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PERIODIC BOUNDARY VALUE PROBLEM FOR FIRST ORDER IMPULSIVE DIFFERENTIAL EQUATION AT RESONANCE

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ABSTRACT. We develop a general theorem concerning the existence of solutions to the periodic boundary value problem for the first-order impulsive differential equation,

$$\begin{cases} x'(t) = f(t, x(t)) & t \in J \setminus \{t_1, t_2, \dots, t_k\} \\ \triangle x(t_i) = I_i(x(t_i)) & i = 1, 2 \dots, k \\ x(0) = x(T). \end{cases}$$

And using it we get a concrete existence result. Moreover, to our knowledge, the coincidence degree method has not been used with first order impulsive differential systems. Besides, our results can also be applied in studying the usual periodic boundary value problem at resonance without impulses.

1. Introduction. In recent years, many authors have discussed impulsive differential equation, see [**1, 3, 6, 7, 9**]. For example, He and Ge [**6**], Bainov and Hristova [**1**] and Liz [**9**] investigated the existence of solutions for first order impulsive equations by use of upper and lower solution methods. Frigon and O'Regan [**3**] investigated the existence of solutions to first order impulsive equations by the alternative theorem and upper and lower solution method. Dong [**2**], Liu and Yu [**8**] researched the existence of solutions to second order impulsive equations by making use of the coincidence degree theory and autonomous curvature bound set. However, to our knowledge, the coincidence degree method developed by Gaines and Mawhin [**5**] has not been used to the first order impulsive differential systems. In this paper, we are concerned with the periodic boundary value problem for

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the nonlinear impulsive differential equation:

$$x'(t) = f(t, x(t)), \quad t \in J' \tag{1.1}$$

$$\Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \dots, k \tag{1.2}$$

associated with the boundary value conditions

$$x(0) = x(T) \tag{1.3}$$

where $T > 0$, $J = [0, T]$, $0 < t_1 < t_2 < \cdots < t_k < T$, $J' = J \setminus \{t_1, t_2 \dots, t_k\}$, $x \in R$, $f : J \times R \to R$, $I_i : R \to R$, $i \in \{1, 2 \dots, k\}$, are continuous. $\triangle x(t_i) = x(t_i + 0) - x(t_i)$.

A map $x : J \to R$ is said to be a solution of (1.1)–(1.3) if it satisfies:

(1) $x(t)$ is continuously differentiable for $t \in J'$, both $x(t+0)$ and $x(t-0)$ exist at $t = t_i$, and $x(t_i) = x(t_i - 0)$, $i = 1, 2 \dots, k$.

(2) $x(t)$ satisfies the relations (1.1)–(1.3).

We shall use the continuation theorem of coincidence degree [**1**] to show a general theorem for the existence of solutions to the problem (1.1)–(1.3) and then use it to get concrete existence conditions in Section 3. This paper is motivated by [**2, 4, 8**].

2. Preliminary lemmas. For the convenience of the readers, we recall at first some notations. Moreover, we present a series of useful lemmas with respect to problem (1.1)–(1.3) that is important in the proof of our results. Consider an operator equation

$$Lx = Nx \tag{2.1}$$

where $L : \operatorname{dom} L \cap X \to Z$ is a linear operator, $N : X \to Z$ is a nonlinear operator, X and Z are Banach spaces. If $\dim \operatorname{Ker} L = \dim (Z/\operatorname{Im} L) < \infty$, and $\operatorname{Im} L$ is closed in Z, then L will be called a Fredholm mapping of index zero. And at the same time, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q$. It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible. We denote the inverse of this map by K_p.

Let Ω be an open and bounded subset of X. The map N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q) : \overline{\Omega} \to$

X is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 1 (Continuation theorem [**5**]). *Suppose that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. If the following conditions are satisfied*:

(i) *For each $\lambda \in (0,1)$, every solution x of*

$$Lx = \lambda Nx$$

is such that $x \notin \partial\Omega$.

(ii) $QNx \neq 0$ *for $x \in \partial\Omega \cap \operatorname{Ker} L$, and* $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where $Q : Z \to Z$ is a continuous projector with* $\operatorname{Im} L = \operatorname{Ker} Q$, *$J : Z/\operatorname{Im} L \to \operatorname{Ker} L$ is an isomorphism. Then the operator equation* (2.1) *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

In the following, in order to obtain the existence theorem of (1.1)–(1.3), we first introduce:

$X = PC[J,R] = \{x : J \to R \mid x(t)$ is continuous for $t \in J'$, $x(t+0)$, $x(t-0)$ exist at $t = t_i$ and $x(t_i) = x(t_i - 0),\ i = 1,2,\ldots,k$ and $x(0) = x(T)\}$

$Z = \{y : J \to R \mid y(t)$ is continuous$\} \times R^k$.

For every $x \in X$, denote its norm by

$$\|x\|_X = \sup_{t\in J} |x(t)|$$

and for every $z = (y,c) \in Z$, denote its norm by

$$\|z\| = \max\{\sup_{t\in J} |y(t)|, \|c\|\}.$$

We can prove that X and Z are Banach spaces. Let

$$\begin{aligned}
&\operatorname{dom} L = \{x : J \longrightarrow R \mid x(t) \text{ is differentiable for } t \in J'\} \bigcap X,\\
&L : \operatorname{dom} L \longrightarrow Z,\ x \longmapsto (x'(t), \triangle x(t_1), \ldots, \triangle x(t_k)),\\
&N : X \longrightarrow Z,\ x \longmapsto (f(t,x(t)), I_1(x(t_1)), \ldots, I_k(x(t_k))).
\end{aligned}$$

Then problem (1.1)–(1.3) can be written as $Lx = Nx$, $x \in \operatorname{dom} L$.

Lemma 2. *Suppose L is defined as above. Then L is a Fredholm mapping of index zero. Furthermore, for the problem* (1.1)–(1.3)

(2.2)
$$\operatorname{Ker} L = \{x(t) \in X, x(t) = c,\ c \in R\}$$
(2.3)
$$\begin{aligned}\operatorname{Im} L &= \big\{(y, a_1, a_2 \dots a_k) \in C[0,T] \times R^k : x'(t) = y(t),\\ &\qquad \triangle x(t_i) = a_i,\ i = 1, 2 \dots k,\ \text{for some } x(t) \in \operatorname{dom} L\big\}\\ &= \left\{(y, a_1, a_2 \dots a_k) \in PC[0,T] \times R^k : \int_0^T y(s)\, ds + \sum_{T > t_i} a_i = 0\right\}\end{aligned}$$

Proof. Firstly, it is easily seen that (2.2) holds. Next we will show that (2.3) holds. Since problem

$$\begin{gathered} x'(t) = y(t), \quad t \in J' \\ \triangle x(t_i) = a_i \end{gathered} \tag{2.4}$$

has solution $x(t)$ satisfying $x(0) = x(T)$ if and only if

$$\int_0^T y(s)\, ds + \sum_{T > t_i} a_i = 0. \tag{2.5}$$

In fact, if (2.4) has solution $x(t)$ such that $x(0) = x(T)$, then from (2.4) we have

$$x(t) = x(0) + \int_0^t y(s)\, ds + \sum_{t > t_i} a_i$$

thus

$$x(T) = x(0) + \int_0^T y(s) ds + \sum_{T > t_i} a_i.$$

In view of $x(0) = x(T)$, we have

$$\int_0^T y(s)\,ds + \sum_{T>t_i} a_i = 0.$$

Hence, (2.5) holds.

On the other hand, if (2.5) holds setting

$$x(t) = c + \int_0^t y(s)\,ds + \sum_{t>t_i} a_i$$

where $c \in R$ is an arbitrary constant, then it is clear that $x(t)$ is a solution of (2.4) and satisfies $x(0) = x(T)$. Hence, (2.3) holds.

Take the projector $Q : Z \to Z$ as follows:

$$Q(y, a_1, a_2, \dots, a_k) = \left(\frac{1}{T}\left[\int_0^T y(t)dt + \sum_{T>t_i} a_i\right], 0 \dots, 0\right) \tag{2.6}$$

and for $(y, a_1, a_2 \dots, a_k) \in Z$. Let

$$z = (y_1, a_1, a_2 \dots, a_k) = (y, a_1, \dots, a_k) - Q(y, a_1, a_2, \dots, a_k).$$

Then $z \in \operatorname{Im} L$. Thus, we have

$$\dim (Z \setminus \operatorname{Im} L) = \dim \operatorname{Im} Q = 1 = \dim \operatorname{Ker} L,$$

moreover by the Ascoli-Arzela theorem, L is a Fredholm mapping of index zero. $\square$

3. Main results. In this section, we shall apply Lemma 1 to obtain a general theorem for the existence of solutions to the problem (1.1)–(1.3) and use the general theorem to get a concrete existence condition of the same problem.

For any subset $G \subset R$, let

$$\Omega = \{x \in X | x(t) \in G, \text{ for all } t \in J',\ x(t_i + 0) \in G,\ i = 1, 2, \dots, k\}$$
$$\Omega \bigcap \operatorname{Ker} L = \{x = c \,|\, c \in R\} := G_1.$$

Theorem 1. *Let the following conditions be satisfied.*

(1) *Let $G \subset R$ be an open bounded subset such that for every $\lambda \in (0,1)$, each possible solution $x(t)$ of the auxiliary system*

$$\begin{cases} x'(t) = \lambda f(t, x(t)) & t \in J' \\ \triangle x(t_i) = \lambda I_i(x(t_i)) & i = 1, 2 \ldots, k, \\ x(0) = x(T) \end{cases} \tag{3.1}$$

satisfies $x(t) \notin \partial\Omega$.

(2) *$h(c) \neq 0$, for $c \in \partial G_1$, $deg(h, G_1, 0) \neq 0$ where h is defined by*

$$h(c) = \frac{1}{T}\bigg[\int_0^T f(t,c)\,dt + \sum_{T>t_i} I_i(c)\bigg], \quad c \in R.$$

Then the PBVP (1.1)–(1.3) *has at least one solution $x(t) \in G$, for $t \in J$.*

Proof. By Lemma 2, we know that L is a Fredholm operator of index zero, and the problem (3.1) can be written as $Lx = \lambda Nx$. Set $\Omega = \{x \in X : x(t) \in G, \text{ for } t \in J,\ x(t_i + 0) \in G, \text{ for } i = 1, \ldots k\}$. Then Ω is open and bounded. To use Lemma 1, we show at first N is L-compact on $\overline{\Omega}$.

Defining a projector

$$P : X \longrightarrow \operatorname{Ker} L, \quad P(x(t)) = x(0),$$

then $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$ can be written in

$$K_p z = \int_0^t y(s)\,ds + \sum_{t>t_i} a_i. \tag{3.2}$$

In fact, we have $K_p L = I - P$; thus, for any $x \in \operatorname{dom} L$, $K_p L x = x - x(0)$, so (3.2)holds.

Again from (2.6) and (3.2), we have

$$QNx = \bigg(\frac{1}{T}\bigg[\int_0^T f(s, x(s))ds + \sum_{T>t_i} I_i(x(t_i))\bigg], 0 \ldots 0\bigg)$$

$$
\begin{aligned}
&K_p(I-Q)Nx \\
&\quad = \int_0^t \left[f(s,x(s)) - \frac{1}{T}\left(\int_0^T f(\tau,x(\tau))\,d\tau + \sum_{T>t_i} I_i(x(t_i)) \right) \right] ds \\
&\qquad + \sum_{t>t_i} I_i(x(t_i)).
\end{aligned}
$$

By using the Ascoli-Arzela theorem, we can prove that $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact, thus N is L-compact on $\overline{\Omega}$.

At last, we will prove that (i) and (ii) of Lemma 1 are satisfied. Note that $x \in \partial\Omega$, if and only if $x(t) \in \overline{G}$, for $t \in J$, and either $x(s) \in \partial G$, for some $s \in J$, or $x(t_{i_0}+0) \in \partial G$, for some $i_0 = \{1, 2 \dots k\}$. Then assumption (i) follows from condition (1).

Let $J : \operatorname{Im} Q \to \operatorname{Ker} L; (c, 0 \dots 0) \to c$ be the isomorphism. Then

$$
JQNx = \frac{1}{T}\left[\int_0^T f(s,x(s))\,ds + \sum_{T>t_i} I_i(x(t_i)) \right].
$$

Since $\operatorname{Ker} L = R$, $\Omega \cap \operatorname{Ker} L = \{c \in R; c \in G\}$, let $JQN = h$, in view of (2), $h(c) \neq 0$, for $c \notin \partial G_1$, $\deg(JQN, \Omega \cap KerL, 0) = \deg(h, G, 0) \neq 0$, i.e., condition (2) implies (ii) of Lemma 1 and the proof is finished. $\square$

Remark 1. Comparing Theorem 1 with Theorem 3.1 in [**3**], we can easily see that

(1) In this paper, L is a Fredholm mapping of index zero. However, in [**3**], L is asked to be invertible. If $a(t) \equiv 0$, Theorem 3.1 in [**3**] requires $m_1, \dots, m_p \neq 1$ whereas in Theorem 1 we are interested in the case $m_1 = m_2 = \cdots = m_p = 1$. So the results obtained are different from each other.

(2) In [**3**], the auxiliary system of Theorem 3.1 is

$$
(3.1_\lambda) \qquad \begin{cases} y'(t) - a(t)y(t) = f(t, y(t), \lambda) & \text{a.e. } t \in [0,T] \\ y(t_k^+) = \lambda I_k(y(t_k^-)) + (1-\lambda)m_k y(t_k^-) & k = 1, 2, \dots, p \\ y(0) = y(T). \end{cases}
$$

Therefore, $(3.1)_1$ is not equivalent to the impulsive periodic problem

$$(1.2)\qquad \begin{cases} y'(t) = f(t, y(t)) & t \neq t_k \\ y(t_k^+) = I_k(y(t_k^-)) & k = 1, 2, \dots, p \\ y(0) = y(T). \end{cases}$$

Since the relation between $f(t, y(t), 1)$ and $f(t, y(t))$ is not confined. In other words, Theorem 3.1 in [**3**] has no relation with (1.2).

Theorem 2. *Let $f : J \times R \to R$ be a continuous function and assume that there exists a constant $M > 0$ such that*

$$(3.3)\qquad xf(t, x) > 0, \quad x(t_i)I_i(x(t_i)) > 0,$$

for $|x| \geq M$, $t \in J$, $i = 1, 2, \dots, k$.

Then the PBVP (1.1)–(1.3) has at least one solution $x(t) \in PC[0, T]$.

Proof. Suppose $x(t)$ is a solution to PBVP (3.1). We show that $\|x\| < M$, when $\lambda \in (0, 1)$. Otherwise, there is $t_0 \in [0, T) \cup \{t_i^+,\ i = 1, 2, \dots, k\}$ such that $\|x\| = |x(t_0)| = \sup_{t \in J} |x(t)| \geq M$.

Without loss of generality we suppose that $x(t_0) \geq M$.

If $t_0 \notin \{t_i, t_i^+, i = 1, 2, \dots, k\} \cup \{0\}$, then one has

$$x(t_0) = \sup_{t \in J} x(t) \geq M, x'(t_0) = 0$$

However, by condition (3.3), $x'(t_0) = \lambda f(t_0, x(t_0)) > 0$, a contradiction.

If $t_0 \in \{t_i,\ i = 1, 2, \dots, k\}$, say $t_0 = t_i$, then $I_i(x(t_i)) > 0$ and hence

$$x(t_i^+) = x(t_i) + \lambda I_i(x(t_i)) > x(t_i)$$

which contradicts the assumption $x(t_i) = \sup_{t \in J} |x(t)|$.

If $t_0 \in \{t_i^+,\ i = 1, 2, \dots, k\}$, say $t_0 = t_i^+$, then there is $\sigma \in (0, t_{i+1} - t_i)$, (if $i = k, t_{i+1}$ is replaced by T), such that $x(t) > M,\ t \in (t_i, t_i + \sigma)$. Since $x'(t) = \lambda f(t, x(t))$, $t \in (t_i, t_i + \sigma)$, $x'(t_i^+) = \lambda f(t_i, x(t_i^+)) > 0$, then

$$x(t_i+\sigma) = x(t_i^+) + \int_{t_i}^{t_i+\sigma} x'(s)\,ds > x(t_i^+)$$

which contradicts $x(t_i^+) = \sup_{t\in J} |x(t)|$.

If $t_0 = 0$, $x(0) = \sup_{t\in J} |x(t)| \geq M$, then $x'(0) = \lambda f(0, x(0)) > 0$. So there is a $\sigma > 0$ small enough, such that $x'(t) > 0$, $t \in (0,\sigma)$ which yields

$$x(\sigma) = x(0) + \int_0^{\sigma} x'(s)\,ds > x(0),$$

a contradiction.

So $\|x\|_X < M$ holds for all cases. Let $\Omega = \{x \in X \mid \|x\|_X < M+1\}$. We have $x \notin \partial\Omega$.

By the proof of Theorem 1, we know that $h(c) = JQNc$

$$h(c) = 0 \Longleftrightarrow JQNc = 0 \Longleftrightarrow QNc = 0 \Longleftrightarrow Nc \in \operatorname{Im} L$$

one has $x = c$, $|c| < M+1$. When $c = M+1$ or $c = -(M+1)$ by condition (3.3), it holds that

$$\operatorname{sgn} c \cdot \left[\int_0^T f(\tau, c)\,d\tau + \sum_{T>t_i} I_i(c) \right] > 0,$$

$c \in \partial G_1 = \{-M-1, M+1\}$.

Obviously

$$\operatorname{sgn} c \cdot h(c) = \operatorname{sgn} c \cdot \frac{1}{T} \cdot \left[\int_0^T f(\tau, c)\,d\tau + \sum_{T>t_i} I_i(c) \right] > 0$$

for $c \in \partial G_1 = \{-M-1, M+1\}$. Then

$$\deg\{JQN, G_1, 0\} = \deg\{h, (-M-1, M+1), 0\} = 1$$

the conditions of Theorem 1 are satisfied, the proof of Theorem 2 is completed. $\square$

Remark 2. Theorem 2 is not included in Theorem 3.1 in [**3**] because M and $-M$ cannot serve as the lower and upper solutions for (1.2).

For example, if $\alpha(t) \equiv M$ is a lower solution for (1.2), it must hold that

$$0 \le f(t, M), \quad M \le I_k(M).$$

However, in our theorem, $I_k(M) \ge M$ is not required but $I_k(x) > 0$ for $x \ge M$.

Finally, we present an example to check our result.

Example. Consider the boundary value problem

$$\begin{cases} x'(t) = x(t)[t^2 + 2 - \sin x(t)] + \sin(x(t) + 1) \\ \qquad\qquad\qquad t \in [0, T],\ t \neq 1/3 \\ \triangle x(1/3) = x(1/3)[4 - \cos x(1/3)] - 1/3 \sin x(1/3) \\ \qquad\qquad\qquad t = 1/3 \\ x(0) = x(T) \end{cases} \tag{3.4}$$

where $f(t, x) = x(t)[t^2 + 2 - \sin x(t)] + \sin(x(t) + 1)$, $I(x) = x(t) \times [4 - \cos x(t)] - t \sin x(t)$.

In this example, we note that $t_k = 1/3$, $k = 1$.

We choose a constant $M > 0$ large enough. When $|x| \ge M$, obviously

$$x \cdot f(t, x) = x^2(t)[t^2 + 2 - \sin x(t)] + x(t)[\sin(x(t) + 1)] > 0$$

$$x \cdot I(x) = x^2(t)[4 - \cos x(t)] - t x(t) \sin x(t) > 0$$

that is to say, the condition of Theorem 2 is satisfied. The BVP (3.4) has at least one solution. $\square$

REFERENCES

1. D.D. Bainov and S.G. Hristova, *The method of quasilinearization for the periodic boundary value problem for systems of impulsive differential equations*, Appl. Math. Comput. **117** (2001), 73–85.

2. Yujun Dong, *Periodic solutions second order impulsive differential systems*, Nonlinear Anal. TMA **27** (1996), 811–820.

3. M. Frigon and D. O'Regan, *Existence results for first-order impulsive differential equations*, J. Math. Anal. Appl. **193** (1995), 96–113.

4. W. Feng and J.R.L. Webb, *Solvability of m-point boundary value problems with nonlinear growth*, J. Math. Anal. Appl. **212** (1997), 481–492.

5. R.E. Gaines and J.L. Mawhin, *Coincidence degree and nonlinear differential equations*, Lecture Notes in Math., vol. 568, Springer, New York, 1997.

6. Xiaoming He and Weigao Ge, *First-order impulsive functional differential equations with periodic boundary value conditions*, Indian J. Pure Appl. Math. **33** (2002), 1257–1273.

7. Lakshmikanthamv, D.D. Bainov and P.S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore 1989.

8. Bing Liu and Jianshe Yu, *Existence of solution for m-point boundary value problem of second-order differential systems with impulses*, Appl. Math. Comput. **125** (2002), 155–175.

9. E. Liz, *Existence and approximation of solutions for impulsive first order problem with nonlinear boundary conditions*, Nonlinear Anal. TMA **25** (1995), 1191–1198.

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