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ON THE LUPAŞ q-ANALOGUE OF THE BERNSTEIN OPERATOR

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ABSTRACT. Let $R_n(f,q;x) : C[0,1] \to C[0,1]$ be q-analogues of the Bernstein operators defined by Lupaş in 1987. If $q = 1$, then $R_n(f,1;x)$ are classical Bernstein polynomials. For $q \neq 1$, the operators $R_n(f,q;x)$ are rational functions rather than polynomials. The paper deals with convergence properties of the sequence $\{R_n(f,q;x)\}$. It is proved that $\{R_n(f,q_n;x)\}$ converges uniformly to $f(x)$ for any $f(x) \in C[0,1]$ if and only if $q_n \to 1$. In the case $q > 0$, $q \neq 1$ being fixed the sequence $\{R_n(f,q;x)\}$ converges uniformly to $f(x) \in C[0,1]$ if and only if $f(x)$ is linear.

1. Introduction. In 1912 Bernstein ([**2**]) found his famous proof of the Weierstrass approximation theorem. Using probability theory he defined polynomials called nowadays *Bernstein polynomials* as follows.

Definition [**2**]. Let $f : [0,1] \to \mathbf{R}$. The *Bernstein polynomial* of f is

$$B_n(f;x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}, \quad n = 1, 2, \dots .$$

Bernstein proved that, if $f \in C[0,1]$, then the sequence $\{B_n(f;x)\}$ converges uniformly to $f(x)$ on $[0,1]$.

Definition. The *Bernstein operator* $B_n : C[0,1] \to C[0,1]$ is given by

$$(B_n)f(x) := B_n(f;x), \quad n = 1, 2, \dots .$$

Later it was found that Bernstein polynomials possess many remarkable properties, which made them an area of intensive research. A systematic treatment of the theory of Bernstein polynomials as it was

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until the 90's is presented, for example, in [**7**] and [**12**]. New papers are constantly coming out, cf., e.g., [**4**], and new applications and generalizations are being discovered, cf., e.g., [**6**] and [**9**]. The aim of these generalizations is to provide appropriate tools for studying various problems of analysis, geometry, statistical inference and computer science. The rapid development of q-calculus has led to the discovery of new generalizations of Bernstein polynomials involving q-integers. The first person to make progress in this direction was Lupaş. In 1987 he introduced, cf. [**8**], a q-analogue of the Bernstein operator and investigated its approximating and shape-preserving properties. In this paper we present new results concerning convergence of the Lupaş operator.

It is worth mentioning that in 1997 Phillips [**10**] introduced another generalization of Bernstein polynomials based on the q-integers called *q-Bernstein polynomials*. The q-Bernstein polynomials attracted a lot of interest and were studied widely by a number of authors. A survey of the obtained results and references on the subject can be found in [**11**]. The Lupaş operators are less known. However, they have an advantage of generating *positive* linear operators for all $q > 0$, whereas q-Bernstein polynomials generate positive linear operators only if $q \in (0, 1)$.

In this paper we would like to draw attention to the Lupaş q-analogue of the Bernstein operator and obtain new results related to the q-analogue.

To present results by Lupaş we recall the following definitions, cf. [**1**, Chapter 10].

Let $q > 0$. For any $n = 0, 1, 2, \ldots$, the *q-integer* $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1} \quad n = 1, 2, \ldots, \quad [0]_q := 0\,;$$

and the *q-factorial* $[n]_q!$ by

$$[n]_q! := [1]_q[2]_q \cdots [n]_q \quad n = 1, 2, \ldots, \quad [0]_q! := 1\,.$$

For integers $0 \le k \le n$ the *q-binomial*, or the Gaussian coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}\,.$$

Clearly, for $q = 1$,

$$[n]_1 = n, \quad [n]_1! = n!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}\,.$$

The q-binomial coefficients are involved in Cauchy's q-binomial theorem, cf. [**1**, Chapter 10, Section 10.2]. We will use the following particular cases of the theorem ([**1**, Chapter 10, Corollary 10.2.2]). The first one is an extension of Newton's binomial formula:

$$(1) \qquad (1+x)(1+qx)\cdots(1+q^{n-1}x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k \,.$$

Another needed formula, which can be derived from (1), is Euler's identity: for $|q| < 1$,

$$(2) \qquad \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{(1-q)^k [k]_q!} = \prod_{k=0}^{\infty} (1+q^k x)\,.$$

Following Lupaş we denote

$$(3) \qquad b_{nk}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(k-1)/2} x^k \,(1-x)^{n-k}}{(1-x+qx)\cdots(1-x+q^{n-1}x)}\,.$$

It follows from (1) that

$$(4) \qquad \sum_{k=0}^{n} b_{nk}(q;x) = 1, \quad x \in [0,1]\,.$$

Indeed, for $x = 1$, equality (4) is obvious. For $x \neq 1$, we get

$$\begin{aligned} \sum_{k=0}^{n} & \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k \,(1-x)^{n-k} \\ &= (1-x)^n \left(1 + \frac{x}{1-x}\right)\left(1 + q\,\frac{x}{1-x}\right)\cdots\left(1 + q^{n-1}\,\frac{x}{1-x}\right) \\ &= (1-x+qx)\cdots(1-x+q^{n-1}x)\,, \end{aligned}$$

and (4) is proved.

Definition (Lupaş [**8**]). Let $f \in C[0,1]$. The linear operator $R_{n,q} : C[0,1] \to C[0,1]$ defined by

$$R_{n,q}(f) = R_n(f,q;x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b_{nk}(q;x) \tag{5}$$

is called the *q-analogue* of the Bernstein operator.

We note that $R_n(f,1;x) = B_n(f;x)$, where $B_n(f;x)$ is a Bernstein polynomial of f. In the case $q \neq 1$ the operators $R_n(f,q;x)$ give *rational functions* rather than polynomials.

It follows directly from the definition that operators $R_n(f,q;x)$ possess the *end-point interpolation* property, that is,

$$\begin{gathered} R_n(f,q;0) = f(0), \quad R_n(f,q;1) = f(1) \\ \text{for all} \quad q > 0 \quad \text{and all} \quad n = 1,2,\dots . \end{gathered} \tag{6}$$

Besides, $R_n(f,q;x)$ are *positive linear operators* on $C[0,1]$ for all $q > 0$ and all $n = 1,2,\dots$.

Lupaş [**8**] investigated approximating properties of the operators $R_n(f,q;x)$ with respect to the uniform norm of $C[0,1]$. In particular, he obtained some sufficient conditions for a sequence $\{R_n(f,q_n;x)\}$ to be approximating for any function $f \in C[0,1]$ and estimated the rate of convergence in terms of the modulus of continuity. He also investigated behavior of the operators $R_n(f,q;x)$ for convex functions.

In this paper we present new results concerning convergence of the sequence $\{R_n(f,q_n;x)\}$ in $C[0,1]$. Our first theorem shows that $\{R_n(f,q_n;x)\}$ is an approximating sequence for any $f \in C[0,1]$, that is, $R_n(f,q_n;x)$ converges uniformly to $f(x)$ on $[0,1]$, if and only if $q_n \to 1$. We establish (Theorem 3) a symmetry between the cases $q \in (0,1)$ and $q \in (1,\infty)$. Finally, we discuss convergence of $\{R_n(f,q;x)\}$ for $q \neq 1$ being fixed. Our results imply that the classical case $q = 1$ is the best for approximation by the Lupaş operators if q is fixed. Therefore, we can expect applications of the q-analogue in the case when the value of q varies, which gives additional flexibility of approximation. Our approach is similar to the one developed in [**5**].

2. Statement of results. The sign $g_n(x) \rightrightarrows g(x)$ means uniform convergence of $\{g_n(x)\}$ to $g(x)$ as $n \to \infty$.

Theorem 1. *The sequence* $\{R_n(f,q_n;x)\}$ *is approximating for all* $f \in C[0,1]$ *if and only if* $q_n \to 1$.

Remark. This is a generalization of Theorem 2 of [**8**].

Theorem 1 implies that, if $q \neq 1$ is fixed, $\{R_n(f,q;x)\}$ may not be approximating for some continuous functions. We will discuss convergence of the sequence $\{R_n(f,q;x)\}$ in the case $q>0$, $q\neq 1$ being *fixed* and state necessary and sufficient conditions for the sequence to be approximating for f.

First, let $q\in(0,1)$. We set

(7)
$$b_{\infty\, k}(q;x) := \frac{q^{k(k-1)/2}\,(x/1-x)^k}{(1-q)^k[k]_q!\prod_{j=0}^{\infty}\left(1+q^j(x/(1-x))\right)}, \quad x\in[0,1)\,.$$

It follows from (2) that, for $q\in(0,1)$ and $x\in[0,1)$,

$$\sum_{k=0}^{\infty} b_{\infty\, k}(q;x) = 1\,. \tag{8}$$

Consider the function

$$\widetilde{R}_{\infty}(f,q;x) = \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)b_{\infty\, k}(q;x) & \text{if } x\in[0,1) \\ f(1) & \text{if } x=1\,. \end{cases} \tag{9}$$

Note that the function $\widetilde{R}_{\infty}(f,q;x)$ is well-defined on $[0,1]$ whenever $f(x)$ is bounded on $[0,1]$.

The following theorem shows that in the case $q\in(0,1)$ the sequence $\{R_n(f,q;x)\}$ is uniformly convergent for any $f\in C[0,1]$.

Theorem 2. *Let $q\in(0,1)$. Then, for any $f\in C[0,1]$,*

$$R_n(f,q;x) \rightrightarrows \widetilde{R}_{\infty}(f,q;x) \quad \text{for} \quad x\in[0,1]\,.$$

Remark. It is worth mentioning that the results above admit a probabilistic interpretation. Indeed, since $b_{nk}(q;x)\geq 0$ for $x\in[0,1]$ and by (4) $\sum_{k=0}^{n} b_{nk}(q;x)=1$, we may consider a sequence of discrete random variables $\{X_n\}$ with the distributions $\mathcal{P}_n$ defined by

$$\mathbf{P}\left\{X_n = \frac{[k]_q}{[n]_q}\right\} = b_{nk}(q;x), \quad k=0,1,\ldots,n\,.$$

Then $R_n(f,q;x) = \mathbf{E}\left[f(X_n)\right]$. For $q \in (0,1)$ consider a discrete random variable X_∞ with the distribution $\mathcal{P}$ defined by

$$\begin{cases} \mathbf{P}\left\{X_\infty = 1-q^k\right\} = b_{\infty\, k}(q;x) & \text{if } x \in [0,1), \\ \mathbf{P}\left\{X_\infty = 1\right\} = 1 & \text{if } x = 1\,. \end{cases}$$

The distribution is well-defined due to (8) and the fact that all $b_{\infty\, k}(q;x) \geq 0$ on $[0,1)$.

Then $R_\infty(f,q;x) = \mathbf{E}\left[f(X_\infty)\right]$ and Theorem 2 means that $\mathcal{P}$ is a limit distribution for the sequence $\{\mathcal{P}_n\}$.

The following theorem allows us to reduce the case $q \in (1,\infty)$ to the case $q \in (0,1)$.

Theorem 3. *Let $f \in C[0,1]$, $g(x) := f(1-x)$. Then for any $q > 0$,*

$$R_n(f,q;x) = R_n(g,1/q;1-x) \quad \text{for} \quad x \in [0,1]\,. \tag{10}$$

Remark. For $q = 1$ this equality coincides with formula (2.16) in [**4**].

Corollary 1. *Let $q \neq 1$ be fixed, $f \in C[0,1]$ and $g(x) := f(1-x)$. Then, for $x \in [0,1]$,*

$$R_n(f,q;x) \rightrightarrows R_\infty(f,q;x) = \begin{cases} \widetilde{R}_\infty(f,q;x) & \text{if } q \in (0,1), \\ \widetilde{R}_\infty(g,1/q;x) & \text{if } q \in (1,\infty)\,. \end{cases}$$

That is, the sequence $\{R_n(f,q;x)\}$ converges uniformly on $[0,1]$ for any $f \in C[0,1]$ and any $q > 0$ being fixed. An explicit form of the limit function for $q \in (0,1)$ is given by (9). *In the case $q \in (1,\infty)$,*

$$R_\infty(f,q;x) = \begin{cases} \sum_{k=0}^{\infty} f\left(1/q^k\right) b_{\infty\, k}\left(1/q;1-x\right) & \text{if } x \in (0,1], \\ f(0) & \text{if } x = 0, \end{cases} \tag{11}$$

where

$$b_{\infty\, k}\left(1/q;x\right) = \frac{q^k\left((1-x)/x\right)^k}{(q-1)\cdots(q^k-1)\prod_{j=0}^{\infty}\left(1+((1-x)/(q^j x))\right)}, \quad x \in (0,1]\,.$$

Using explicit forms (9) *and* (11) *we derive a necessary and sufficient condition for* $\{R_n(f,q;x)\}$ *to be an approximating sequence for* $q \neq 1$ *being fixed.*

Theorem 4. *Let* $q > 0$, $q \neq 1$ *be fixed and* $f \in C[0,1]$. *Then*

$$R_\infty(f,q;x) = f(x) \quad \text{for all} \quad x \in [0,1]$$

if and only if $f(x) = ax + b$ *for some* $a, b \in \mathbf{R}$.

That is, in contrast to the case $q = 1$, when $\{R_n(f,1;x)\} = \{B_n(f;x)\}$ is an approximating sequence for any $f \in C[0,1]$, the sequence $\{R_n(f,q;x)\}$, $q \neq 1$ is not approximating for f unless f is linear.

3. Some auxiliary results. It will be convenient to use for $x \in [0,1)$ the substitution

$$u := \frac{x}{1-x}, \quad u \in [0,\infty)\,. \tag{12}$$

We may express b_{nk} for $x \in [0,1)$ as follows:

$$\begin{aligned} b_{nk}(q;x) &= \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(k-1)/2}(1-x)^n \left(x/(1-x)\right)^k}{(1-x)^n \prod_{j=0}^{n-1}\left(1+q^j(x/(1-x))\right)} \\ &= \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(k-1)/2}u^k}{w_n(q;u)} =: \rho_{nk}(q;u), \end{aligned} \tag{13}$$

where

$$w_n(q;x) = \prod_{j=0}^{n-1}(1+q^j u).$$

Clearly,

$$\rho_{nk}(q;u) \;=\; b_{nk}\left(q; \frac{u}{u+1}\right)$$

and

$$R_n(f,q;x) = R_n\left(f,q;\frac{u}{u+1}\right) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)\rho_{nk}(q;u).$$

It follows from (4) that

$$\sum_{k=0}^{n} \rho_{nk}(q;u) = 1 \quad \text{for} \quad u \in [0,\infty). \tag{14}$$

Similarly we get from (7) that, for $q \in (0,1)$,
(15)

$$b_{\infty\, k}(q;x) = b_{\infty\, k}\left(q; \frac{u}{u+1}\right) = \frac{q^{k(k-1)/2}u^k}{(1-q)^k[k]_q!\, w_\infty(q;u)} := \rho_{\infty\, k}(q;u),$$

where

$$w_\infty(q;x) = \prod_{j=0}^{\infty}(1+q^j u).$$

Obviously, (8) implies that, if $q \in (0,1)$, then

$$\sum_{k=0}^{\infty} \rho_{\infty\, k}(q;u) = 1 \quad \text{for} \quad u \in [0,\infty). \tag{16}$$

We need the following fact stated in [**8**]. For the reader's favor we present its proof below.

Lemma 1 (Lupaş). *The following equalities are true*:

$$\begin{aligned} R_n(1,q;x) &= 1, \\ R_n(t,q;x) &= x\,; \end{aligned} \tag{17}$$

$$R_n(t^2,q;x) = x^2 + \frac{x(1-x)}{[n]_q} - \frac{x^2(1-x)(1-q)}{1-x+xq}\left(1-\frac{1}{[n]_q}\right). \tag{18}$$

Corollary 1. *Operators $R_n(f,q;x)$ reproduce linear functions, that is*

$$R_n(at+b,q;x) = ax+b \quad \text{for all} \quad q>0 \quad \text{and all} \quad n=1,2,\ldots. \tag{19}$$

Proof. Obviously, $R_n(1,q;x) = \sum_{k=0}^{n} b_{nk}(q;x) = 1$ according to (4). It suffices to prove (17) and (18) for $x \in [0,1)$, because for $x = 1$ they hold due to (6). Using the substitution (12) we get

$$\begin{aligned} R_n\left(t, q; \frac{u}{u+1}\right) &= \sum_{k=0}^{n} \frac{[k]_q}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(k-1)/2} u^k}{w_n(q;u)} \\ &= \frac{u}{u+1} \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{q^{(k-1)(k-2)/2} (qu)^{k-1}}{w_{n-1}(q;qu)} \\ &= \frac{u}{u+1} \sum_{k=0}^{n-1} \rho_{n-1,k}(q;qu) = \frac{u}{u+1}\,, \end{aligned}$$

and (17) is proven.

Likewise,

$$\begin{aligned} R_n\left(t^2, q; \frac{u}{u+1}\right) &= \sum_{k=0}^{n} \frac{[k]_q^2}{[n]_q^2} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(k-1)/2} u^k}{w_n(q;u)} \\ &= \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[k+1]_q}{[n]_q} \rho_{n-1,k}(q;qu) \\ &= \frac{u}{u+1} \frac{[n-1]_q}{[n]_q} \sum_{k=0}^{n-1} \left(\frac{1+q[k]_q}{[n-1]_q} \right) \rho_{n-1,k}(q;qu)\,. \end{aligned}$$

Using (14) and(17) we obtain

$$\begin{aligned} R_n\left(t^2, q; \frac{u}{u+1}\right) &= \frac{u}{u+1} \cdot \frac{[n-1]_q}{[n]_q} \left(\frac{1}{[n]_q} + q \cdot \frac{qu}{qu+1} \right) \\ &= \frac{u}{u+1} \frac{1}{[n]_q} + \frac{u}{u+1} \frac{qu}{qu+1} \left(1 - \frac{1}{[n]_q} \right), \end{aligned}$$

or, equivalently,

$$\begin{aligned} R_n(t^2, q; x) &= \frac{x}{[n]_q} + \frac{qx^2}{1-x+qx} \left(1 - \frac{1}{[n]_q} \right) \\ &= x^2 \left(1 - \frac{1}{[n]_q} \right) + \frac{x}{[n]_q} - \left(x^2 - \frac{qx^2}{1-x+qx} \right) \\ &\quad \times \left(1 - \frac{1}{[n]_q} \right) \\ &= x^2 + \frac{x(1-x)}{[n]_q} - \frac{x^2(1-x)(1-q)}{1-x+qx} \left(1 - \frac{1}{[n]_q} \right). \qquad \square \end{aligned}$$

Remark. The statement of Lemma 1 can also be derived from the following recurrence formula:

$$R_n(t^m, q; x) = \frac{x}{[n]_q^{m-1}} \sum_{r=0}^{m-1} \binom{m-1}{r} ([n]_q - 1)^r R_{n-1}\left(t^r, q; \frac{qx}{1-x+qx}\right),$$
$$m = 1, 2 \ldots .$$

Lemma 2. *Let $q \in (0,1)$ and $b_{nk}(q;x)$, $b_{\infty\, k}(q;x)$ be given by* (3) *and* (7), *respectively.*

Then

$$b_{nk}(q;x) \rightrightarrows b_{\infty\, k}(q;x) \quad \text{for} \quad x \in [0,1), \quad k = 0, 1, 2, \ldots .$$

Proof. After we apply the substitution (12), we consider the functions $\rho_{nk}(q;u)$ and $\rho_{\infty\, k}(q;u)$ defined by (13) and (15), respectively.

The lemma will be proven if we show that

$$\rho_{nk}(q;u) \rightrightarrows \rho_{\infty\, k}(q;u) \quad \text{for} \quad u \in [0,\infty) .$$

Since
$$\begin{bmatrix} n \\ k \end{bmatrix}_q \to \frac{1}{(1-q)^k [k]_q!}$$
and $u^k / w_\infty(q;u)$ is bounded on $[0,\infty)$, it suffices to prove that

$$\frac{u^k}{w_n(q;u)} \rightrightarrows \frac{u^k}{w_\infty(q;u)} \quad \text{for} \quad u \in [0,\infty) . \tag{20}$$

To prove this we use Dini's theorem on uniform convergence of a monotone sequence of continuous functions. We apply this theorem to the functions

$$\frac{u^k}{w_n(q;u)} \quad n > k \quad \text{and} \quad \frac{u^k}{w_\infty(q;u)} \tag{21}$$

on the compact set $[0,\infty]$. (We define all of the functions to be 0 at ∞.) $\square$

4. Proofs of the theorems.

Proof of Theorem 1. Since $R_n(f,q;x)$, define positive linear operators, the Korovkin theorem, cf. [**3**, Chapter 3, Section 3] implies that $R_n(f,q_n;x) \rightrightarrows f(x)$ for any $f \in C[0,1]$ if and only if

$$R_n(t^m,q_n;x) \rightrightarrows x^m \quad \text{for} \quad x\in[0,1] \quad \text{and} \quad m=0,1,2\,.$$

For $m=0,1$ this is true for any sequence $\{q_n\}$ due to (19).

It follows from (18) that $R_n(t^2,q_n;x) \rightrightarrows x^2$ for $x \in [0,1]$ if and only if

$$\frac{x(1-x)}{[n]_{q_n}} - \frac{x^2(1-x)(1-q_n)}{1-x+xq_n}\left(1-\frac{1}{[n]_{q_n}}\right) \rightrightarrows 0 \quad \text{for} \quad x\in[0,1]\,. \tag{22}$$

i) Suppose that $q_n \to 1$. Then, for any fixed positive integer k, we have $[n]_{q_n} \geq [k]_{q_n}$ when $n \geq k$. Therefore, $\liminf_{n\to\infty}[n]_{q_n} \geq \lim_{n\to\infty}[k]_{q_n} = k$. Since k has been chosen arbitrarily, it follows that $[n]_{q_n} \to \infty$. Hence,

$$\frac{x(1-x)}{[n]_{q_n}} \rightrightarrows 0 \quad \text{for} \quad x\in[0,1]\,.$$

At the same time, for $q \geq 1/2$, we have

$$\frac{x^2(1-x)}{1-x+qx} \leq \frac{1/4}{1-x+qx} \leq \frac{1/4}{1-x/2} \leq \frac{1}{2} \quad \text{for all} \quad x\in[0,1]\,.$$

Therefore, (22) is true.

ii) Suppose that, for any $f \in C[0,1]$, $R_n(f,q_n;x) \rightrightarrows f(x)$ for $x \in [0,1]$. Then $R_n(t^2,q_n;x) \rightrightarrows x^2$ for $x\in[0,1]$, and by (22),

$$\frac{x(1-x)}{[n]_{q_n}} - \frac{x^2(1-x)(1-q_n)}{1-x+xq_n}\left(1-\frac{1}{[n]_{q_n}}\right) \rightrightarrows 0 \quad \text{for} \quad x\in[0,1]\,.$$

Taking $x=1/2$, we conclude that

$$\frac{1/4}{[n]_{q_n}} - \frac{1/8(1-q_n)}{1/2(1+q_n)}\left(1-\frac{1}{[n]_{q_n}}\right) \longrightarrow 0 \quad \text{as} \quad n\to\infty\,,$$

or
$$\frac{1}{[n]_{q_n}} + \left(1 - \frac{2}{1+q_n}\right)\left(1 - \frac{1}{[n]_{q_n}}\right) \longrightarrow 0 \quad \text{as} \quad n \to \infty\,.$$
Suppose that $\{q_n\}$ does not tend to 1. Then it contains a subsequence $\{q_m\} \to t \neq 1$. If $t < 1$, then $[m]_{q_m} \to 1/(1-t)$, so
$$\begin{aligned}\frac{1}{[m]_{q_m}} + \left(1 - \frac{2}{1+q_m}\right)\left(1 - \frac{1}{[m]_{q_m}}\right) \longrightarrow 1 - t + \left(1 - \frac{2}{1+t}\right)t \\ = \frac{1-t}{1+t} \neq 0\,.\end{aligned}$$
For $t > 1$, we get $[m]_{q_m} \to \infty$ and
$$\frac{1}{[m]_{q_m}} + \left(1 - \frac{2}{1+q_m}\right)\left(1 - \frac{1}{[m]_{q_m}}\right) \longrightarrow 1 - \frac{2}{1+t} = \frac{t-1}{t+1} \neq 0\,.$$
(In particular, for $t = \infty$, the limit equals 1.)

The contradiction shows that $q_n \to 1$. □

Proof of Theorem 2. Due to (6) it suffices to prove that $R_n(f,q;x) \rightrightarrows R_\infty(f,q;x)$ for $x \in [0,1)$. Consider
$$\Delta := |R_n(f,q;x) - R_\infty(f,q;x)|\,.$$
For $x \in [0,1)$,
$$\Delta = \left| \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b_{nk}(q;x) - \sum_{k=0}^{\infty} f(1-q^k) b_{\infty\, k}(q;x) \right|.$$
Let $\varepsilon > 0$ be given. We choose $a \in (0,1)$ in such a way that $\omega_f(1-a) < \varepsilon/3$, where ω_f denotes the modulus of continuity of f. Let R be a positive integer satisfying the condition $1 - q^{R+1} \geq a$. Then $[k]_q/[n]_q \geq a$ for all $k \geq R+1$. Using (4) and (8), we get
$$\begin{aligned}\Delta &= \left| \sum_{k=0}^{n} \left(f\left(\frac{[k]_q}{[n]_q}\right) - f(1)\right) b_{nk}(q;x) - \sum_{k=0}^{\infty} \big(f(1-q^k) - f(1)\big) b_{\infty\, k}(q;x) \right| \\ &\leq \left| \sum_{k=0}^{R} \left(f\left(\frac{[k]_q}{[n]_q}\right) - f(1)\right) b_{nk}(q;x) - \sum_{k=0}^{R} \big(f(1-q^k) - f(1)\big) b_{\infty\, k}(q;x) \right|\end{aligned}$$

$$+\sum_{k=R+1}^{n}\left|f\left(\frac{[k]_q}{[n]_q}\right)-f(1)\right|b_{nk}(q;x)+\sum_{k=R+1}^{\infty}\left|f(1-q^k)-f(1)\right|b_{\infty\, k}(q;x)$$
$$=: \delta_1+\delta_2+\delta_3\,.$$

Since $f\left([k]_q/[n]_q\right)\to f(1-q^k)$ as $n\to\infty$, we get by applying Lemma 2 that $\delta_1<\varepsilon/3$ for n large enough.

Due to the fact that $b_{nk}(q;x)\geq 0$ for $x\in[0,1]$, we get the following estimate for δ_2:

$$\delta_2\leq\ \omega_f\,(1-a)\sum_{k=R+1}^{n}b_{nk}(q;x)\leq\ \omega_f\,(1-a)\sum_{k=0}^{n}b_{nk}(q;x)$$
$$=\ \omega_f\,(1-a)<\varepsilon/3$$

because of (4). Similarly, using (8) we get $\delta_3\leq\omega_f\,(1-a)<\varepsilon/3$. Thus, $\Delta<\varepsilon$ for n large enough. □

Proof of Theorem 3. For $x=0$ and $x=1$, the statement is obvious due to (6). So, we assume that $x\neq 0$.

Clearly,

$$R_n(f,q;x)\ =\ \sum_{k=0}^{n}f\left(\frac{[n-k]_q}{[n]_q}\right)b_{n,n-k}(q;x)\,.$$

Consider

$$\begin{aligned}b_{n,n-k}(q;x)&=\begin{bmatrix}n\\n-k\end{bmatrix}_q\frac{q^{(n-k)(n-k-1)}x^{n-k}(1-x)^k}{q^{n(n-1)/2}x^n\prod_{j=0}^{n}\left(1+((1-x)/q^jx)\right)}\\&=\begin{bmatrix}n\\k\end{bmatrix}_{1/q}\frac{(1/q)^{k(k-1)/2}\,(1-x)^k\,x^{n-k}}{\prod_{j=0}^{n}(x+((1-x)/q^j))}\\&=b_{nk}\left(\frac{1}{q};1-x\right).\end{aligned}$$

On the other hand,

$$\frac{[n-k]_q}{[n]_q}=\frac{[n]_{1/q}-[k]_{1/q}}{[n]_{1/q}}=1-\frac{[k]_{1/q}}{[n]_{1/q}}\,.$$

Therefore,

$$\begin{aligned} R_n(f,q;x) &= \sum_{k=0}^{n} f\left(1-\frac{[k]_{1/q}}{[n]_{1/q}}\right) b_{nk}\left(\frac{1}{q};1-x\right) \\ &= \sum_{k=0}^{n} g\left(\frac{[k]_{1/q}}{[n]_{1/q}}\right) b_{nk}\left(\frac{1}{q};1-x\right) \\ &= R_n\left(g,\frac{1}{q};1-x\right). \qquad \square \end{aligned}$$

Proof of Theorem 4. If $f(x)=ax+b$, then by (19) $R_n(f,q;x)=ax+b$ for all $n=1,2\ldots$, and therefore

$$R_\infty(f,q;x)=\lim_{n\to\infty} R_n(f,q;x)=ax+b=f(x)\,.$$

Now, suppose that $f\in C[0,1]$ and $R_\infty(f,q;x)=f(x)$ for all $x\in[0,1]$. Due to Theorem 3 it suffices to prove the statement in the case $q\in(0,1)$.

Consider the function

$$\varphi(x):=f(x)-(f(1)-f(0))x\,.$$

Obviously, $\varphi(0)=\varphi(1)$ and $R_\infty(\varphi,q;x)=\varphi(x)$. We will prove that $\varphi(x)=\varphi(0)=\varphi(1)$ for all $x\in[0,1]$. Let

$$M:=\max_{x\in[0,1]}\varphi(x)\,.$$

Assume that $M>\varphi(1)$. Then $M=\varphi(z)$ for some $z\in(0,1)$ and $\varphi(1-q^k)<M$ for $k>N_0$. Using (2) and positivity of $b_{\infty\,k}(q;x)$, $k=0,1,\ldots$, on $(0,1)$, we get

$$M=\varphi(z)=\sum_{k=0}^{\infty}\varphi(1-q^k)b_{\infty\,k}(q;z)<M\,.$$

The contradiction shows that $\varphi(x)\le\varphi(1)$ for all $x\in[0,1]$. Likewise, we prove that $\varphi(x)\ge\varphi(1)$ for all $x\in[0,1]$. Thus, $\varphi(x)\equiv\varphi(1)\equiv b$ for some $b\in\mathbf{R}$ and finally $f(x)=ax+b$ with $a=f(1)-f(0)$. $\square$

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