \in K$. To prove $\{f_n\}$ is equicontinuous on K, let $\varepsilon > 0$ be given. By Theorem 4.1 $\{f_n\}$ is equicontinuous at each point x of K. Therefore, for every fixed x, there exists $\delta_x > 0$ such that $|f_n(t) - f_n(x)| \leq \varepsilon/2$ whenever $\|x - t\| \leq 2\delta_x$ for all n. Then $\{B_{\delta_x}(x)\}_{x\in K}$ is an open cover for K having a finite subcover, say $B_{\delta_{x_1}}(x_1), \dots, B_{\delta_{x_k}}(x_k)$. Let $\delta = \min\{\delta_{x_1}, \dots, \delta_{x_k}\}$. Then $\delta > 0$. Let $s, t \in K$ be such that $\|s - t\| < \delta$. Then $s \in B_{\delta_{x_i}}(x_i)$ for some i. Thus $\|s - x_i\| < \delta_{x_i}$, and hence $|f_n(s) - f_n(x_i)| < \varepsilon/2$ for all n. Now $\|s - t\| < \delta$ implies $\|t - x_i\| < 2\delta_{x_i}$, and hence $|f_n(t) - f_n(x_i)| < \varepsilon/2$ for all n. Thus we get $|f_n(s) - f_n(t)| < \varepsilon$ for all n whenever $\|s - t\| < \delta$. Hence $\{f_n\}$ is equicontinuous on K. □

Theorem 5.2. *Let* F *be a Boehmian and* $K \subseteq \mathbf{R}^N$ *a compact set. If* F *has a representation* $[f_n/\phi_n]$ *such that*

i) $\lim_{n\to\infty} f_n(x)$ *exists for all* $x \in K$

ii) $\{f_n\}$ *is equicontinuous on* K.

Then F *has a value at each point of the interior of* K.

Proof. Let $x_0 \in K^0$, the interior of K. Then, by (i), $\lim_{n\to\infty} f_n(x_0)$ exists. Let $\varepsilon > 0$ be given. Since $x_0 \in K^0$, there exists $\delta_1 > 0$ such that $x \in K$ whenever $\|x - x_0\| < \delta_1$. By (ii), there exists $\delta_2 > 0$ such that, for all n, $|f_n(x) - f_n(x_0)| < \varepsilon$ whenever $\|x - x_0\| < \delta_2$ and $x \in K$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and we have $|f_n(x) - f_n(x_0)| < \varepsilon$ whenever $\|x - x_0\| < \delta$. Thus $\{f_n\}$ is equicontinuous at x_0. Hence, by Theorem 4.2, F has a value at x_0. □

Remark 5.3. A delta sequence (ϕ_n), where each ϕ_n has the form $\phi_n(x) = (\alpha_n)^N \phi(\alpha_n x)$, is called a *delta sequence made of one function*. In the concluding remarks of [**4**], it is asked whether the condition in the definition of value of a Boehmian be replaced by the condition "$\lim_{n\to\infty} f_n(x) = a$ for all representation $[f_n/\phi_n]$ with (ϕ_n) made of one function." As every delta sequence made of one function is ρ-regular, the counter example given in Section 3 shows that the answer to this question is negative.

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