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A MASSERA THEOREM FOR QUASI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

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ABSTRACT. Massera-type criteria are derived for the existence of periodic wave solutions of quasi-linear partial differential equations of first order. Results generalize a theorem of Massera for first order scalar ordinary differential equations.

1. Introduction. In 1950, Massera [**8**] first established the following results, which now are often referred to as Massera theorems.

Theorem A. *For a scalar differential equation*

$$\dot{x} = f(t, x), \tag{1}$$

where $f \in C(\mathbf{R}_+ \times \mathbf{R} \to \mathbf{R})$ *is* ω*-periodic in* t *for some* $\omega > 0$*, the existence of a solution that is bounded in the future implies the existence of a nonconstant* ω*-periodic solution.*

Theorem B. *Consider a linear system of differential equation*

$$\dot{x} = A(t)x + b(t), \tag{2}$$

where $A \in C(\mathbf{R} \to \mathbf{R}^{n\times n})$ *and* $b \in C(\mathbf{R} \to \mathbf{R}^n)$ *are* ω*-periodic for some* $\omega > 0$*. System* (2) *admits a nonconstant* ω*-periodic solution if and only if it admits a solution that is bounded in the future.*

In 1973, Chow [**1**] extended Theorem B to linear scalar functional differential equations with finite delay of retarded type under a "small

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delay" assumption. Similar results are obtained by Li and Lin [**5**] for linear functional differential equations. In 1995, Makay [**7**] extended Chow's results to general linear retarded functional differential equation with finite delay, infinite delay and also to integral equations. In 1999, Li et al. [**6**] proved several Massera-type criteria for linear periodic evolution equations. In 2000, Fan and Wang [**2, 3**] established Massera-type criteria for linear and convex neutral functional differential equations with finite delay, infinite delay and also for hyperneutral functional differential equations with finite delay.

In the present paper, we establish Massera-type criteria for the existence of periodic wave solutions of scalar quasi-linear partial differential equations of first order. Our main result (Theorem 3) generalizes Theorem A of Massera.

The paper is organized as follows. Section 2 provides basic terminologies and preliminary results. In Section 3, for two-dimensional quasi-linear partial differential equations, we give, in Theorem 2, a Massera-type result that establishes the equivalence between the existence of a bounded solution and the existence of periodic wave solutions. In Section 4, the n-dimensional version of the result is given in Theorem 3. Since the proofs for Theorems 2 and 3 are essentially the same, for notational simplicity, we provide a detailed proof only for the case $n = 2$. Some examples are given in Section 4 to illustrate our results.

2. Preliminaries. Consider a quasi-linear partial differential equation of first order

$$(E_n) \qquad \sum_{k=1}^{n} a_k(x_1, \dots, x_n, u) \frac{\partial u}{\partial x_k} = c(x_1, \dots, x_n, u),$$

where $c, a_k \in C(\mathbf{R}^n \times \mathbf{R} \to \mathbf{R})$, $k = 1, \dots, n$. The *characteristic differential equations* of equation (E_n) are

$$(3) \qquad \begin{aligned} \dot{x}_k &= a_k(x_1, \dots, x_n, u), \quad k = 1, \dots, n, \\ \dot{u} &= c(x_1, \dots, x_n, u). \end{aligned}$$

The vector field defined by (3) in $(x_1, \dots, x_n, u)$ space are called the *characteristic vector field* of (E_n), and integral curves of (3) are called *characteristic curves*. We assume that coefficients a_k and c are such that solutions to (3) are uniquely determined by their initial conditions.

Definition 1. Let $U \subset \mathbf{R}^n$ be open. A function $u \in C(U \to \mathbf{R})$ is said to be a *weak solution* of equation (E_n) over U if, for every P on the graph of u, there exists a neighborhood $N(P)$ of P such that the characteristic curve of (E_n) through P lies on the graph of u when it is restricted to $N(P)$.

Definition 2. Let $U \subset \mathbf{R}^n$ be open. A weak solution u of equation (E_n) over U is said to be a *bounded weak solution of Lipschitz type*, if there exists a positive constant $M > 0$ such that

$$|u(X)| \leq M, \quad \text{and} \quad |u(X_1) - u(X_2)| \leq M|X_1 - X_2|,$$

for all $X, X_1, X_2 \in U$.

For $1 \leq i \leq n$ and $\omega_i > 0$, a function $g \in C(\mathbf{R}^n \to \mathbf{R}^1)$ is said to be ω_i*-periodic* with respect to x_i, if

$$g(x_1, \dots, x_{i-1}, x_i + \omega_i, x_{i+1}, \dots, x_n) = g(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n),$$

for all $(x_1, \dots, x_n) \in \mathbf{R}^n$.

Definition 3. Let $1 \leq m \leq n$ and $1 \leq k_1 < k_2 < \cdots < k_m \leq n$ be integers, and $\omega_i > 0$, $i = 1, \dots, m$. A weak solution $u(x_1, \dots, x_n)$ of equation (E_n) in $\mathbf{R}^n$ is said to be a *periodic wave solution* of period ω_i with respect to x_{k_i}, if u is periodic of period ω_i with respect to x_{k_i}, $i = 1, \dots, m$.

Lemma 1. *Suppose that* $a_k(x_1, \dots, x_n, u)$, $k = 1, \dots, n$, *and* $c(x_1, \dots, x_n, u)$ *are periodic functions of period* ω_i *with respect to* x_i. *Let* $W \subset \mathbf{R}^{n-1}$ *be open and* $u(x_1, \dots, x_n)$ *be a weak solution to* (E_n) *over the region*

$$\Omega = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_i > 0, (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in W\}.$$

Then $u_k(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = u(x_1, \dots, x_{i-1}, x_i + k\omega_i, x_{i+1}, \dots, x_n)$ *is also a weak solution to* (E_n) *over* Ω.

Proof. Without loss of generality, we may assume that $i = 1$, and denote $(x_1, \dots, x_n) = (x, y)$, where $x = x_1 \in \mathbf{R}$ and $y = (x_2, \dots, x_n) \in$

$\mathbf{R}^{n-1}$. Let (x_0, y_0, u_0) be a point on the graph of $u_k(x, y)$, i.e.,

$$(x_0, y_0, u_0) \in \{(x, y, u_k(x, y)) : (x, y) \in \Omega\}.$$

Then $u_0 = u_k(x_0, y_0) = u(x_0 + k\omega, y_0)$, and hence $(x_0 + k\omega, y_0, u_0)$ is on the graph of $u(x, y)$. Since $u(x, y)$ is a weak solution of equation (E_2), there exists a neighborhood V of $(x_0 + k\omega, y_0, u_0)$ such that the characteristic curve of (E_2) through $(x_0 + k\omega, y_0, u_0)$ is on the graph of $u(x, y)$ when restricted to V. Let $(x(t), y(t), u(t))$ be the solution of the characteristic equations with $(x(0), y(0), u(0)) = (x_0 + k\omega, y_0, u_0)$.

Define a shift operator $P_{k\omega} : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ as follows

$$P_{k\omega}(x, y, u) = (x - k\omega, y, u).$$

Then $P_{k\omega}(x_0 + k\omega, y_0, u_0) = (x_0, y_0, u_0)$, $P_{k\omega}V$ is a neighborhood of (x_0, y_0, u_0), and

$$P_{k\omega}(x(t), y(t), u(t)) = (x(t) - k\omega, y(t), u(t)).$$

Set $x_1(t) = x(t) - k\omega$. Then

$$\begin{aligned}
\dot{x}_1(t) &= \dot{x}(t) = a(x(t), y(t), u(t)) \\
&= a(x_1(t) + k\omega, y(t), u(t)) \\
&= a(x_1(t), y(t), u(t)), \\
\dot{y}(t) &= b(x_1(t), y(t), u(t)), \\
\dot{u}(t) &= c(x_1(t), y(t), u(t)), \\
x_1(0) &= x(0) - k\omega = x_0, \ y(0) = y_0, \ u(0) = u_0.
\end{aligned}$$

Therefore $(x_1(t), y(t), u(t))$ is the characteristic curve of (E_n) through (x_0, y_0, u_0).

Since $(x(t), y(t), u(t))$ is on the graph of $u(x, y)$ when restricted to V, and the graph of $u(x, y)$ is transformed into that of $u_k(x, y)$ under the action of $P_{k\omega}$, $(x_1(t), y(t), u(t))$ is on the graph of $u_k(x, y)$ when restricted to $P_{k\omega}V$. This completes the proof. $\square$

Lemma 2. *Let $V \subset \mathbf{R}^n$ be open. Suppose that $u_1(x_1, \dots, x_n)$ and $u_2(x_1, \dots, x_n)$ are weak solutions of (E_n) over V. Let G_1 and G_2 be the graphs of u_1 and u_2, respectively. If $G_1 \cap G_2 \neq \varnothing$,* *then, for any* $(x_1^0, \dots, x_n^0, u^0) \in G_1 \cap G_2$*, there exists a neighborhood* V^* *of* $(x_1^0, \dots, x_n^0, u_0)$ *such that the characteristic curve through* $(x_1^0, \dots, x_n^0, u^0)$ *lies on* $G_1 \cap G_2$ *when restricted to* V^**.*

Proof. Let $X = (x_1, \dots, x_n)$. Since $u_1(X)$ and $u_2(X)$ are weak solutions of (E_n), there exists a neighborhood of (X_0, u_0) such that the characteristic curve through (X_0, u_0) is on the graph G_i of u_i when restricted to V_i, $i = 1, 2$. Let $V^* = V_1 \cap V_2$. Then the characteristic curve through (X_0, u_0) is on $G_1 \cap G_2$ when restricted to V^*. $\square$

Lemma 3. *Let* $u_1(x_1, \dots, x_n)$ *and* $u_2(x_1, \dots, x_n)$ *be weak solutions of equation* (E_n) *over an open set* $U \subset \mathbf{R}^n$*. Then*

$$u(x_1, \dots, x_n) = \min\{u_1(x_1, \dots, x_n), u_2(x_1, \dots, x_n)\}$$

is a weak solution of (E_n) *over* U*.*

Proof. Let G_1 and G_2 denote the graphs of $u_1(X)$ and $u_2(X)$, respectively. For any point (X_0, u_0) on the graph of $u(X)$, if $(X_0, u_0) \notin G_1 \cap G_2$, then $(X_0, u_0) \in G_1$ or $(X_0, u_0) \in G_2$, and hence there exists a neighborhood V of (X_0, u_0) such that the characteristic curve through (X_0, u_0) is on the graph of $u(X)$ when restricted to V. If $(X_0, u_0) \in G_1 \cap G_2$, then the existence of such a neighborhood V follows from Lemma 2. $\square$

The following lemma is often referred to as the Dini theorem and can be found in standard analysis texts, see e.g. [**9**].

Lemma 4 (Dini theorem). *Let* E *be a compact metric space, and let* $\{f_n\}$ *be a monotone sequence of real-valued continuous functions on* E*. If* $\{f_n\}$ *converges pointwisely to a continuous function* g*, then* $\{f_n\}$ *converges uniformly to* g*.*

3. Massera criteria for $n = 2$**.** Let $x = (x, y) \in \mathbf{R}^2$. We consider the quasi-linear differential equation (E_n) with $n = 2$

$$(E_2) \qquad a(x, y, u)\,\frac{\partial u}{\partial x} + b(x, y, u)\,\frac{\partial u}{\partial y} = c(x, y, u),$$

where $a, b, c \in C(\mathbf{R}^3 \to \mathbf{R})$ are such that solutions to the characteristic equations

$$\begin{aligned} \dot{x} &= a(x, y, u) \\ \dot{y} &= b(x, y, u) \\ \dot{u} &= c(x, y, u). \end{aligned} \tag{4}$$

are uniquely determined by their initial conditions. Let

$$\mathbf{R}_+^2 = \{(x, y) \in \mathbf{R}^2 : x > 0\}$$

denote the right half plane.

Theorem 1. *Suppose that a, b, c are periodic of period ω with respect to x and equation (E_2) has a bounded weak solution of Lipschitz type over $\mathbf{R}_+^2$. Then (E_2) has a periodic wave solution of period ω with respect to x.*

Proof. Let $u(x, y)$ be the bounded weak solution of Lipschitz type of (E_2) defined on $\mathbf{R}_+^2$, and let its bound and Lipschitz constant be $M > 0$. Define

$$u_i(x, y) = u(x + i\,\omega, y), \quad x, y \in \mathbf{R}_+^2, \quad i = 1, 2, \dots .$$

By Lemma 1, $u_i(x, y)$, $i = 1, 2, \dots$, are bounded weak solutions of Lipschitz type of equation (E_2) over $\mathbf{R}_+^2$, and their bounds and Lipschitz constants are M.

Define

$$w_k(x, y) = \min\{u_0(x, y), u_1(x, y), \dots , u_k(x, y)\}, \\ x, y \in \mathbf{R}_+^2, \quad k = 0, 1, 2, \dots .$$

By Lemma 3, for $k = 0, 1, 2, \dots , w_k(x, y)$ is also a bounded weak solution of Lipschitz type to (E_2) over $\mathbf{R}_+^2$, and its bound and Lipschitz constant are M. Furthermore, $\{w_k(x, y)\}$ is uniformly bounded and equicontinuous on $\mathbf{R}_+^2$. By the Arzela-Ascoli theorem, there exist a subsequence $\{w_m\}$ of $\{w_k\}$ and a continuous function $w(x, y)$ defined on

$\mathbf{R}_+^2$ such that $\{w_m(x,y)\}$ converge uniformly to $w(x,y)$ on any compact subset of $\mathbf{R}_+^2$.

By the definition of $w_k(x,y)$, we have

$$w_k(x,y) \geq w_{k+1}(x,y), \quad \text{for} \quad (x,y) \in \mathbf{R}_+^2, \quad k = 0,1,2,\dots .$$

Then the Dini theorem (Lemma 4) implies that $\{w_k(x,y)\}$ converges uniformly to $w(x,y)$ on any compact region D of $\mathbf{R}_+^2$, and hence $w(x,y)$ is bounded and satisfies Lipschitz condition, and its bound and Lipschitz constant are M.

Next, we show that $w(x,y)$ is a weak solution of (E_2) over $\mathbf{R}_+^2$. Let W denote the graph of $w(x,y)$, and let $(x(t),y(t),u(t))$ be the solution of (4) such that $(x(0),y(0),u(0)) = (x_0,y_0,u_0) \in W$. Set $u_0^{(k)} = w_k(x_0,y_0)$. Then

$$(x_0,y_0,u_0^{(k)}) \longrightarrow (x_0,y_0,u_0), \quad \text{as} \quad k \to \infty.$$

Let $(x_k(t),y_k(t),u_k(t))$ denote the solution of (4) such that

$$(x_k(0),y_k(0),u_k(0)) = (x_0,y_0,u_0^{(k)}).$$

Since $w_k(x,y)$ is a weak solution of (E_2), $(x_k(t),y_k(t),u_k(t))$ is on the graph of $w_k(x,y)$ for sufficiently small t. The fact that $w_k(x,y)$ uniformly converges to $w(x,y)$ on any compact domain of $\mathbf{R}_+^2$ implies that, for sufficiently small t, the distance from $(x_k(t),y_k(t),u_k(t))$ to W

$$d((x_k(t),y_k(t),u_k(t)),W) \longrightarrow 0, \quad \text{as} \quad k \to \infty.$$

By continuous dependence on initial conditions,

$$d\big((x_k(t),y_k(t),u_k(t)),(x(t),y(t),u(t))\big) \longrightarrow 0 \quad \text{as} \quad k \to \infty,$$

for sufficiently small t, which implies that there exists a neighborhood V of (x_0,y_0,u_0) such that if $(x(t),y(t),u(t)) \in V$, then it must be on W, and hence $u(x,y)$ is a weak solution of (E_2).

Define

$$z_k(x,y) = w(x+k\omega,y), \quad (x,y) \in \mathbf{R}_+^2, \quad k = 0,1,2,\dots .$$

Then $z_k(x,y)$ is also a bounded weak solution of Lipschitz type of (E_2) over $\mathbf{R}_+^2$, and its bound and Lipschitz constant can be taken as $M>0$.

By the definitions of $z_k(x,y)$ and $w(x,y)$, one can show that $z_k(x,y)$ is the limit function of the function sequence $\{w_i(x+k\omega,y)\}_{i=1}^{\infty}$. Since

$$\begin{aligned} w_i(x+k\omega,y) &= \min\{u_0(x+k\omega,y),u_1(x+k\omega,y),\dots,u_i(x+k\omega,y)\} \\ &= \min\{u_k(x,y),u_{k+1}(x,y),\dots,u_{k+i}(x,y)\}, \\ &\qquad i=0,1,2,\dots, \end{aligned}$$

we have

$$z_k(x,y)=\inf\{u_k(x,y),u_{k+1}(x,y),\dots\},$$

and

$$z_k(x,y)\geq z_{k+1}(x,y),\quad x,y\in\mathbf{R}_+^2,\quad k=0,1,2,\dots .$$

Applying the Arzela-Ascoli theorem and Dini theorem again, we deduce that there exists a continuous function $z(x,y)$ defined on $\mathbf{R}_+^2$ such that $\{z_k(x,y)\}$ converges uniformly to $z(x,y)$ on any compact region of $\mathbf{R}_+^2$, and $z(x,y)$ is a weak solution of (E_2) over $\mathbf{R}_+^2$. For any $(x,y)\in\mathbf{R}_+^2$, we have

$$z(x,y)=\lim_{k\to+\infty}z_k(x,y)=\lim_{k\to+\infty}z_{k-1}(x+\omega,y)=z(x+\omega,y).$$

Define

$$z^*(x,y):=z(x-(k-1)\omega,y),\ x\in[k\,\omega,(k+1)\omega],\ k=0,\pm1,\pm2,\dots .$$

Since $\omega\leq x-(k-1)\omega\leq 2\,\omega$, $z^*(x,y)$ is defined on $\mathbf{R}^2$ and $z^*(x,y)\equiv z(x,y)$ for any $(x,y)\in\mathbf{R}_+^2$. Moreover, $z^*(x+\omega,y)=z^*(x,y)$ for $(x,y)\in\mathbf{R}^2$. We have shown that $z(x,y)$ can be extended to $\mathbf{R}^2$.

To complete the proof, it is left to show that $z^*(x,y)$ is a weak solution of (E_2). Let (x_0,y_0,u_0) be an arbitrary point on the graph of $z^*(x,y)$; without loss of generality, we can assume that $x_0\leq 0$, $x_0\in[k\omega,(k+1)\omega]$, $k\leq -1$.

Let $P_{k\omega}$ be the shift operator defined above; then

$$P_{(k-1)\omega}(x_0,y_0,u_0)=(x_0-(k-1)\omega,y_0,u_0).$$

Since $(x_0-(k-1)\omega, y_0, u_0)$ is on the graph of $z(x,y)$ and $z(x,y)$ is a weak solution, there exists a neighborhood V of $(x_0-(k-1)\omega, y_0, u_0)$ such that the characteristic curve through $(x_0-(k-1)\omega, y_0, u_0)$ is on the graph of $z(x,y)$ when restricted to V. By the definition of $z^*(x,y)$, the graph of $z(x,y)$ lying in V becomes the graph of $z^*(x,y)$ lying in the neighborhood $P_{(k-1)\omega}V$ of (x_0,y_0,u_0) under the action of $P_{(k-1)\omega}$. By a similar argument as above, one can show that the characteristic curve through $(x_0-(k-1)\omega, y_0, u_0)$ restricted to V is transformed into the characteristic curve through (x_0,y_0,u_0) restricted to $P_{(k-1)\omega}V$, and hence on the graph of $z^*(x,y)$. This proves that $z^*(x,y)$ is a weak solution of (E_2), completing the proof. $\square$

Let

$$\mathbf{R}^2_{++} := \{(x,y)\in\mathbf{R}^2 : x>0,\ y>0\}$$

denote the positive quadrant of $\mathbf{R}^2$.

Theorem 2. *Suppose that a,b,c are periodic of period ω_1 and ω_2 with respect to x and y, respectively. If equation (E_2) has a bounded weak solution of Lipschitz type defined on $\mathbf{R}^2_{++}$, then equation (E_2) has a periodic wave solution over $\mathbf{R}^2$, with period ω_1 and ω_2 with respect to x and y, respectively.*

Proof. Let $u(x,y)$ be a bounded weak solution of Lipschitz type of (E_2) over $\mathbf{R}^2_{++}$. Define

$$\begin{gathered} u_i(x,y) = u(x, y+i\omega_2), \quad i=0,1,2,\dots, \\ w_k(x,y) = \min\{u_0(x,y), u_1(x,y), \dots, u_k(x,y)\}, \\ (x,y)\in\mathbf{R}^2_{++}, \quad k=0,1,2,\dots. \end{gathered}$$

A similar argument as in the proof of Theorem 1 yields that there exists a continuous function $w(x,y)$ defined on $\mathbf{R}^2_{++}$ such that $\{w_k(x,y)\}$ converges uniformly to $w(x,y)$ on any compact region of $\mathbf{R}^2_{++}$, and $w(x,y)$ is a bounded weak solution of Lipschitz type to (E_2).

Let

$$z_k(x,y) := w(x, y+k\omega_2), \quad (x,y)\in\mathbf{R}^2_{++}, \quad k=0,1,2,\dots.$$

Then $z_k(x,y)$ is also a bounded weak solution of Lipschitz type to (E_2) over $\mathbf{R}^2_{++}$. In addition, we have

$$z_{k+1}(x,y) \le z_k(x,y), \quad \text{for} \quad (x,y) \in \mathbf{R}^2_{++}, \quad k = 0,1,2,\dots .$$

Using a similar argument as above, one can prove that there exists a continuous function $z(x,y)$ defined on $\mathbf{R}^2_{++}$ such that $\{z_k(x,y)\}$ uniformly converges to $z(x,y)$ on any compact region of $\mathbf{R}^2_{++}$, and $z(x,y)$ is a bounded weak solution of Lipschitz type of (E_2). By the monotonicity of $\{z_k(x,y)\}$, we have

$$z(x,y+\omega_2) = z(x,y), \quad \text{for any} \quad (x,y) \in \mathbf{R}^2_{++}.$$

Moreover, $z(x,y)$ can be extended to a bounded weak solution of Lipschitz type to (E_2) over $\mathbf{R}^2_{+}$, say $z^*(x,y)$ and $z^*(x,y+\omega_2) = z^*(x,y)$ for any $(x,y) \in \mathbf{R}^2_{+}$.

Since $z^*(x,y)$ is a bounded weak solution of Lipschitz type to (E_2) over $\mathbf{R}^2_{+}$, Theorem 1 implies that there exists a periodic wave solution $z^{**}(x,y)$ of (E_2) with respect to the first variable x, which is defined on $\mathbf{R}^2$. From the proof of Theorem 1, one can see that the periodicity of $z^*(x,y)$ with respect to the second variable y is intact during the derivation of $z^{**}(x,y)$, and hence $z^{**}(x,y)$ is a periodic wave solution of (E_2) over $\mathbf{R}^2$ of period ω_1 and ω_2 with respect to x and y, respectively. $\square$

4. Massera criteria for general n. Using the proof of Theorem 2 inductively, we can establish the following Massera criteria for equation (E_n) for any finite n. We omit the proof.

Theorem 3. *Let $1 \le m \le n$ and $1 \le k_1 < k_2 < \cdots < k_m \le n$ be integers and $\omega_i > 0$, $i = 1,\dots,m$. For $1 \le k \le n$, assume that $a_k(x_1,\dots,x_n,u)$ and $c(x_1,\dots,x_n,u)$ are periodic with respect to x_i of period ω_i, $i = 1,\dots,m$. If equation (E_n) admits a bounded weak solution of Lipschitz type over the cone*

$$\{(x_1,\dots,x_n) \in \mathbf{R}_n : x_{k_i} > 0,\ i = 1,\dots,m\},$$

then (E_n) admits a periodic wave solution of period $\omega_1,\dots,\omega_m$ with respect to $x_{k_1},\dots,x_{k_m}$, respectively.

When $n = 1$, Theorem 3 gives Theorem A.

Example 1. Consider the following quasi-linear partial differential equation

$$u_x - u_y = -(\sin x + 2\cos 2y)u. \tag{5}$$

It can be verified that

$$u(x, y) = e^{\cos x + \sin 2y}\left(1 - e^{-(x+y)}\right)$$

is a bounded solution of Lipschitz type of equation (5) defined on $\mathbf{R}^2_{++}$. Theorem 2 implies that equation (5) admits a periodic wave solution of period 2π and π with respect to x and y, respectively. In fact,

$$u^*(x, y) = e^{\cos x + \sin 2y}$$

is such a periodic wave solution of equation (5).

Solutions to a quasi-linear equation (E_n) can be obtained by finding first integrals of its characteristic equations (3). A nonconstant Lipschitz function $V(x_1, \dots, x_n, u)$ is said to be a *first integral* of system (3) in a region $U \subset \mathbf{R}^{n+1}$ if

$$V(x_1(t), \dots, x_n(t), u(t)) = \text{const.}$$

for all solutions $(x_1(t), \dots, x_n(t), u(t))$ to (3) in U. A necessary and sufficient condition for V to be a first integral of (3) is

$$\sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\, a_i + \frac{\partial V}{\partial u}\, c = 0$$

for all $(x_1, \dots, x_n, u) \in U$. The following result is standard, see [**4**].

Lemma 5. *Let $V(x_1, \dots, x_n, u)$ be a first integral of* (3) *in $U \subset \mathbf{R}^{n+1}$. Then, for any Lipschitz function $\Phi(\xi)$, the equation*

$$\Phi(V(x_1, \dots, x_n, u)) = 0 \tag{6}$$

defines an implicit weak solution to (E_n).

Theorem 3 and Lemma 5 give the following result.

Corollary 1. *Under the assumptions of Theorem* 3, *suppose that system* (3) *admits a first integral* $V(x_1, \dots, x_n, u)$ *such that, for some Lipschitz function* $\Phi(\xi)$, *equation* (6) *defines a weak solution*

$$u = u(x_1, \dots, x_n)$$

of Lipschitz type that is bounded in

$$\{(x_1, \dots, x_n) \in \mathbf{R}^n : x_{k_i} > 0, \ i = 1, \dots, m\}.$$

Then (E_n) *admits a periodic wave solution of period* $\omega_1, \dots, \omega_m$ *with respect to* $x_{k_1}, \dots, x_{k_m}$, *respectively.*

Example 2. Consider a quasi-linear partial differential equation

$$(7) \qquad \begin{aligned} (\sin x + u^2)u_x + (\sin 2y + u^2)u_y + (\sin 3z - 2u^2)u_z \\ = -u(\sin x + \sin 2y + \sin 3z). \end{aligned}$$

Its characteristic equations

$$\begin{aligned} x' &= \sin x + u^2 \\ y' &= \sin 2y + u^2 \\ z' &= \sin 3z - 2u^2 \\ u' &= -u(\sin x + \sin 2y + \sin 3z) \end{aligned}$$

have a first integral

$$V = x + y + z + \ln|u|.$$

Then, by Lemma 4, $\Phi(x+y+z+\ln|u|) = 0$ defines an implicit solution to (7) for any Φ. In particular,

$$u = e^{-x-y-z}$$

is a weak solution to (7) of Lipschitz type and is bounded in the positive cone $\mathbf{R}^3_{+++}$ of $\mathbf{R}^3$. Therefore, by Theorem 3, equation (7) has a wave solution that is periodic in x, y, and z of period $2\pi, \pi$ and $2\pi/3$, respectively.

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