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THE DE LA VALLÉE POUSSIN THEOREM FOR VECTOR VALUED MEASURE SPACES

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ABSTRACT. The purpose of this paper is to extend the de la Vallée Poussin theorem to $\mathrm{cabv}(\mu, X)$, the space of measures defined in Σ with values in the Banach space X which are countably additive, of bounded variation and μ-continuous, endowed with the variation norm.

1. Introduction. The celebrated theorem of de la Vallée Poussin, VPT in brief, characterizes the Lebesgue uniform integrability in the following way.

Let $\mathcal{F}$ be a family of scalar measurable functions on a finite measure space (Ω, Σ, μ). Then the following are equivalents.

(i) $\sup_{f\in\mathcal{F}} \int_\Omega |f|\, d\mu < \infty$ and $\mathcal{F}$ is uniformly integrable, i.e., the set $\{\int_E |f|\, d\mu, f \in \mathcal{F}\}$ converges uniformly to zero in A if $\mu(E) \to 0$.

(ii) If $E_{nf} = \{\omega \in \Omega : |f(\omega)| > n\}$, then $\lim_{n\to\infty} \int_{E_{nf}} |f|\, d\mu = 0$, uniformly in $\mathcal{F}$.

(iii) There is a Young function Φ with the property that $\Phi(x)/x$ is an increasing function: $\lim_{x\to\infty}(\Phi(x)/x) = \infty$, and there is a constant $0 < C < \infty$ such that $\sup_{f\in\mathcal{F}} \int_\Omega \Phi(|f|)\, d\mu = C$.

The theorem of Dunford states that the uniform integrability of a subset K of $L_1(\mu)$ is equivalent to the relative weak compactness of K, and in [**1**, subsection 2.1] Alexopoulos gives more information on the structure of K in the corresponding Orlicz space $L_\Phi(\mu)$. The uniform integrability also is essential in the study of the relative weak compactness in $L_1(\mu, X)$, in fact every conditionally weakly compact subset of $L_1(\mu, X)$ is uniformly integrable, [**3**, Theorem IV]. The purpose of this paper is to extend the VPT to $\mathrm{cabv}(\mu, X)$. This result allows us to characterize, in terms of the Orlicz theory, a condition in $\mathrm{cabv}(\mu, X)$ which plays the role of the uniform integrability in $L_1(\mu, X)$.

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2. Definitions, notation and basic facts. The notation is standard. We remit to [**4**] and [**7**] for details.

A Young function is a convex function $\Phi : \mathbf{R} \to \mathbf{R}^+$ such that $\Phi(-x) = \Phi(x)$, $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. In this paper we are interested in the class of Young functions Φ which appears in the VPT, i.e., $(\Phi(x)/x)$ is increasing and $\lim_{x\to\infty}(\Phi(x)/x) = \infty$. Basically there are two types of Young functions in the VPT: a) If $\mathcal{F}$ is a bounded subset of $\mathcal{L}_\infty$, then we can take a Young function Φ that jumps to infinity in some $a > 0$ and that $\Phi(x) = 0$ if $0 \le x < a$. b) Otherwise, Φ can be taken continuous with $\Phi(x) = 0$ if and only if $x = 0$. We denote by YVP the class of Young functions Φ of a) or b).

From now on, (Ω, Σ, μ) will represent an atomless abstract finite measure space, where Σ is a σ-algebra on which μ is a σ-additive and nonnegative measure. Let F be a countably additive X-valued and μ-continuous measure in (Ω, Σ, μ). The Φ-variation of F

$$I_\Phi(F) := \sup_\pi \left\{ \sum_n \Phi\left(\frac{\|F(A_n)\|}{\mu(A_n)}\right)\mu(A_n)\right\}$$

where the supremum is taken over all partitions $\pi = \{A_n\}$ of Ω in Σ and the convention $0/0 = 0$ is employed. If $I_\Phi(F) < \infty$, F is said to be of bounded Φ-variation. We denote by $\mathrm{cabv}_\Phi(\mu, X)$ the space of countably additive and μ-continuous X-valued measures F such that there is a $K > 0$ with $I_\Phi(F/K) \le 1$ which is a Banach space with the norm $V_\Phi(F) := \inf\{K > 0 : I_\Phi(F/K) \le 1\}$. In particular, $\mathrm{cabv}_\infty(\mu, X) = \{F : \Sigma \to X : \exists K > 0, \forall A \in \Sigma, \|F(A)\| \le k\mu(A)\}$ with $V_\infty(F) = \inf\{K > 0 : \forall A \in \Sigma, \|F(A)\| \le \mu(A)\}$, and if $\Phi(x) = |x|$ we will write $\mathrm{cabv}(\mu, X)$ and $|\cdot|$, the variation norm, to the corresponding space and norm.

$L_\Phi(\mu, X)$ is an isometric subspace of $\mathrm{cabv}_\Phi(\mu, X)$ under the map $f \to F$ such that $F(A) = \int_A f\, d\mu$. We denote by $\chi(\mu, X)$ the subset of the X-valued step functions defined in (Ω, Σ, μ) and by $\mathcal{M}_\Phi(\mu, X)$ the closed linear span of $\chi(\mu, X)$ in $L_\Phi(\mu, X)$. Given $\mathcal{F}$ a set of functions $f : \Omega \to X$ such that $\|f(\cdot)\| \in \mathcal{L}_1(\mu)$, we say that $\mathcal{F}$ is uniformly integrable if and only if $\{\|f(\cdot)\|, f \in \mathcal{F}\}$ is uniformly integrable in $\mathcal{L}_1(\mu)$. In particular, from the VPT we know that if $\Phi \in$ YVP, then the bounded sets of $L_\Phi(\mu, X)$ are uniformly integrable.

If $F \in \mathrm{cabv}\,(\mu, X)$ and $A \in \Sigma$, $F \cdot A$ is the countably additive and μ-continuous X-valued measure of bounded variation such that

$F \cdot A(E) = F(A \cap E)$ for all $E \in \Sigma$. With this notation, the variational measure of F is the countably additive and μ-continuous scalar measure μ_F on Σ such that $\mu_F(A) = |F \cdot A|$.

In [**8**], Ülger gives a characterization of the bounded sets of $L_\infty(\mu, X)$ which are relatively weakly compact in $L_1(\mu, X)$ using the Talagrand's results [**6**]. In [**2**] Diestel, Ruess and Schachermayer remove the $L_\infty(\mu, X)$-boundedness condition giving, with an independent and easier proof, the best characterization of the relatively weakly compact subsets of $L_1(\mu, X)$. In this setting, the characterization of the relative weak compactness in $L_1(\mu, X)$ of [**2**] can be reformulated in the following way.

Theorem A. *Let K be a bounded subset of $L_1(\mu, X)$. Then the following are equivalents.*

(i) *K is weakly relatively compact.*

(ii) *K is a bounded subset of $L_\Phi(\mu, X)$ for some $\Phi \in$ VPT, and for every sequence $(f_n) \in K$ there exists a sequence $(\hat{f}_n)$ with $f_n \in$ co $\{f_m, m \geq n\}$ such that $(\hat{f}_n(\omega))$ is norm convergent in X for almost everywhere.*

(iii) *K is a bounded subset of $L_\Phi(\mu, X)$ for some $\Phi \in$ VPT, and for every sequence $(f_n) \in K$ there exists a sequence $(\hat{f}_n)$ with $\hat{f}_n \in$ co $\{f_m, m \geq n\}$ such that $(\hat{f}_n(\omega))$ is weakly convergent in X for almost everywhere.*

The notion of complementary Young functions is essential in the Orlicz theory, especially in the characterization of duals of Orlicz spaces. Given a Young function Φ, we say that a Young function Ψ is the complementary of Φ if $\Psi(x) := \sup\{t|x| - \Phi(t), t > 0\}$. We recall that if Φ is a continuous Young function of YVP, Ψ has the same properties. In general, if Φ is continuous with $\Phi(x) = 0$ if and only if $x = 0$, then $(\mathcal{M}_\Phi(\mu, X))' = \mathrm{cabv}_\Psi(\mu, X')$, see [**5**]. Therefore a suitable adaptation of the proof of [**2**, Theorem 2.1] produces the following extension of Corollary 3.4 of [**2**].

Theorem B. *Let Φ be a continuous function of* YVP*, and let K be a bounded subset of $\mathcal{M}_\Phi(\mu, X)$. Then the following are equivalent*:

(i) *K is relatively weakly compact in $\mathcal{M}_\Phi(\mu, X)$.*

(ii) *For every sequence $(f_n) \in K$ there exists a sequence $(\hat{f}_n)$ with $\hat{f}_n \in \mathrm{co}\{f_m, m \geq n\}$ such that $(\hat{f}_n(\omega))$ is norm convergent in X for almost everywhere.*

(iii) *For every sequence $(f_n) \in K$, there exists a sequence $(\hat{f}_n)$ with $\hat{f}_n \in \mathrm{co}\{f_m, m \geq n\}$ such that $(\hat{f}_n(\omega))$ is weakly convergent in X for almost everywhere.*

In $\mathrm{cabv}_\Phi(\mu, X)$ it is possible to also define the Orlicz norm $\|\cdot\|_\Phi$:

$$\|F\|_\Phi := \sup\left\{\sup_\pi \sum_{A_i\in\pi} \frac{\|F(A_i)\|\|H(A_i)\|}{\mu(A_i)},\right.$$
$$\left. H \in \mathrm{cabv}_\Psi(\mu, X') : V_\psi(H) \leq 1\right\}$$

where Ψ is the complementary of Φ and π is a partition of Ω in Σ. This norm is equivalent to $V_\Phi(\cdot)$ and $V_\Phi(F) \leq \|F\|_\Phi \leq 2V_\Phi(F)$ for every $F \in \mathrm{cabv}_\Phi(\mu, X)$, see [**7**, p. 29]. Moreover, if $f \in L_\Phi(\mu, X)$ and $g \in L_\Psi(\mu, X')$, then from [**7**, p. 33], $\int_\Omega \|f(\omega)\|\|g(\omega)\|\,d\mu \leq 2V_\Phi(f)V_\Psi(g)$. Inspired in the Orlicz norm, consider the functional $V_\Phi^v(F)$ defined for F in $\mathrm{cabv}_\Phi(\mu, X)$ by

$$V_\Phi^v(F) := \sup\left\{\sup_\pi \sum_{A_i\in\pi} \frac{\mu_F(A_i)\|H(A_i)\|}{\mu(A_i)},\right.$$
$$\left. H \in \mathrm{cabv}_\Psi(\mu, X') : V_\Phi(H) \leq 1\right\}$$

$V_\Phi^v(F)$ is unambiguously defined as a finite number or as $+\infty$. If $V_\Phi^v(F)$ is finite, F is said to be of bounded Φ^v-variation. Then

Definition 2.1. $\mathrm{cabv}_\Phi^v(\mu, X)$ is the linear subspace of $\mathrm{cabv}_\phi(\mu, X)$ consisting of all measures of bounded Φ^v-variation, with the norm $V_\Phi^v(\cdot)$.

It is evident that

Corollary 2.1. *$L_\Phi(\mu, X)$ is an isometric subspace of* $\mathrm{cabv}_\Phi^v(\mu, X)$.

3. Main results. The aim of this paper is to prove the following extension of the VPT.

Theorem 3.1. *Let* $\mathcal{F}$ *be a subset of* $\operatorname{cabv}(\mu, X)$. *Then the following are equivalents*:

(i) $\mathcal{F}$ *is bounded in* $\operatorname{cabv}(\mu, X)$, *and the set of measures* $\{\mu_F, F \in \mathcal{F}\}$ *is uniformly* μ*-continuous, i.e.,* $\lim_{\mu(A)\to 0} \mu_F(A) = 0$ *uniformly in* $\mathcal{F}$.

(ii) $\mathcal{F}$ *is a bounded subset of* $\operatorname{cabv}^v_\Phi(\mu, X)$ *for some* $\Phi \in \mathrm{YVP}$.

If X has the Radon-Nikodym property, this is exactly the de la Vallée-Poussin theorem. If not, the nice thing is that every $F \in \operatorname{cabv}(\mu, X'')$ has a weak*-derivative. For a good exposition of this space of weak*-derivative functions, with range in a Banach dual space Y', in this case in X'', see Schlüchtermann [**5**]. We extend the definition [**5**, 1.2.6] to our setting.

Definition 3.1. Given a Young function Φ and a Banach space Y, let $\mathcal{L}^{\omega^*}_\Phi(\mu, Y')$ be the space of functions $f : \Omega \to Y'$ such that

(a) f is weak*-measurable, i.e., $\langle f(\cdot), y\rangle$ is measurable for all $y \in Y$.

(b) There exists $h \in \mathcal{L}_1(\mu)$, and there exists $H > 0 : \Phi(H\|f(\omega)\|) \le h(\omega)$, $\omega \in \Omega$.

In $\mathcal{L}^{\omega^*}_\Phi(\mu, Y')$ we define the semi-norm

$$\|f\|_{\Phi^*} := \sup\left\{ \int_\Omega |\langle f(\omega), g(\omega)\rangle|\, d\mu,\ g \in \chi(\mu, Y) : \|g\|_{L_\Psi} \le 1\right\}$$

where Ψ is the complementary Young function of Φ.

The identification of functions f, g such that for all $x \in Y$, $\langle x, f(\omega) - g(\omega)\rangle = 0$ almost everywhere in $\omega \in \Omega$, produces the corresponding Banach space $L^{\omega^*}_\Phi(\mu, Y')$. The relationship between $L^{\omega^*}_1(\mu, Y')$ and $\operatorname{cabv}(\mu, Y')$ is clearly exposed in [**5**, Lemma 1.2.7 and Theorem 1.2.8]: For every $f \in L^{\omega^*}_1(\mu, Y')$, the measure F_f such that $F_f(A)$ is the Gelfand integral $\int_A f(\omega)\, d\mu$ and F_f belongs to $\operatorname{cabv}(\mu, Y')$ with $\|f\|_{1^*} = |F_f|$. Conversely, if $F \in \operatorname{cabv}(\mu, Y')$ there is an $f_F \in L^{\omega^*}_1(\mu, Y')$ such that $F(A)$ is the Gelfand integral $\int_A f_F\, d\mu$,

$\|f_F(\cdot)\| \in L_1(\mu), \|f_F\|_{1_*} = \int_\Omega \|f_F(\omega)\|\, d\mu$ and $\|f_F\|_{1^*} = |F|$. It is clear that for every $F \in \mathrm{cabv}\,(\mu, Y')$, $\mu_F(A) = \|f_F \cdot \chi_A\|_{1^*}$.

We set

Definition 3.2. $\mathrm{cabv}^s_\Phi(\mu, X') := \{F \in \mathrm{cabv}_\Phi(\mu, X') : \|f_F(\cdot)\| \in L_\Phi(\mu)\}$, with the norm $V^s_\Phi(F) = V_\Phi(\|f_F(\cdot)\|)$.

It is straightforward that

Corollary 3.1. $\mathrm{cabv}^s_\Phi(\mu, X')$ *is an isometric subspace of* $\mathrm{cabv}^v_\Phi(\mu, X')$ *which contains to* $L_\Phi(\mu, X)$.

Proof of Theorem 3.1. (i) → (ii). We identify isometrically every $F \in \mathrm{cabv}\,(\mu, X)$ with $iF \in \mathrm{cabv}\,(\mu, X'')$ where $i : X \to X''$ is the natural isometry. It is clear that the set

$$\mathcal{G} := \{\langle f_{iF}(\cdot), g(\cdot)\rangle, F \in \mathcal{F}, g \in \chi(\mu, X') : \|g\|_{L_\infty} \le 1\}$$

is bounded in $L_1(\mu)$ and uniformly integrable, and then from VPT there is a Young function $\Phi' \in \mathrm{YVP}$ and a $C > 0$ such that

$$\sup\left\{ \int_\Omega \Phi'(|\langle f_{iF}(\omega), g(\omega)\rangle|)\, d\mu, F \in \mathcal{F}, g \in \chi(\mu, X') : \|g\|_{L_\infty} \le 1 \right\} \le C$$

and then

$$\sup\{V_\Phi(\langle f_{iF}(\omega), g(\omega)\rangle), F \in \mathcal{F}, g \in \chi(\mu, X') : \|g\|_{L_\infty} \le 1\} \le 1$$

where $\Phi = \Phi'/C$. Let Ψ be the complementary Young function of Φ. For every partition $\pi = \{A_j\}$ of Ω in Σ, and for every $H \in \mathrm{cabv}_\Psi(\mu, X''')$ with $V_\Psi(H) \le 1$, we construct the class of the function $h_\pi : \Omega \to X'''$ such that $h_\pi = \sum_{A_j \in \pi}(H(A_j)/\mu(A_j))\chi A_j$, which belongs to the closed unit ball of $L_\Psi(\mu, X''')$. Moreover, for every $A \in \Sigma$, $\mu_F(A) = \|f_{iF} \cdot \chi_A\|_{1^*}$. Then, for every $F \in \mathcal{F}$, and every partition π

$$\begin{aligned}\sum_{A_j \in \pi} \frac{\mu_F(A_j)\|H(A_j)\|}{\mu(A_j)} &= \sum_{A_j \in \pi} \frac{\|f_{iF} \cdot \chi_{A_j}\|_{1^*}\|H(A_j)\|}{\mu(A_j)} \\ &= \sup\bigg\{ \int_\Omega |\langle f_{iF}(\omega), g(\omega)\rangle| \|h_\pi(\omega)\| \, d\mu, \\ &\qquad F \in \mathcal{F}, g \in \chi(\mu, X') : \|g\|_{L_\infty} \le 1\bigg\} \\ &\le \sup\{2V_\Phi(\langle f_{iF}(\cdot), g(\cdot)\rangle)V_\Psi(h_\pi), \\ &\qquad g \in \chi(\mu, X') : \|g\|_{L_\infty} \le 1\} \le 2.\end{aligned}$$

Then $\mathcal{F} \subset \mathrm{cabv}_\Phi^v(\mu, X)$ with $V_\Phi^v(F) \le 2$ for every $F \in \mathcal{F}$.

(ii) $\to$ (i). If $\mathcal{F}$ is a bounded subset of $\mathrm{cabv}_\Phi^v(\mu, X)$ for some $\Phi \in \mathrm{YVP}$, there is a $C > 0$ such that for all $F \in \mathcal{F}$, $V_\Phi^v(F) \le C$. Then for every $F \in \mathcal{F}$, for every $g \in \chi(\mu, X')$ such that $\|g\|_{L_\infty} \le 1$ and for every $H \in \mathrm{cabv}_\Psi(\mu, X')$ such that $V_\Psi(H) \le 1$, we have

$$\int_\Omega |\langle f_{iF}(\omega), g(\omega)\rangle| \|h_\pi(\omega)\| \, d\mu \le C$$

and also for every $h \in \mathcal{M}_\Psi(\mu) : V_\Psi(h) \le 1$,

$$\int_\Omega \left| \left\langle \frac{f_{iF}(\omega)}{C}, g(\omega) \right\rangle \right| |h(\omega)| \, d\mu \le 1.$$

In consequence, $V_\Phi(\langle (f_{iF}(\cdot)/C), g(\cdot)\rangle) \le \|\langle (f_{iF}(\cdot)/C), g(\cdot)\rangle\|_\Phi \le 1$. Therefore,

$$\sup\bigg\{ \int_\Omega \Phi\bigg(\bigg|\bigg\langle \frac{f_{iF}(\omega)}{C}, g(\omega)\bigg\rangle\bigg|\bigg) \, d\mu, F \in \mathcal{F}, g \in \chi(\mu, X') : \\ \|g\|_{L_\infty} \le 1\bigg\} \le 1$$

and from the VPT the set

$$\mathcal{G} := \{\langle f_{iF}(\cdot), g(\cdot)\rangle, F \in \mathcal{F}, g \in \chi(\mu, X') : \|g\|_{L_\infty} \le 1\}$$

is uniformly integrable in $L_1(\mu)$. The result follows easily. $\square$

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