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GERMS OF HOLOMORPHIC FUNCTIONS ON TOPOLOGICAL VECTOR SPACES AND INVARIANT RINGS

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ABSTRACT. Let V be a locally convex and Hausdorff topological vector space and G a finite group of holomorphic automorphisms of $O_{V,0}$. Here we prove that the ring $O_{V,0}^G$ of all invariant germs is a C.M.$_\infty$-local ring.

1. Introduction. Let (X, O_X) be a Hausdorff reduced complex space locally embedded in a locally convex topological vector space, i.e., a Cartan space with the terminology of [**2**, p. 65] and G a finite group of holomorphic automorphisms of X. Let X/G be the set of all G-orbits equipped with the quotient topology, and let $f : X \to X/G$ be the quotient map. Since G is finite, X/G is Hausdorff. For any open subset Ω of X/G, let $H^0(\Omega, O_\Omega) := H^0(\pi^{-1}(\Omega), O_{\pi^{-1}(\Omega)})^G$ be the set of all G-invariant holomorphic functions on $\pi^{-1}(\Omega)$. In this way we obtain a sheaf $O_{X/G}$ of local **C**-algebras on X/G. In general the local rings $O_{X/G,P}$ are not Noetherian. Here we study the Cohen-Macaulyness of the local rings of X/G, X smooth, in the non-Noetherian case. For a theory of grade in the non-Noetherian case, see [**1**] or [**2**, Chapter 1]. We recall here the definition of C.M.$_\infty$-local ring given in [**2**, pp. 34–35]. Let A be a unitary commutative ring and n a nonnegative integer. For any A-module M, let $T_n(M)$ denote the set of all $x \in M$ such that the annihilator of x has grade at least n. It is easy to see that $T_n(M/T_n(M)) = 0$ and $T_nT_n = T_n$. Thus the functor T_n defines a torsion theory, i.e., for all submodules N of M we may define the n-closure of N in M as the inverse image in M of $T_n(M/N)$. The ring A, respectively the module M, is said to be n-Noetherian if each increasing sequence of n-closed ideals of A, respectively n-closed submodules of M, is stationary and ∞-Noetherian if it is n-Noetherian for all n. The

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ring A is said to be C.M.$_n$ if it is n-Noetherian and grade and height is the same function on all prime n-closed ideals of A. The ring A is said to be C.M.$_\infty$ if it is C.M.$_n$ for all n.

This definition is a very good extension to non-Noetherian local rings of the notion of the Cohen-Macaulay ring. In this note we prove the following result.

Theorem 1.1. *Let V be a locally convex and Hausdorff topological vector space and G a finite group of holomorphic automorphisms of $O_{V,0}$. Then the ring $O_{V,0}^G$ of all invariant germs is a* C.M.$_\infty$*-local ring.*

Remark 1.2. The C.M.$_\infty$-ness of the invariant ring $O_{V,0}^G$ has the following consequences, see [**2**, pp. 40–41].

(a) Every prime ideal of $O_{V,0}^G$ has height equal to its grade.

(b) If J is an ideal of $O_{V,0}^G$ generated by n elements, then the prime ideals of $O_{V,0}^G$ which are minimal among those containing J are finite in number and have height at most n.

(c) For any ideal I of $O_{V,0}^G$ and any integer $n \geq 1$ there are only finitely many ideals of height n associated to $O_{V,0}^G/I$.

(d) The localization of $O_{V,0}^G$ at any prime of finite length is a Cohen-Macaulay ring.

2. The proof.

Proof of Theorem 1.1. Set $R := O_{V,0}$ and $A := O_{V,0}^G = R^G$. For any $x \in A$, $y \in R$, $I \subseteq A$ and $J \subseteq R$, set $(I : x)_A := \{z \in A : zx \in I\}$ and $(J : y)_R := \{z \in R : zy \in J\}$. For any $x \in R$, set $\alpha(x) = (\sum_{g\in G} g \circ x)/\mathrm{card}\,(G)$ and $\beta(x) = \prod_{g\in G} g \circ x$. Hence, $\beta(x) \in A$ for every $x \in R$, $\beta(x) = x^{\mathrm{card}\,(G)}$ if $x \in A$, $\alpha(x) \in A$ for every $x \in R$ and $\alpha(x) = x$ if and only if $x \in A$, i.e., $\alpha : R \to A$ is a retraction of the inclusion of A in R. Thus R is a flat A-module. Use α to show that for every ideal J of A and any $x \in A$ we have $JR \cap A = J$ and $(JR : x)_R \cap A = (J : x)_R$. Hence we obtain that n elements $f_1, \dots, f_n$ of the maximal ideal of A form a regular sequence in A if and only if they form a regular sequence in R. For any ideal I of A, respectively

J of R, $\mathrm{gr}_A(I)$, respectively $\mathrm{gr}_R(J)$, will denote its grade [**2**, Chapter 1]. By [**2**, Corollary, p. 42; Proposition 5.5, p. 144], we may use regular sequences instead of the Koszul complex to compute the grade of proper ideals of R. For any integer $n \geq 1$ and any ideal I of A, respectively J of R, set $T_{n,A}(I) := \{x \in A : \mathrm{gr}_A((I : x)) > n\}$, respectively $T_{n,R}(J) := \{x \in R : \mathrm{gr}_R((J : x)) > n\}$). As in [**2**] we will say that I, respectively J, is n-closed if $T_{n,A}(I) = I$, respectively $T_{n,R}(J) = J$, and that a ring is n-Noetherian if every increasing sequence of n-closed ideals is stationary. By [**2**, Proposition 5.5, p. 144], R is n-Noetherian for every $n \geq 1$.

First claim. *For every integer $n \geq 1$, there are $f_1, \dots, f_n \in A$ such that $f_1(0) = \cdots = f_n(0) = 0$ and the germ at 0 of the analytic set $\{f_1 = \cdots = f_n = 0\}$ has pure codimension n in V at 0.*

Proof of first claim. The case $n = 1$ is true, and we may take as f_1 any element of $A \setminus \{0\}$ because R is factorial [**2**, Proposition 5.15, p. 157; Proposition on p. 221]. Fix $n \geq 2$ and assume that the first claim is true for the integer $n' = n - 1$. Take $f_1, \dots, f_{n-1} \in A$ with $f_1(0) = \cdots = f_{n-1}(0) = 0$ such that $\{f_1 = \cdots = f_{n-1} = 0\}$ has pure codimension $n-1$ in V at 0. Since R is a C.M. ring, there is an $h \in R$ with $h(0) = 0$ and such that $\{f_1 = \cdots = f_{n-1} = h = 0\}$ has pure codimension n in V at 0. Since each germ f_i is G-invariant, for every $g \in G$ we have $g^*(h)(0) = 0$, and the analytic set $Z(g) := \{f_1 = \cdots = f_{n-1} = g^*(h) = 0\}$ has pure codimension n in V at 0. Set $f_n := \beta(h)$. We have $f_n \in R$, $f_n(0) = 0$ and $\{f_1 = \cdots = f_n = 0\} = \cup_{g \in G} Z(g)$, proving the first claim. □

Second claim. *Take $f_1, \dots, f_n \in A$ as in the first claim. Then the sequence $f_1, \dots, f_n$ is a regular sequence in A.*

Proof of second claim. The case $n = 1$ is obvious because A is an integral domain. The sequence $f_1, \dots, f_n$ is a regular sequence in R. Apply the equality $(JR : x)_R \cap A = (J : x)_A$ to $J := (f_1, \dots, f_{n-1})$ and use induction on the integer n.

By the second claim for every $n \geq 1$ the maximal ideal of A has a regular sequence of length n. □

Third claim. *Let I be the proper ideal of A. If the extended ideal IR contains an R-regular sequence of length $n \geq 1$, then I contains an A-regular sequence of length n.*

Proof of third claim. The assumption on IR implies that the zero-set $Z(IR)$ of IR has codimension at least n in V at the origin. Since $Z(IR)$ is G-invariant, we may copy the proof of the first claim to obtain $f_1, \ldots, f_n \in I$ with common zero-set of codimension n and then we are finished by the second claim. □

Fourth claim. *For any proper ideal I of A we have $T_{n,R}(IR) \cap A = T_{n,A}(I)$.*

Proof of fourth claim. Take $x \in A$. By the third claim we have $\mathrm{gr}_A((I : x)) = \mathrm{gr}_R((IR : x))$. Hence the fourth claim holds. □

Completion of the proof. Let I_i, $i \geq 1$, be an increasing sequence of n-closed ideals of A. Since R is n-Noetherian [**2**, Proposition 5.5, p. 144], the sequence $T_{n,R}(I_iR)$, $i \geq 1$, is stationary. Since $T_{n,R}(I_iR) \cap A = I_i$ by the n-closedness of I_i and the fourth claim, the sequence I_i, $i \geq 1$, is stationary. Thus, for every positive integer n, the ring A is n-Noetherian. By the third claim and the C.M.$_\infty$-ness of R, we obtain that height and grade agree for any n-closed prime ideal of A, i.e., that for every $n \geq 1$, A is C.M.$_n$ [**2**, Definition 3.2], i.e., that A is C.M.$_\infty$. □

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