ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 2, 2005

THE HYPERSPACES $K(X)$

PATRICIA PELLICER-COVARRUBIAS

ABSTRACT. Let $C(X)$ denote the hyperspace of subcontinua of a continuum X. For $p \in X$, define the hyperspaces $C(p,X) = \{A \in C(X) : p \in A\}$ and $K(X) = \{C(p,X) : p \in X\}$. The hyperspace $K(X)$ is not always a continuum. Conditions under which it is compact, connected, arcwise connected and locally connected are studied. A characterization of hereditarily indecomposable continua is also given.

1. Introduction. All spaces are assumed to be metric. Let X be a continuum. Throughout this paper $C(X)$ will denote the hyperspace of subcontinua of X equipped with the Hausdorff metric, see Definitions 1.6 and 2.1 in [**6**]. Also, for $p \in X$ define the hyperspace $C(p,X) = \{A \in C(X) : p \in A\}$. Further, define $K(X) = \{C(p,X) : p \in X\}$.

It is known that several properties of a continuum X can be determined in terms of the topological properties of $C(X)$, and vice versa. For more information on this subject, we refer the reader to [**6**]. Following this idea, the aim of this paper is to investigate and present some relations between topological properties of a continuum X and those of its hyperspaces $K(X)$.

There are not many results in the literature concerning the hyperspaces $C(p,X)$. For a continuum X and $A \subset X$ in [**5**, Theorem 2], Eberhart proves that the hyperspaces $C(p,X)$ are absolute retracts. In particular, $K(A,X) = \{C(p,X) : p \in A\} \subset C(C(X))$.

In [**8**], the author characterizes, among other classes of continua, the arc and the simple closed curve in terms of their hyperspace $K(X)$. In [**9**] she also characterizes atriodic continua in terms of the hyperspaces $C(p,X)$. Moreover, in [**11**] she determines when the hyperspaces $C(p,X)$ contain n-cells. Finally, in [**3, 4, 10**] retractions and contractibility related to these hyperspaces are studied.

AMS *Mathematics Subject Classification.* Primary 54B20, 54F15.
Key words and phrases. Continuum, hyperspace, property of Kelley, hereditarily indecomposable continuum.
Received by the editors on July 9, 2002, and in revised form on September 27, 2002.

Further in [**2**], Charatonik considers a mapping $F : 2^X \to C(C(X))$ given by $F(A) = \{K \in C(X) : A \subset K\}$. In that paper relations between continuity of F and some topological properties of the continuum X are studied, such as smoothness and connectedness. This mapping is related to a particular mapping that we will use later.

In this paper characterizations are given for continua whose hyperspace $K(X)$ is compact or arcwise connected. Also conditions are given under which $K(X)$ is connected or locally connected. Finally, a characterization of hereditarily indecomposable continua is given.

2. Preliminaries. A *continuum* means a compact, connected metric space. We denote by I the unit interval, by I^∞ the Hilbert cube and by $\mathbf{N}$ the set of all positive integers. By $\mathbf{C}$ we mean the set of all complex numbers, equipped with the natural topology, and by S^1 the unit circle, i.e., $S^1 = \{z \in \mathbf{C} : |z| = 1\}$. A *ray* is a homeomorphic copy of the space $[0, 1)$ with its usual topology.

Further, for a continuum X and $A \subset B \subset X$, we denote by $\mathrm{cl}_B(A)$ the closure of A with respect to B. In case $B = X$, we shall simply write $\mathrm{cl}\,(A)$. Let $\mathrm{diam}\,(A)$ denote the diameter of A. Also $X \approx Y$ means that X is homeomorphic to Y. Moreover, H will denote the Hausdorff metric for $C(X)$ and by H^2 we will mean the Hausdorff metric for $C(C(X))$. Finally, if the continuum X has a metric d, $x \in X$ and A is a closed subset of X, let $d(x, A) = \inf\{d(x, a) : a \in A\}$. Moreover, $N_d(\varepsilon, A)$ denotes the set $\{x \in X : d(x, A) < \varepsilon\}$. In case it is clear the metric we are using, we will simply write $N(\varepsilon, A)$.

Let $A, B \in C(X)$. An *order arc* from A to B is a mapping $\alpha : I \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$ and $\alpha(r) \subsetneq \alpha(s)$ whenever $r < s$, see [**7**].

A *Whitney map* for $C(X)$ is a mapping $\mu : C(X) \to [0, \infty)$ such that $\mu(X) = 1$, $\mu(\{p\}) = 0$ for each $p \in X$ and $\mu(A) < \mu(B)$ whenever $A \subsetneq B$, see [**6**, p. 105]. Similarly, we define a Whitney map for $C(p, X)$ as a mapping $\mu : C(p, X) \to [0, \infty)$ such that $\mu(X) = 1$, $\mu(\{p\}) = 0$ and $\mu(A) < \mu(B)$ whenever $A \subsetneq B$.

Let X be a continuum, and let $x \in X$. Then, X is said to have the *property of Kelley* at x provided that for each sequence $\{x_n\}_{n=1}^\infty$ converging to x, and for each $A \in C(x, X)$, there exists a sequence $\{A_n\}_{n=1}^\infty$ which converges to A and such that $A_n \in C(x_n, X)$ for each

$n \in \mathbf{N}$. Furthermore, X has the property of Kelley if it has the property of Kelley at every point.

A subcontinuum A of a continuum X is said to be *terminal* in X provided that for any $B \in C(X)$ such that $A \cap B \neq \varnothing$, we have that either $A \subset B$ or $B \subset A$. A continuum X is *indecomposable* if it cannot be written as the union of two of its proper subcontinua. Moreover, X is *hereditarily indecomposable* provided every subcontinuum of X is indecomposable.

Let (X, T) be a topological space, and let $\{A_i\}_{i=1}^{\infty}$ be a sequence of subsets of X. We define the *limit inferior* of $\{A_i\}_{i=1}^{\infty}$ and the *limit superior* of $\{A_i\}_{i=1}^{\infty}$ as follows:

(1) $\liminf A_i = \{x \in X :$ for any $U \in T$ such that $x \in U$, $U \cap A_i \neq \varnothing$ for all but finitely many $i\}$.

(2) $\limsup A_i = \{x \in X :$ for any $U \in T$ such that $x \in U$, $U \cap A_i \neq \varnothing$ for infinitely many $i\}$.

We will say that $\lim A_i = A$ if $\limsup A_i = A = \liminf A_i$. For more information on this subject, we refer the reader to [**6**, pp. 20–24].

3. General properties. We devote this section to recall and present some basic results on continua and the hyperspaces $C(p, X)$.

Definition 3.1. Let X be a continuum. We say X is a *triod* provided there exists $K \in C(X)$ such that $X \setminus K$ has at least three components.

Definition 3.2. Let X be a continuum and $A_1, A_2, A_3 \in C(X)$. We say that A_1, A_2 and A_3 form a *weak triod* if $A_1 \cap A_2 \cap A_3 \neq \varnothing$ and $A_i \setminus (A_j \cup A_k) \neq \varnothing$ whenever $\{i, j, k\} = \{1, 2, 3\}$.

Theorem 3.3 [**12**, Theorem 1.8]. *Let X be a continuum, and let $A, B, C \in C(X)$ be such that they form a weak triod. Then X contains a triod.*

Observation 3.4. If X is a continuum such that $C(p, X)$ is an arc, then there exists a *unique* order arc $\alpha : I \to C(p, X)$ from $\{p\}$ to X such that $\alpha(I) = C(p, X)$. By *unique* we mean that if α' is an order arc from $\{p\}$ to X, then $\alpha(I) = \alpha'(I)$. Since order arcs are monotone, any two elements of $C(p, X)$ are comparable.

Lemma 3.5. *Let X be a continuum. If $p \in X$ is such that for any $A, B \in C(p, X)$, A and B are comparable, then $C(p, X)$ is an arc.*

Proof. Let $\alpha_1 : I \to C(p, X)$ be an order arc from p to X. Suppose that $C(p, X)$ is not an arc. Then there exists $K \in C(p, X) \setminus \alpha_1(I)$. Taking an order arc from $\{p\}$ to K and then another one from K to X, we obtain an order arc α_2 from $\{p\}$ to X containing K. Hence we have two order arcs α_1 and α_2 from $\{p\}$ to X such that $\alpha_1(I) \neq \alpha_2(I)$. Then, it is not difficult to see that there exist $s, t \in I$ such that $\alpha_1(s) \setminus \alpha_2(t) \neq \varnothing$ and $\alpha_2(t) \setminus \alpha_1(s) \neq \varnothing$. $\square$

Lemma 3.6. *Let X be a continuum. Then X is hereditarily indecomposable if and only if $C(p, X)$ is an arc for each $p \in X$.*

Proof. Suppose that there exists $p \in X$ such that $C(p, X)$ is not an arc. Then by Lemma 3.5 there exist $K, K' \in C(p, X)$ such that $K \setminus K' \neq \varnothing \neq K' \setminus K$. However, $p \in K \cap K'$, thus $K \cup K' \in C(X)$ and $K \cup K'$ is decomposable. Therefore X is not hereditarily indecomposable.

Conversely, if $C(p, X)$ is an arc for each $p \in X$, let $K \in C(X)$ and suppose that $K = A \cup B$ for some $A, B \in C(X)$. Let $p \in A \cap B$. Using Observation 3.4 we get that either $A \subset B$ or $B \subset A$. Hence $K = A$ or $K = B$. Therefore, K is indecomposable and X is hereditarily indecomposable. $\square$

Lemma 3.7. *Let Y be a compactification of the ray with remainder X. If $p \in X$ is such that $C(p, X)$ is an arc, then $C(p, Y)$ is an arc.*

Proof. Let $A, B \in C(p, Y)$. If $A, B \subset X$, according to Observation 3.4 we get that A and B are comparable. Now, since X is terminal in Y, in any other case it is easy to see that A and B are comparable. Thus, by Lemma 3.5 we conclude that $C(p, Y)$ is an arc. $\square$

As a consequence of Lemma 3.5, we have the following result.

Corollary 3.8. *Let X be an arc. If p is an endpoint of X, then $C(p,X)$ is an arc.*

4. Compactness.

Definition 4.1. Let X be a continuum and $A \subset X$. We define the hyperspace $K(A,X) = \{C(p,X) : p \in A\}$. We will denote $K(X,X)$ simply by $K(X)$.

It is well known that the hyperspaces $C(X)$ are continua, so a natural question that arises, regarding the hyperspaces $K(X)$ is whether they are continua too. We shall see that this is not always the case. In this section we will find conditions for a continuum X, which are equivalent to the compactness of $K(X)$.

Definition 4.2. For a continuum X and $A \subset X$, let $\tau_A : A \to K(A,X)$ be given by $\tau_A(p) = C(p,X)$. For convenience, we will denote τ_X simply by τ.

Observation 4.3. Let X be a continuum and $A \subset X$. Note that the function $\tau_A : A \to K(A,X)$ is bijective.

Lemma 4.4. *If X is a continuum, then the function τ_A^{-1} : $K(A,X) \to A$ is continuous for every $A \subset X$.*

Proof. Let $C(p,X) \in K(A,X)$ and a sequence $\{C(p_n,X)\}_{n=1}^{\infty} \subset K(A,X)$ which converges to $C(p,X)$.

We may suppose that there exists $B \in C(p,X)$ such that the sequence $\{\{p_n\}\}_{n=1}^{\infty}$ converges to B. Then B cannot have more than one point so $B = \{p\}$ and we can conclude that $p_n \to p$. $\square$

Lemma 4.5. *Let X be a continuum, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence which converges to x. Then $\limsup C(x_n,X) \subset C(x,X)$.*

Proof. Let $A \in \limsup C(x_n, X)$. Then for every $k \in \mathbf{N}$, there exist $n_k \in \mathbf{N}$ and $A_{n_k} \in C(x_{n_k}, X)$ such that $H(A, A_{n_k}) < 1/k$. In particular, the sequence $\{A_{n_k}\}_{k=1}^{\infty}$ converges to A. Since $x_{n_k} \in A_{n_k}$ for every k, we get that $x \in A$, thus $A \in C(x, X)$. Hence $\limsup C(x_n, X) \subset C(x, X)$. □

Lemma 4.6. *Let X be a continuum. Then the function τ is continuous if and only if X has the property of Kelley.*

Proof. Suppose that X has the property of Kelley. Let $x \in X$ and a sequence $\{x_n\}_{n=1}^{\infty}$ which converges to x. Since X has the property of Kelley, for every $B \in C(x, X)$ and for every $n \in \mathbf{N}$, there exists $B_n \in C(x_n, X)$ such that the sequence $\{B_n\}_{n=1}^{\infty}$ converges to B. Thus $C(x, X) \subset \liminf C(x_n, X)$. From Lemma 4.5 it follows that $C(x_n, X) \to C(x, X)$. Whence τ is continuous.

Conversely, suppose that τ is continuous and take $x \in X$, $A \in C(x, X)$ and a sequence $\{x_n\}_{n=1}^{\infty}$ which converges to x. Then

$$C(x_n, X) = \tau(x_n) \longrightarrow \tau(x) = C(x, X).$$

Hence, for every $n \in \mathbf{N}$, there exists $A_n \in C(x_n, X)$ such that the sequence $\{A_n\}_{n=1}^{\infty}$ converges to A. In other words X has the property of Kelley at x. □

The following lemma provides necessary and sufficient conditions on a continuum X under which $K(X)$ is compact.

Lemma 4.7. *The following conditions are equivalent for a continuum X:*

i) *$K(X)$ is compact*;

ii) *$\tau : X \to K(X)$ is continuous*;

iii) *X has the property of Kelley and*

iv) *$X \approx K(X)$.*

Proof. Clearly ii) $\Rightarrow$ i) and iv) $\Rightarrow$ i). Moreover, from Lemma 4.4 and Observation 4.3 it follows that i) $\Rightarrow$ ii) and i) $\Rightarrow$ iv). Finally, by Lemma 4.6 conditions ii) and iii) are equivalent. □

5. Local and arcwise connectedness.

Lemma 5.1. *Let X be a continuum and B a locally connected subspace of X. Then the function $\tau_B : B \to K(B, X)$ is continuous.*

Proof. Let $p \in B$ and $\varepsilon > 0$. Since B is locally connected, we can take a connected open subset U of B which contains p and such that $\operatorname{diam}(U) < \varepsilon/2$. Let $q \in U$, $K \in C(p, X)$ and $D = K \cup \operatorname{cl}(U)$. Then clearly $D \subset N(\varepsilon, K)$ and $K \subset N(\varepsilon, D)$. Hence $H(K, D) < \varepsilon$. Now it is easy to see that $D \in C(q, X)$. Thus $C(p, X) \subset N_H(\varepsilon, C(q, X))$. Similarly, $C(q, X) \subset N_H(\varepsilon, C(p, X))$. Therefore $H^2(\tau_B(p), \tau_B(q)) = H^2(C(p, X), C(q, X)) < \varepsilon$. $\square$

Corollary 5.2. *Let X be a continuum and B a locally connected subspace of X. Then $B \approx K(B, X)$. In particular, $K(B, X)$ is locally connected.*

Proof. By Observation 4.3, Lemma 4.4 and Lemma 5.1, we have that τ_B is a homeomorphism. In particular, $K(B, X)$ is locally connected. $\square$

Lemma 5.3. *Let X be a continuum. Then $B \subset X$ is arcwise connected if and only if $K(B, X)$ is arcwise connected.*

Proof. Suppose that $K(B, X)$ is arcwise connected. Then, by means of Lemma 4.4, $\tau_B^{-1}(K(B, X)) = B$ is arcwise connected.

Conversely, suppose that B is an arcwise connected subset of X. Consider $\{C(p, X), C(q, X)\} \subset K(B, X)$ and take an arc A in B which contains p and q. Since A is a locally connected subspace of X, applying Lemma 5.1 we get that τ_A is a mapping. In particular, $\tau_A(A)$ is an arcwise connected subcontinuum of $K(B, X)$, which contains $C(p, X)$ and $C(q, X)$. $\square$

Example 5.4. There exists a continuum X which is not locally connected but such that $K(X)$ is locally connected.

Let $A = \{(0, y) \in \mathbf{R}^2 : y \in [-2, 2]\}$ and $B = \{(x, \sin\frac{1}{x}) \in \mathbf{R}^2 : x \in (0, \pi]\}$. Define $X = A \cup B$. Clearly X is a continuum, though it is not locally connected. Let's analyze $K(X)$.

Since both A and B are locally connected, by Corollary 5.2 we have that $K(A, X)$ is an arc and $K(B, X)$ is a ray. Clearly $K(A, X)$ is closed in $K(X)$. If $K(B, X)$ is not closed, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset B$ converging to x for some $x \in A$ such that the sequence $\{C(x_n, X)\}_{n=1}^{\infty}$ converges to $C(x, X)$. Let $M = \{0\} \times [-1, 1]$. Consider an arc $K \in C(x, X)$ such that neither K is contained in M nor M is contained in K. It is easy to see that K is not a limit point of the sets $C(x_n, X)$, a contradiction. Hence $K(B, X)$ is closed in $K(X)$. Finally, since $K(X)$ is the disjoint union of the closed sets $K(A, X)$ and $K(B, X)$, we get that $K(X)$ is locally connected. Note, however, that $K(X)$ is *not* connected.

Example 5.5. There exists a continuum X which is not locally connected but such that $K(X)$ is connected and locally connected.

Let W be the continuum discussed in Example 5.4 and consider the following equivalence relation in W:

$$(x, y) \sim (z, w) \quad \text{if and only if} \quad \begin{cases} (x, y) = (z, w) & \text{or} \\ (x, y) \in \{(0, -2), (\pi, \sin(1/\pi))\}. \end{cases}$$

Let $X = W/_{\sim}$ and $\rho : W \to X$ be the natural projection. Note that $\rho(A)$ and $\rho(B)$ are locally connected while X is not. Hence, according to Corollary 5.2 we get that $K(A, X)$ is an arc and $K(B, X)$ is a ray.

Consider the point $w \in \rho(A) \cap \rho(B)$ and a small, open arc U containing it. In the last example we saw that $K(X \backslash U)$ is disconnected, whence $K(\rho(A), X) \cap \mathrm{cl}\,(K(\rho(B), X)) = \{C(w, X)\}$. In other words, since $K(X) = K(\rho(A), X) \cup K(\rho(B), X)$, then $K(X)$ is the one-point union of an arc and a ray, joined in such a way that $K(X)$ is a ray. In particular, $K(X)$ is connected and locally connected, though X is not locally connected.

6. Compactifications and connectedness. In this section we will analyze connectedness of the hyperspace $K(Y)$, when Y is a compactification of the ray.

Lemma 6.1. *If any of the following conditions holds*:

i) Y *is a compactification of the ray with remainder* X *and* X *has a point* p *such that* $C(p, X)$ *is an arc, or*

ii) Y *is a continuum and* $p \in Y$ *is such that* $C(p, Y)$ *is an arc,*

and $\{p_n\}_{n=1}^{\infty} \subset Y$ *is a sequence converging to* p, *then* $\lim_{n\to\infty} C(p_n, Y) = C(p, Y)$.

Proof. By Lemma 3.7, i) implies ii), so we will assume that ii) holds. Let $A \in C(p, Y)$ and $\mu : C(Y) \to I$ be a Whitney map.

For every $n \in \mathbf{N}$ take $A_n \in C(p_n, Y)$ such that $\mu(A_n) = \mu(A)$. Suppose that there exists $B \in C(Y)$ such that a subsequence $\{A_{n_i}\}_{i=1}^{\infty}$ of $\{A_n\}_{n=1}^{\infty}$ converges to B. We know that $p_{n_i} \in A_{n_i}$ for every $i \in \mathbf{N}$ and $p_{n_i} \to p$, thence $p \in B$. Thus, by Observation 3.4, A and B are comparable. On the other hand, since μ is continuous, $\mu(A) = \mu(A_{n_i}) \to \mu(B)$. Since μ is strictly monotone, we obtain that $A = B$. Therefore, $A_n \to A$ and $A \in \liminf C(p_n, Y)$. Thus $C(p, Y) \subset \liminf C(p_n, Y)$. According to this and Lemma 4.5, we get that $C(p_n, Y) \to C(p, Y)$. $\square$

Corollary 6.2. *Let* Y *be a compactification of the ray with an arc* X *as remainder. Then* $K(Y)$ *is connected.*

Proof. By Corollary 5.2 we know that $K(X, Y)$ and $K(Y \setminus X, Y)$ are connected. Furthermore, if p is an endpoint of X and $\{p_n\}_{n=1}^{\infty} \subset Y \setminus X$ is a sequence converging to p, using Corollary 3.8 and Lemma 6.1 we have that $C(p_n, Y) \to C(p, Y)$. Thus $C(p, Y) \in \operatorname{cl}(K(Y \setminus X, Y)) \cap K(X, Y)$. Hence $K(Y) = K(X, Y) \cup K(Y \setminus X, Y)$ is connected. $\square$

Proposition 6.3. *Let* Y *be a compactification of the ray with remainder* X. *Then* $K(X, Y)$ *is closed in* $K(Y)$.

Proof. Consider a sequence $\{C(x_n, Y)\}_{n=1}^{\infty} \subset K(X, Y)$ which converges to $C(x, Y)$ for some $x \in Y$. By Lemma 4.4 we have that $x_n = \tau^{-1}(C(x_n, Y)) \to \tau^{-1}(C(x, Y)) = x$. Whence $x \in X$ and $K(X, Y)$ is closed. $\square$

In what follows we shall use the following notation. For a metric space R, we shall denote by $C(R)$ the family of compact, connected and nonempty subsets of R.

Lemma 6.4. *Let Y be a compactification of the ray R with remainder X. If for every point $x \in X$ there exists $A \in C(x, X)$ such that $A \notin \mathrm{cl}\,(C(R))$, then $K(R, Y)$ is closed in $K(Y)$. In particular, $K(Y)$ is not connected.*

Proof. If we suppose that $K(R, Y)$ is not closed in $K(Y)$, then there exists a sequence $\{C(y_n, Y)\}_{n=1}^{\infty} \subset K(R, Y)$ converging to $C(x, Y)$ for some $x \in X$. However, by hypothesis there exists $A \in C(x, X) \setminus \lim C(y_n, Y)$, a contradiction. Thence, $K(R, Y)$ is closed in $K(Y)$. According to this and Proposition 6.3, we can write $K(Y)$ as the disjoint union of the closed, nonempty subsets $K(X, Y)$ and $K(R, Y)$. $\square$

Example 6.5. A compactification Y of the ray, such that the remainder is a simple closed curve and $K(Y)$ is connected.

Let $Y = S^1 \cup \{(1 + \frac{1}{t})e^{it} \in \mathbf{C} : t \in [1, \infty)\}$. Then it is easy to see that Y has the property of Kelley. Thus, by Lemma 4.7, $Y \approx K(Y)$. In particular, $K(Y)$ is connected.

Example 6.6. A compactification Y of the ray, having a simple closed curve S as a remainder and such that $K(Y)$ is not connected.

Consider the space Y in Figure 1. Let $x \in S$ and A be a subarc of $C(x, S)$ which contains the point a of the figure in its interior relative to S. It is easy to see that A cannot be approximated by subcontinua of R. Hence, according to Lemma 6.4, $K(Y)$ is not connected.

Definition 6.7. Let $n \in \mathbf{N}$. An *n-od* is a continuum X with the property that there exists $K \in C(X)$ such that $X \setminus K$ has at least n components. The continuum K is called a *core* of the n-od.

Definition 6.8. Let $n \in \mathbf{N}$. We say that X is a *simple n-od* provided that X is an n-od with the following properties. The core of X is a one-

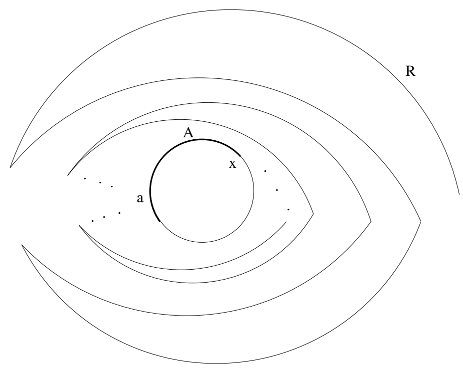

FIGURE 1. Compactification Y of the ray such that $K(Y)$ is not connected.

point set $\{x\}$ and $X \setminus \{x\}$ has exactly n components $A_1, A_2, \ldots, A_n$, which satisfy that $A_i \cup \{x\}$ is an arc for each $i \in \{1, 2, \ldots, n\}$.

Example 6.9. A compactification Y of the ray, with a 4-od as a remainder and such that $K(Y)$ is not connected.

Consider the space Y represented in Figure 2. Note that for every point x in the remainder X, there exists a proper subtriod T of X in such a way that x is contained in the core of T. It is easy to see that T can be chosen in such a way that it cannot be approximated by subcontinua of the ray R. Applying Lemma 6.4 we obtain that $K(Y)$ is not connected.

We have seen that, even when $K(Y)$ is connected for every compactification Y of the ray whose remainder is an arc, this is not always the case for finite graphs. In fact, we shall see that, if the remainder is almost any finite graph, then $K(Y)$ is not connected.

Definition 6.10. Let G be a finite graph, $x \in G$ and $n \in \mathbf{N} \setminus \{1, 2\}$. A point x is a *ramification point* of G, if x is the core of a simple n-od in G. In case n is maximal we say that the order of x in G is n and we will write ord$\,(x, G) = n$.

Recall that a *noose* is the one-point union of an arc A and a simple closed curve S such that $A \cap S$ is an endpoint of A.

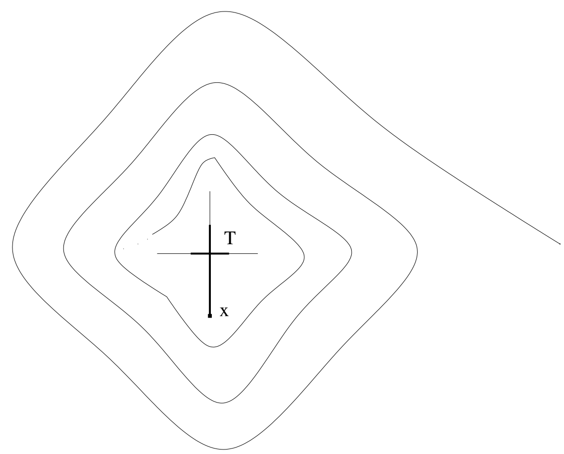

FIGURE 2. Compactification Y of the ray such that $K(Y)$ is not connected.

Proposition 6.11. *Let G be a finite graph which is different from an arc, a simple closed curve, a simple triod and a noose. Then all the points of G are contained in the core of a triod.*

Proof. Let $x \in G$. We will analyze two cases.

Case 1. G has a unique ramification point r.

It is easy to see that if $\operatorname{ord}(r, G) = 3$, then G is either a simple triod or a noose. Thus we may suppose that $\operatorname{ord}(r, G) = n \geq 4$ and $\{r\}$ is the core of a simple n-od W for some $n \geq 4$, i.e., $W = \cup\{A_i : i \in \{1, 2, \dots, n\}\}$, where each A_i is an arc, r is an endpoint of each A_i and $A_i \cap A_j = \{r\}$ whenever $i \neq j$. Take an arc A in G such that $x \in A$ and $A \cap W = \{a_i\}$, where $a_i \in A_i$ for some $i \in \{1, 2, \dots, n\}$ (A may be degenerate).

Let $K = A \cup A_i$. Then since $n \geq 4$, we have that $(W \cup A) \setminus K = W \setminus A_i$ has at least three components. Hence $W \cup A$ is a triod with core K and x is contained in the core of the triod.

Case 2. G has at least two ramification points.

In this case we can consider an arc L in G, containing exactly two ramification points, r, s of G in such a way that these are the endpoints of L. Thus, r and s are the core of a simple m-od M and of a simple n-od N, respectively. We can suppose that $M \cap N = \varnothing$. Thus $M = \cup \{M_i : i \in \{1,2,\dots,m\}\}$ and $N = \cup \{N_j : j \in \{1,2,\dots,n\}\}$, where each M_i and N_j is an arc. Besides $M_k \cap M_l = \{r\}$, $N_k \cap N_l = \{s\}$ and $M_v \cap N_w = \varnothing$ whenever $k \neq l$, $v \in \{1,2,\dots,m\}$ and $w \in \{1,2,\dots,n\}$.

On the other hand, it is easy to see that L intersects M and N in some $M_\alpha \setminus \{r\}$ and in some $N_\beta \setminus \{s\}$ and only in one for each of them. Take an arc A in G (it may be degenerate) such that $x \in A$ and $A \cap (M \cup N \cup L) = \{a\}$ where $a \in M_i$ for some $i \in \{1,2,\dots,m\}$ or $a \in N_j$ for some $j \in \{1,2,\dots,n\}$. We will suppose that $a \in M_i$ for some $i \in \{1,2,\dots,m\}$.

Now let $K = A \cup M_i \cup L$ and $T = M \cup N \cup L \cup A$. We shall see that T is a triod, with core K, and $x \in K$. We have

$$\begin{aligned} T \setminus K &= (M \cup N \cup L \cup A) \setminus (A \cup M_i \cup L) \\ &= (M \cup N) \setminus (M_i \cup L) \\ &= \left(\bigcup \{M_j \cup N_k : j \notin \{i,\alpha\}, k \neq \beta\} \right) \setminus \{r,s\}. \end{aligned}$$

Since r and s are ramification points, $m \geq 3$ and $n \geq 3$. Then $T \setminus K$ has at least three components, which means that T is a triod with core K. Finally it is clear that x is contained in the core of the triod. ◻

Theorem 6.12. *Let Y be a compactification of the ray with remainder X, $x \in X$ and a sequence $\{x_n\}_{n=1}^\infty \subset Y \setminus X$ which converges to x. If x is contained in the core of a triod T in X, then* $\liminf C(x_n, Y) \subsetneq C(x, Y)$.

Proof. Take $K \in C(x,T)$ such that $T \setminus K = A_1 \cup A_2 \cup A_3$ where $A_i \neq \varnothing$ for every $i \in \{1,2,3\}$ and A_i and A_j are separated whenever $i \neq j$. For every $i \in \{1,2,3\}$ take $a_i \in A_i$ and $\delta > 0$ such that $\delta < d(a_i, K \cup A_j \cup A_k)$ whenever $\{i,j,k\} = \{1,2,3\}$.

If we suppose that the conclusion of the theorem does not hold, using Lemma 4.5 it follows that $\liminf C(x_n, Y) = C(x, Y)$. Therefore, there exist $n \in \mathbf{N}$ and $C_i \in C(x_n, Y)$ such that $H(A_i \cup K, C_i) < \delta/2$ for every $i \in \{1, 2, 3\}$. Letting $T' = C_1 \cup C_2 \cup C_3$, then it is easy to see that T' is a weak triod in $Y \setminus X$. However, according to Theorem 3.3, the ray contains a triod, which is absurd. □

We present the following consequences of the previous results.

Corollary 6.13. *Let X be a continuum such that each point of X is contained in the core of a triod. If Y is a compactification of the ray R, with remainder X, then $K(Y)$ is not connected.*

Proof. Let $\{C(y_n, Y)\}_{n=1}^{\infty} \subset K(R, Y)$ be a sequence converging to $C(y, Y)$ for some $y \in Y$. Then by Theorem 6.12, $y \notin X$ and $C(y, Y) \in K(R, Y)$. Therefore $K(R, Y)$ is closed in $K(Y)$. According to this and Proposition 6.3, we can write $K(Y)$ as the union of the two nonempty and disjoint closed sets $K(X, Y)$ and $K(R, Y)$. □

The following results follow from Proposition 6.11 and Corollary 6.13.

Corollary 6.14. *Let G be a finite graph which is neither an arc nor a simple closed curve, nor a simple triod, nor a noose. If Y is a compactification of the ray R with remainder G, then $K(Y)$ is not connected.*

Corollary 6.15. *Let $n \in \mathbf{N} \setminus \{1\}$ and X be an n-cell. If Y is a compactification of the ray R with remainder X, then $K(Y)$ is not connected.*

Lemma 6.16. *Let Y be a compactification of the ray with remainder X. Suppose that $K(X, Y)$ is connected and that X contains a point p such that Y has the property of Kelley at p. Then $K(Y)$ is connected.*

Proof. Let $\{p_n\}_{n=1}^{\infty} \subset Y \setminus X$ be a sequence which converges to p. According to Lemma 4.5 and the fact that Y has the property of Kelley at p, we have that $C(p_n, Y) \to C(p, Y)$. Whence $C(p, Y) \in$

$\mathrm{cl}\,(K(Y \setminus X, Y)) \cap K(X, Y)$. Moreover, by Corollary 5.2, $K(Y \setminus X, Y)$ is connected. Therefore $K(Y) = K(X, Y) \cup K(Y \setminus X, Y)$ is connected. $\square$

Lemma 6.17. *Let Y be a compactification of the ray with remainder X. If X has the property of Kelley, then $X \approx K(X, Y)$.*

Proof. According to Observation 4.3 and Lemma 4.4, it suffices to show that τ_X is continuous. Let $x \in X$ and $\{x_n\}_{n=1}^{\infty} \subset X$ be a sequence which converges to x. Take $A \in C(x, Y)$. Since X is terminal in Y, either $A \subset X$ or $X \subset A$. By hypothesis X has the property of Kelley, so in either case it is easy to see that $A \in \liminf C(x_n, Y)$. Hence, $C(x, Y) \subset \liminf C(x_n, Y)$. The result follows from Lemma 4.5. $\square$

Corollary 6.18. *Let Y be a compactification of the ray with remainder X. Suppose that X has the property of Kelley and X has a point x such that for any $A, B \in C(x, X)$, A and B are comparable. Then $K(Y)$ is connected.*

Proof. By Lemma 6.17 we have that $X \approx K(X, Y)$ and by Lemma 3.5 we know that $C(x, X)$ is an arc. Let $\{x_n\}_{n=1}^{\infty} \subset Y \setminus X$ be a sequence which converges to x. From Lemma 6.1, we deduce that $C(x, Y) \in \mathrm{cl}\,(K(Y \setminus X, Y)) \cap K(X, Y)$. Thus $K(Y) = K(X, Y) \cup K(Y \setminus X, Y)$ is connected. $\square$

As applications of the corollary above, we present some examples.

Example 6.19. Let X be one of the following continua: $\sin \frac{1}{x}$ curve, the continuum X presented in Example 6.5, the Buckethandle continuum, see [**6**, p. 193] or a pseudoarc, see [**1**, p. 44]. If Y is a compactification of the ray with remainder X, then $K(Y)$ is connected.

It is easy to see that in every continuum X that is mentioned above, there exists a point w such that any two subcontinua of X containing w are comparable. On the other hand, all of them have the property of Kelley. Therefore, by Corollary 6.18, we get that $K(Y)$ is connected in all the cases.

Theorem 6.20. *Let Y be a compactification of the ray with remainder X. If X is a hereditarily indecomposable continuum, then Y has the property of Kelley.*

Proof. Note that Y has the property of Kelley at y for every $y \in Y \setminus X$. The result follows from Lemma 3.6 and Lemma 6.1. $\square$

The following theorem follows directly from Theorem 6.20 and Lemma 4.7.

Theorem 6.21. *If Y is a compactification of the ray with remainder X and X is a hereditarily indecomposable continuum, then $K(Y)$ is connected.*

7. A characterization. Finally, as an application, we develop a result in order to give a characterization of hereditarily indecomposable continua. In what follows, the symbol $\mathrm{Im}\,(f)$ will represent the image of a mapping f.

Theorem 7.1. *Let X be a continuum. If for any compactification Y of the ray, with remainder X, Y has the property of Kelley, then X is hereditarily indecomposable.*

Proof. Suppose that X is not hereditarily indecomposable. Then there exist $A, B \in C(X)$ such that $A \setminus B \neq \varnothing \neq B \setminus A$ and $A \cap B \neq \varnothing$. We may assume that $X \subset I^\infty$. Take a point $a \in A \setminus B$ and, for each $n \in \mathbf{N}$, consider two finite sets $K_n \subset A$ and $F_n \subset X$ such that $A \subset N_{d_1}(\frac{1}{n}, K_n)$ and $X \subset N_{d_1}(\frac{1}{n}, F_n)$, where d_1 represents the metric in the Hilbert cube I^∞. Since $N_{d_1}(\frac{1}{n}, K_n)$ and $N_{d_1}(\frac{1}{n}, F_n)$ are arcwise connected subsets of I^∞, we can take two mappings $\alpha_n : [\frac{1}{2n+1}, \frac{1}{2n}] \to N_{d_1}(\frac{1}{n}, K_n)$ and $\beta_n : [\frac{1}{2n}, \frac{1}{2n-1}] \to N_{d_1}(\frac{1}{n}, F_n)$ such that

i) $\alpha_n\Big(\dfrac{1}{2n+1}\Big) = a = \alpha_n\Big(\dfrac{1}{2n}\Big),\ K_n \subset \mathrm{Im}\,(\alpha_n),$

ii) $\beta_n\Big(\dfrac{1}{2n}\Big) = a = \beta_n\Big(\dfrac{1}{2n-1}\Big)$ and $F_n \subset \mathrm{Im}\,(\beta_n).$

Moreover, we define

$$\mathcal{P} = \bigcup \Big\{ (\alpha_n(t), t) \in I^\infty \times I : t \in \Big[\frac{1}{2n+1}, \frac{1}{2n}\Big], n \in \mathbf{N} \Big\}$$
$$\mathcal{Q} = \bigcup \Big\{ (\beta_n(t), t) \in I^\infty \times I : t \in \Big[\frac{1}{2n}, \frac{1}{2n-1}\Big], n \in \mathbf{N} \Big\}$$

and $Y = (X \times \{0\}) \cup \mathcal{P} \cup \mathcal{Q}$.

We shall prove in a series of steps that Y is a compactification of the ray with remainder $X \times \{0\}$, which does not have the property of Kelley at some point $y \in Y$.

Step 1. $\mathcal{P} \cup \mathcal{Q}$ is a ray.

Let $f : (0,1] \to \mathcal{P} \cup \mathcal{Q}$ be given by

$$f(t) = \begin{cases} (\alpha_n(t), t) & \text{if } t \in [1/(2n+1), 1/(2n)], \\ (\beta_n(t), t) & \text{if } t \in [1/(2n), 1/(2n-1)]. \end{cases}$$

Note that $\alpha_m(\frac{1}{2m}) = a = \beta_m(\frac{1}{2m})$; thus f is well defined and clearly f is continuous and one-to-one. Furthermore, let $(x,t) \in \mathcal{P} \cup \mathcal{Q}$. Then $t \in [\frac{1}{m+1}, \frac{1}{m}]$ for some $m \in \mathbf{N}$. Thus if $m = 2k$ for some $k \in \mathbf{N}$, then $(x,t) = (\alpha_k(t), t) = f(t)$. If $m = 2k-1$ for some $k \in \mathbf{N}$, then $t \in [\frac{1}{2k}, \frac{1}{2k-1}]$, whence $(x,t) = (\beta_k(t), t) = f(t)$. Therefore, f is surjective. Hence, f has an inverse f^{-1}.

Moreover, let $\{(x_n, t_n)\}_{n=1}^\infty \subset \mathcal{P} \cup \mathcal{Q}$ be a sequence which converges to $(x,t) \in \mathcal{P} \cup \mathcal{Q}$. Thus $f^{-1}(x_n, t_n) = t_n \to t = f^{-1}(x,t)$ which means that f^{-1} is continuous. Therefore, f is a homeomorphism and $\mathcal{P} \cup \mathcal{Q}$ is a ray.

Step 2. $\mathcal{P} \cup \mathcal{Q}$ is dense in Y.

It is enough to prove that $X \times \{0\} \subset \operatorname{cl}(\mathcal{P} \cup \mathcal{Q})$. Let $(x,0) \in X \times \{0\}$. For each $n \in \mathbf{N}$, take $x_n \in F_n$ such that $d_1(x_n, x) < \frac{1}{n}$. Further, since $x_n \in F_n \subset \operatorname{Im}(\beta_n)$, there exists $t_n \in [\frac{1}{2n}, \frac{1}{2n-1}]$ such that $x_n = \beta_n(t_n)$. In particular, $(x_n, t_n) \in \mathcal{Q}$. Notice that $t_n \to 0$. Hence $(x_n, t_n) \to (x, 0)$. Therefore, we can conclude that $(x,0) \in \operatorname{cl}(\mathcal{P} \cup \mathcal{Q})$.

Step 3. Y is compact.

It suffices to show that Y is closed in $I^\infty \times I$. Let $\{(x_n, t_n)\}_{n=1}^\infty \subset Y$ be a sequence converging to a point $(x, t) \in I^\infty \times I$. If $t_n = 0$ for infinitely many n's, then clearly $(x, t) \in X \times \{0\} \subset Y$. However, if there exists $\varepsilon > 0$ such that $t_n > \varepsilon$ for every $n \in \mathbf{N}$, then $\{(x_n, t_n)\}_{n=1}^\infty \subset f([\varepsilon, 1])$ where f is the function defined in Step 1. Hence $(x, t) \in f([\varepsilon, 1]) \subset \mathcal{P} \cup \mathcal{Q} \subset Y$. Thus we only need to consider the case in which the sequence $\{t_n\}_{n=1}^\infty$ converges to zero. Let $n \in \mathbf{N}$; then either $x_n = \alpha_{r_n}(t_n)$ or $x_n = \beta_{r_n}(t_n)$ for some $r_n \in \mathbf{N}$. Thus $t_n \in [\frac{1}{2r_n+1}, \frac{1}{2r_n}]$ or $t_n \in [\frac{1}{2r_n}, \frac{1}{2r_n-1}]$. Since $t_n \to 0$, then $r_n \to \infty$. On the other hand, we know that $x_n \in N_{d_1}(\frac{1}{r_n}, F_{r_n} \cup K_{r_n}) \subset N_{d_1}(\frac{1}{r_n}, X)$. Therefore, $x = \lim x_n \in X$. Hence $(x, t) \in X \times \{0\} \subset Y$.

Step 4. If $\{(x_n, t_n)\}_{n=1}^\infty \subset \mathcal{P}$ is a sequence which converges to $(x, 0)$, then $x \in A$.

Let $n \in \mathbf{N}$. Since $(x_n, t_n) \in \mathcal{P}$, there exists $r_n \in \mathbf{N}$ such that $t_n \in [\frac{1}{2r_n+1}, \frac{1}{2r_n}]$ and $x_n = \alpha_{r_n}(t_n)$. Note that $t_n \to 0$ and so $r_n \to \infty$. Now we have that $\alpha_{r_n}(t_n) \in N_{d_1}(\frac{1}{r_n}, K_{r_n}) \subset N_{d_1}(\frac{1}{r_n}, A)$. Therefore, since $\alpha_{r_n}(t_n) = x_n \to x$, we conclude that $x \in A$.

Step 5. If $y \in A \cap B$, then Y does not have the property of Kelley at $(y, 0)$.

For each $n \in \mathbf{N}$ we can take $y_n \in K_n$ such that $d_1(y_n, y) < \frac{1}{n}$. Besides, by construction we have that $K_n \subset \operatorname{Im}(\alpha_n)$. Thence $y_n = \alpha_n(t_n)$ for some $t_n \in [\frac{1}{2n+1}, \frac{1}{2n}]$. Thus, it is easy to see that $t_n \to 0$. Hence $(y_n, t_n) \to (y, 0)$ and, moreover, $\{(y_n, t_n)\}_{n=1}^\infty \subset \mathcal{P}$.

Consider the element $B \times \{0\}$ of $C((y, 0), Y)$ and suppose that $\{B_n\}_{n=1}^\infty \subset C(Y)$ is a sequence which converges to $B \times \{0\}$ and is such that for each $n \in \mathbf{N}$, $B_n \in C((y_n, t_n), Y)$. We will analyze two cases to see that this is not possible.

Case 1. There exists $N \in \mathbf{N}$ such that if $n > N$, then $B_n \subset \mathcal{P}$.

By Step 4, in this case $\limsup B_n \subset A$, whence $B \subset A$, a contradiction. Therefore, this case is impossible.

Case 2. There exists a subsequence $\{B_{n_i}\}_{i=1}^{\infty}$ of $\{B_n\}_{n=1}^{\infty}$ such that for every $i \in \mathbf{N}$, we have that $B_{n_i} \cap \mathcal{Q} \neq \varnothing$.

Let $i \in \mathbf{N}$. We know that $y_{n_i} = \alpha_{n_i}(t_{n_i})$ for some $t_{n_i} \in [\frac{1}{2n_i+1}, \frac{1}{2n_i}]$, and that $(y_{n_i}, t_{n_i}) \in B_{n_i}$. Further, in this case we can take a point $(x_i, t_i) \in B_{n_i} \cap \mathcal{Q}$. In particular, $t_i \in [\frac{1}{2i}, \frac{1}{2i-1}]$ for some $i \in \mathbf{N}$. We will suppose that $t_{n_i} \leq t_i$. Thus $n_i \geq i$.

Since B_{n_i} is connected and contains (x_i, t_i) and (y_{n_i}, t_{n_i}), it follows that B_{n_i} contains $(z_i, \frac{1}{2i})$ for some $z_i \in I^{\infty}$. However, by construction we know that $z_i = \alpha_i(\frac{1}{2i})$ and that $z_i = a$. Hence, $(a, \frac{1}{2i}) \subset B_{n_i}$ for every $i \in \mathbf{N}$. Thus $(a, 0) \in \liminf B_{n_i}$ which means that $(a, 0) \in B \times \{0\}$, a contradiction. Therefore, this case is impossible too.

In any case we obtained a contradiction so we can conclude that Y does not have the property of Kelley at $(y, 0)$.

As a result of Steps 1–5 we get the conclusion of the theorem. $\square$

By Theorem 6.20 and Theorem 7.1, we get the following characterization.

Corollary 7.2. *Let X be a continuum. Then X is hereditarily indecomposable if and only if for any compactification Y of the ray, with remainder X, Y has the property of Kelley.*

Questions. 1. Give necessary and sufficient conditions under which $K(Y)$ is connected, when Y is a compactification of the ray.

2. For a continuum X, give necessary and sufficient conditions under which $K(X)$ is connected.

REFERENCES

1. R.H. Bing, *Concerning hereditarily indecomposable continua*, Pacific J. Math. **1** (1951), 43–51.

2. W.J. Charatonik, *Some functions on hyperspaces of continua*, Topology Appl. **22** (1986), 211–221.

3. J.J. Charatonik and P. Pellicer-Covarrubias, *Retractions and local contractibility in the hyperspaces $C_n(x)$*, preprint.

4. ———, *Retractions and contractibility in hyperspaces*, preprint.

5. C. Eberhart, *Intervals of continua which are Hilbert cubes*, Proc. Amer. Math. Soc. **68** (1978), 220–224.

6. A. Illanes and S.B. Nadler, Jr., *Hyperspaces*: *Fundamentals and recent advances*, Marcel Dekker, Inc., New York, 1999.

7. S.B. Nadler, Jr., *Hyperspaces of sets*, Marcel Dekker, Inc., New York, 1978.

8. P. Pellicer-Covarrubias, *The hyperspaces* $C(p,X)$, Topology Proc. **27** (2003), 259–285.

9. ———, *The hyperspaces* $C(p,X)$ *for atriodic continua*, Houston J. Math., to appear.

10. ———, *Retractions in hyperspaces*, Topology Appl. **135** (2004), 277–291.

11. ———, *Cells in hyperspaces*, preprint.

12. R.H. Sorgenfrey, *Concerning triodic continua*, Amer. J. Math. **66** (1944), 439–460.

Departamento de Matematicas, Facultad de Ciencias, Circuito exterior S/N, Ciudad Universitaria, CP 04510, Mexico DF, Mexico
E-mail address: paty@ciencias.unam.mx