

TWO-POINT DISTORTION THEOREMS FOR SPHERICALLY CONVEX FUNCTIONS

WILLIAM MA AND DAVID MINDA

ABSTRACT. One-parameter families of sharp two-point distortion theorems are established for spherically convex functions f , that is, meromorphic univalent functions f defined on the unit disk \mathbf{D} such that $f(\mathbf{D})$ is a spherically convex subset of the Riemann sphere \mathbf{P} . These theorems provide for $a, b \in \mathbf{D}$ sharp lower bounds on $d_{\mathbf{P}}(f(a), f(b))$, the spherical distance between $f(a)$ and $f(b)$, in terms of $d_{\mathbf{D}}(a, b)$, the hyperbolic distance between a and b , and the quantities $(1 - |a|^2)f^{\sharp}(a)$, $(1 - |b|^2)f^{\sharp}(b)$, where $f^{\sharp} = |f'|/(1 + |f|^2)$ is the spherical derivative. The weakest lower bound obtained is an invariant form of a known growth theorem for spherically convex functions. Each of the two-point distortion theorems is necessary and sufficient for spherical convexity. These two-point distortion theorems are equivalent to sharp two-point comparison theorems between hyperbolic and spherical geometry on a spherically convex region Ω on \mathbf{P} . Each of these two-point comparison theorems characterize spherically convex regions.

1. Introduction. We begin by surveying the relatively brief history of two-point distortion theorems for univalent functions in order to set the stage for our work. The classical theory of univalent functions often deals with the family S of normalized ($g(0) = 0, g'(0) = 1$) univalent functions g defined on the unit disk $\mathbf{D} = \{z : |z| < 1\}$. Sharp growth and distortion theorems for functions in S are well-known; these results are necessary but not sufficient for univalence. In 1978 Blatter [1] established a sharp two-point distortion theorem for non-normalized univalent functions f defined on \mathbf{D} which is also sufficient for univalence. Blatter's result gives a sharp lower bound on the Euclidean distance $|f(a) - f(b)|$ in terms of $d_{\mathbf{D}}(a, b)$, the hyperbolic distance between a and b relative to \mathbf{D} , and the quantities $(1 - |a|^2)|f'(a)|$, $(1 - |b|^2)|f'(b)|$. Later, Kim and Minda [4] extended the method of Blatter and obtained a one-parameter family of sharp two-point distortion theorems that were both necessary and sufficient for univalence. An invariant version of the classical growth theorem for

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univalent functions is the weakest result in the one-parameter family. These two-point distortion theorems yield sharp two-point comparison theorems between Euclidean geometry and hyperbolic geometry on simply connected regions. But the two-point comparison theorems do not characterize simply connected regions.

There are two-point distortion theorems for some subclasses of univalent functions. Kim and Minda [4] derived a one-parameter family of sharp two-point distortion theorems for convex univalent functions which characterize convex univalent functions. Interestingly, the associated two-point comparison theorems for Euclidean and hyperbolic geometry on convex regions do characterize convex regions. The authors [8] established sharp two-point distortion theorems which characterize strongly close-to-convex functions of order $\alpha \in [0, 1]$. That paper provides both sharp upper and lower bounds on $|f(a) - f(b)|$ in terms of $d_{\mathbf{D}}(a, b)$, $(1 - |a|^2)|f'(a)|$ and $(1 - |b|^2)|f'(b)|$. The case $\alpha = 0$ corresponds to convex univalent functions.

Sharp two-point distortion theorems were recently obtained for bounded univalent functions [9]. In this context one considers univalent functions f which map \mathbf{D} into itself. Two-point distortion theorems give upper and lower bounds on the hyperbolic distance $d_{\mathbf{D}}(f(a), f(b))$ in terms of $d_{\mathbf{D}}(a, b)$, $(1 - |a|^2)|f'(a)|/(1 - |f(a)|^2)$ and $(1 - |b|^2)|f'(b)|/(1 - |f(b)|^2)$. Invariant versions of the classical growth theorems for bounded univalent functions are contained as special instances of the two-point distortion theorems. In this setting the two-point distortion theorems yield two-point comparison theorems between hyperbolic geometry on a simply connected region $\Omega \subset \mathbf{D}$ and the restriction to Ω of the ambient hyperbolic geometry on \mathbf{D} . These comparison theorems do not characterize simply connected subregions of \mathbf{D} .

Another natural context in which to consider two-point distortion theorems is the family of meromorphic univalent functions f defined on \mathbf{D} . Here one seeks upper and lower bounds on the spherical distance $d_{\mathbf{P}}(f(a), f(b))$ in terms of $d_{\mathbf{D}}(a, b)$, $(1 - |a|^2)f^{\#}(a)$ and $(1 - |b|^2)f^{\#}(b)$, where $f^{\#} = |f'|/(1 + |f|^2)$ denotes the spherical derivative. Families of two-point distortion theorems for meromorphic univalent functions which would be analogous to those for univalent or bounded univalent functions cannot be valid. The reason for this is that there do not exist growth theorems for the class of normalized, $g(0) = 0$, $g'(0) = 1$,

meromorphic univalent functions g on \mathbf{D} . Precisely, nontrivial upper or lower bounds do not exist on $|g(z)|$, $0 < |z| < 1$, over the class of normalized meromorphic univalent functions. In fact, for $0 < |p| < 1$, $g_p(z) = z/[(1 - z/p)(z - p)]$ is a normalized meromorphic univalent function on \mathbf{D} and $g_p(z) \rightarrow 0$ as $p \rightarrow 0$ (when $0 < |z| < 1$) and $g_p(z) = \infty$ when $z = p$.

In spite of this, two-point distortion theorems might exist for some subclasses of meromorphic univalent functions. For nonnormalized spherically convex functions f defined on \mathbf{D} we obtain a one-parameter family of sharp two-point lower distortion theorems, that is, sharp lower bounds on $d_{\mathbf{P}}(f(a), f(b))$ in terms of the quantities mentioned earlier. Each of these two-point distortion theorems also characterizes spherically convex functions. The weakest one in the family is an invariant version of the known growth theorem for spherically convex functions [7]. These results are actually equivalent to two-point comparison theorems between hyperbolic geometry and the restriction of spherical geometry to spherically convex regions. Interestingly, each of these two-point comparison theorems do characterize spherically convex regions.

2. Preliminaries. The hyperbolic metric on \mathbf{D} is defined by $\lambda_{\mathbf{D}}(z)|dz| = |dz|/(1 - |z|^2)$. The induced hyperbolic distance between $a, b \in \mathbf{D}$ is given by

$$d_{\mathbf{D}}(a, b) = \inf_{\delta} \int_{\delta} \lambda_{\mathbf{D}}(z)|dz|,$$

where the infimum is taken over all rectifiable paths δ in \mathbf{D} joining a and b . A path γ connecting a and b is called a hyperbolic geodesic if

$$d_{\mathbf{D}}(a, b) = \int_{\gamma} \lambda_{\mathbf{D}}(z)|dz|.$$

The unique hyperbolic geodesic between a and b is the arc of the circle through a and b that is orthogonal to the unit circle $\partial\mathbf{D}$. Explicitly,

$$d_{\mathbf{D}}(a, b) = \tanh^{-1} \left| \frac{a - b}{1 - \bar{a}b} \right|.$$

The hyperbolic metric and distance are both invariant under $\text{Aut}(\mathbf{D})$, the group of conformal automorphisms of \mathbf{D} .

A region Ω on the Riemann sphere \mathbf{P} is called hyperbolic if $\mathbf{P} \setminus \Omega$ contains at least three points. When Ω is hyperbolic, there is a meromorphic universal covering projection f of \mathbf{D} onto Ω . The set of all such coverings of \mathbf{D} onto Ω is given by $f \circ T$, where T ranges over $\text{Aut}(\mathbf{D})$. The hyperbolic metric $\lambda_\Omega(w)|dw|$ on Ω is determined from $f^*(\lambda_\Omega(w)|dw|) = \lambda_{\mathbf{D}}(z)|dz|$, where

$$f^*(\lambda_\Omega(w)|dw|) = \lambda_\Omega(f(z))|f'(z)||dz|$$

denotes the pull-back via f of $\lambda_\Omega(w)|dw|$. This determines $\lambda_\Omega(w)|dw|$ independent of the choice of the covering projection. The hyperbolic distance function induced by this metric is

$$d_\Omega(A, B) = \inf \int_\delta \lambda_\Omega(w)|dw|,$$

where the infimum is taken over all locally rectifiable paths δ in Ω joining A and B . If Ω is simply connected, then a covering $f : \mathbf{D} \rightarrow \Omega$ is a conformal mapping and $d_\Omega(f(a), f(b)) = d_{\mathbf{D}}(a, b)$ for all $a, b \in \mathbf{D}$. When Ω is not simply connected, then $d_\Omega(f(a), f(b)) \leq d_{\mathbf{D}}(a, b)$ for $a, b \in \mathbf{D}$.

The spherical metric on \mathbf{P} is the conformal metric $\lambda_{\mathbf{P}}(z)|dz| = |dz|/(1 + |z|^2)$. It is invariant under $\text{Rot}(\mathbf{P})$, the group of rotations of the sphere, which consists of $R(z) = e^{i\theta}(z - a)/(1 + \bar{a}z)$, where $a \in \mathbf{C}$ and $\theta \in \mathcal{R}$, or $R(z) = e^{i\theta}/z$, where $\theta \in \mathcal{R}$. The invariance property is $R^*(\lambda_{\mathbf{P}}(w)|dw|) = \lambda_{\mathbf{P}}(z)|dz|$ for $R \in \text{Rot}(\mathbf{P})$. The distance function induced on \mathbf{P} by the spherical metric is

$$d_{\mathbf{P}}(a, b) = \inf \int_\delta \lambda_{\mathbf{P}}(z)|dz|,$$

where the infimum is taken over all locally rectifiable paths δ on \mathbf{P} connecting a and b . A path γ joining a and b is called a spherical geodesic if

$$d_{\mathbf{P}}(a, b) = \int_\gamma \lambda_{\mathbf{P}}(z)|dz|.$$

The points a and b are antipodal on \mathbf{P} precisely when $b = -1/\bar{a}$. For distinct $a, b \in \mathbf{P}$ spherical geodesics always exist. If a and b are antipodal, then any of the infinitely many great circular arcs connecting

a and b is a spherical geodesic. If a and b are not antipodal, then the unique spherical geodesic is the shorter arc between a and b of the single great circle determined by a and b . The explicit formula for spherical distance is

$$d_{\mathbf{P}}(a, b) = \begin{cases} \arctan \left| \frac{a - b}{1 + \bar{a}b} \right| & \text{if } a, b \in \mathbf{C}, \\ \arctan \frac{1}{|a|} & \text{if } a \in \mathbf{C}, b = \infty. \end{cases}$$

Spherical distance is invariant under $\text{Rot}(\mathbf{P}) : d_{\mathbf{P}}(R(a), R(b)) = d_{\mathbf{P}}(a, b)$ for all $a, b \in \mathbf{P}$ and $R \in \text{Rot}(\mathbf{P})$. Note that $d_{\mathbf{P}}(a, b) \leq \pi/2$ with equality if and only if a and b are antipodal points. For a region Ω on \mathbf{P} , $\Omega \neq \mathbf{P}$, and $w \in \Omega$, let

$$\varepsilon_{\Omega}(w) = \min\{d_{\mathbf{P}}(w, c) : c \in \partial\Omega\}$$

denote the spherical distance from w to $\partial\Omega$. Note that $0 < \varepsilon_{\Omega}(w) \leq \pi/2$. Sometimes it is more convenient to work with the quantity $E_{\Omega}(w) = \tan \varepsilon_{\Omega}(w)$.

For a hyperbolic region Ω on \mathbf{P} it is natural to consider the spherical density

$$\mu_{\Omega}(w) = \frac{\lambda_{\Omega}(w)|dw|}{\lambda_{\mathbf{P}}(w)|dw|} = (1 + |w|^2)\lambda_{\Omega}(w)$$

of the hyperbolic metric. The function μ_{Ω} is a real-valued function on Ω , even if $\infty \in \Omega$, and it is invariant under $\text{Rot}(\mathbf{P}) : \mu_{R(\Omega)}(R(z)) = \mu_{\Omega}(z)$ for any $R \in \text{Rot}(\mathbf{P})$. The spherical density was employed in [13] and [15]. The fact that $\mu_{\Omega}(w) \rightarrow \infty$ whenever $w \rightarrow \partial\Omega$ was observed in [15]. If $f : \mathbf{D} \rightarrow \Omega$ is a meromorphic universal covering, then

$$\mu_{\Omega}(f(z)) = \frac{1 + |f(z)|^2}{(1 - |z|^2)|f'(z)|} = \frac{1}{(1 - |z|^2)f^{\#}(z)}.$$

A region Ω on \mathbf{P} is called spherically convex if for each pair a, b of points in Ω any spherical geodesic connecting a and b also lies in Ω . Trivially, \mathbf{P} itself is spherically convex. If Ω is spherically convex and contains a pair of antipodal points, then $\Omega = \mathbf{P}$. Henceforth, whenever we discuss spherically convex regions Ω , we always assume $\Omega \neq \mathbf{P}$. A spherical disk $D_{\mathbf{P}}(a, r) = \{w \in \mathbf{P} : d_{\mathbf{P}}(a, w) < r\}$ is spherically convex

provided $0 < r \leq \pi/4$. For $r = \pi/4$ the spherical disk $D_{\mathbf{P}}(a, \pi/4)$ is a hemisphere with spherical center a . Note that the unit disk \mathbf{D} is a hemisphere. Also, given any hemisphere Ω , there is a rotation R of \mathbf{P} with $R(\Omega) = \mathbf{D}$. The rotation R carries the spherical center of Ω to the origin. For a spherically convex region Ω , $\varepsilon_{\Omega}(w) \leq \pi/4$ for $w \in \Omega$ with equality if and only if Ω is a hemisphere with spherical center w . Observe that $\mu_{\mathbf{D}}(w) = (1 + |w|^2)/(1 - |w|^2) \geq 1$ so $\mu_{\Omega} \geq 1$ for any hemisphere since the spherical density is invariant under rotations. A meromorphic univalent function f defined on \mathbf{D} is called spherically convex if the image $f(\mathbf{D})$ is spherically convex.

We make use of several invariant differential operators for meromorphic functions f defined on \mathbf{D} . Set

$$\begin{aligned} D_{s1}f(z) &= \frac{(1 - |z|^2)f'(z)}{1 + |f(z)|^2}, \\ D_{s2}f(z) &= \frac{(1 - |z|^2)^2 f''(z)}{1 + |f(z)|^2} - \frac{2\bar{z}(1 - |z|^2)f'(z)}{1 + |f(z)|^2} \\ &\quad - \frac{2(1 - |z|^2)^2 \overline{f(z)}f'(z)^2}{(1 + |f(z)|^2)^2}, \end{aligned}$$

and

$$\begin{aligned} D_{s3}f(z) &= \frac{(1 - |z|^2)^3 f'''(z)}{1 + |f(z)|^2} - \frac{6(1 - |z|^2)^3 \overline{f(z)}f'(z)f''(z)}{1 + |f(z)|^2} \\ &\quad - \frac{6\bar{z}(1 - |z|^2)^2 f''(z)}{1 + |f(z)|^2} + \frac{6\bar{z}^2(1 - |z|^2)f'(z)}{1 + |f(z)|^2} \\ &\quad + \frac{12\bar{z}(1 - |z|^2)^2 \overline{f(z)}f'(z)^2}{(1 + |f(z)|^2)^2} + \frac{6(1 - |z|^2)^3 \overline{f(z)}^2 f'(z)^3}{(1 + |f(z)|^2)^3} \end{aligned}$$

at any point which is not a pole of f . For simplicity we write $D_j f$ in place of $D_{sj} f$, $j = 1, 2, 3$. The reader should note that $D_j f$ has a different meaning in [4], [8] and [9]. If $f(0) = 0$, then $D_j f(0) = f^{(j)}(0)$, $j = 1, 2, 3$. These operators are invariant in the sense that $|D_j(R \circ f \circ T)| = |D_j f| \circ T$ for all $R \in \text{Rot}(\mathbf{P})$ and $T \in \text{Aut}(\mathbf{D})$. For a locally univalent meromorphic function f on \mathbf{D} set

$$Q_f(z) = \frac{D_2 f(z)}{D_1 f(z)} = \frac{(1 - |z|^2)f''(z)}{f'(z)} - 2\bar{z} - \frac{2(1 - |z|^2)\overline{f(z)}f'(z)}{1 + |f(z)|^2}$$

and note that

$$\frac{D_3 f(z)}{D_1 f(z)} - \frac{3}{2} \left(\frac{D_2 f(z)}{D_1 f(z)} \right)^2 = (1 - |z|^2)^2 S_f(z),$$

where

$$S_f = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$$

is the Schwarzian derivative of f . The quantities $D_1 f(z)$, $D_2 f(z)$ and $D_3 f(z)$ are not defined at a pole of f , but $|D_j f(z)|$, $Q_f(z)$ and $S_f(z)$ are all defined and continuous at a simple pole. For more information on these differential operators, see [7].

Here we gather together some basic properties of spherically convex functions that we will use.

(i) If f is spherically convex, then for $z \in \mathbf{D}$, $|D_1 f(z)| \leq 1$. Equality holds at $z_0 \in \mathbf{D}$ if and only if $f(\mathbf{D})$ is a hemisphere and $f(z_0)$ is the spherical center [13, Corollary 1]. The inequality $|D_1 f(z)| \leq 1$ means geometrically that f is a contraction from \mathbf{D} with hyperbolic geometry to the image with spherical geometry.

(ii) If f is spherically convex, then for $z \in \mathbf{D}$

$$|Q_f(z)|^2 \leq 4(1 - |D_1 f(z)|^2).$$

If $f(\mathbf{D})$ is a hemisphere, then equality holds at every point of \mathbf{D} . If equality holds at a single point of \mathbf{D} , then $f(\mathbf{D})$ is a hemisphere. This inequality is in [7, Theorem 4]. The sharpness result is stated in [14, Corollary 3] only when $f(0) = 0$. The general case is reduced to this situation by considering $R \circ f \circ T$ for appropriate $R \in \text{Rot}(\mathbf{P})$ and $T \in \text{Aut}(\mathbf{D})$. Note that (ii) implies (i).

(iii) If f is spherically convex, then for $z \in \mathbf{D}$

$$(1 - |z|^2)^2 |S_f(z)| + \frac{1}{2} |Q_f(z)|^2 \leq 2(1 - |D_1 f(z)|^2)$$

(see [16] or [7]). Observe that the inequality in (ii) follows from (iii).

3. Differential inequalities. We obtain integral inequalities from second-order linear differential inequalities.

Proposition 1. Suppose $u, v \in \mathcal{C}^2[a, b]$, $k, p > 0$, $v'' \leq k^2 p^2 v$ and $u'' = k^2 p^2 u$. If $v(a) \geq u(a)$ and $v(b) \geq u(b)$, then either $v \equiv u$ or else $v > u$ on (a, b) .

Proof. See [8]. \square

Proposition 2. Suppose $v \in \mathcal{C}^2[-L, L]$, $0 < v \leq 1$, $k > 0$, $p \geq 1$, $|v'| \leq kp v$ and $v'' \leq k^2 p^2 v$. Then

$$\begin{aligned} \int_{-L}^L \frac{v(s)^{1/p}}{1 + v(s)^{2/p}} ds \\ \geq \frac{1}{k} \arctan \frac{2 \sinh(kL) ((v(L) + v(-L)) / (2 \cosh(kpL)))^{1/p}}{1 + ((v(L) + v(-L)) / (2 \cosh(kpL)))^{2/p}} \end{aligned}$$

and equality holds if and only if $v(s) = Ae^{\pm kps}$, $A > 0$.

Proof. The general solution of $u'' = k^2 p^2 u$ which satisfies the boundary conditions $u(-L) = v(-L)$ and $u(L) = v(L)$ is

$$u(s) = A[\cosh(kps) + \tau \sinh(kps)],$$

where

$$A = \frac{v(L) + v(-L)}{2 \cosh(kpL)}$$

and

$$\tau = \frac{v(L) - v(-L)}{v(L) + v(-L)} \cdot \frac{\cosh(kpL)}{\sinh(kpL)} \in [-1, 1]$$

(see [9]). Since $t \mapsto t/(1 + t^2)$ is strictly increasing for $t \in (0, 1]$, Proposition 1 implies that

$$\begin{aligned} \int_{-L}^L \frac{v(s)^{1/p}}{1 + v(s)^{2/p}} ds &\geq \int_{-L}^L \frac{u(s)^{1/p}}{1 + u(s)^{2/p}} ds \\ &= \int_{-L}^L \frac{A^{1/p} [\cosh(kps) + \tau \sinh(kps)]^{1/p}}{1 + A^{2/p} [\cosh(kps) + \tau \sinh(kps)]^{2/p}} ds \\ &= I(\tau) \end{aligned}$$

with strict inequality unless $v = u$. Direct calculation shows that

$$I(1) = I(-1) = \frac{1}{k} \arctan \left(\frac{2A^{1/p} \sinh(kL)}{1 + A^{2/p}} \right)$$

and

$$\begin{aligned} & I''(\tau) \\ &= \frac{A^2}{p^2} \int_{-L}^L \frac{\sinh^2(kps) u(s)^{(1/p)-2} [1 - p - 6u(s)^{2/p} + (1+p)u(s)^{4/p}]}{[1 + u(s)^{2/p}]^3} ds. \end{aligned}$$

Because $p \geq 1$ and $0 < u(s) \leq v(s) \leq 1$, it follows that $1 - p - 6u(s)^{2/p} + (1+p)u(s)^{4/p} < 0$. Thus, $I''(\tau) < 0$ and so $I(\tau)$ is strictly concave down on $[-1, 1]$ which implies that $I(\tau) \geq I(\pm 1)$ with strict inequality unless $\tau = \pm 1$. This proves the desired inequality and shows that strict inequality holds unless $u(s) = Ae^{\pm kps}$.

4. A characterization of spherically convex regions. If Ω is an Euclidean convex region in \mathbf{C} , $\Omega \neq \mathbf{C}$, then $\lambda_\Omega(w) \geq 1/(2\delta_\Omega(w))$, for $w \in \Omega$, where $\delta_\Omega(w) = \text{dist}(w, \partial\Omega)$ is the Euclidean distance from w to $\partial\Omega$ [12]. Conversely, if this inequality holds on a hyperbolic region Ω , then Ω is convex. This result is due to Hilditch [2]; another proof was given in [10]. We obtain a spherical analog of this result. This spherical analog is needed in Section 5.

The next result makes precise the intuitive notion that a region which is not spherically convex must possess a boundary point near which the region is not spherically convex. Euclidean analogs of this result are due to Keogh [3] and Hilditch [2].

Lemma 1. *Suppose Ω is a proper subregion of \mathbf{P} which is not spherically convex. Then $c \in \partial\Omega$, $d \in \mathbf{P}$ and $0 < r < R < \pi/4$ exist such that $d_{\mathbf{P}}(c, d) = R$ and $D_{\mathbf{P}}(c, r) \setminus \overline{D_{\mathbf{P}}(d, R)} \subset \Omega$.*

Proof. We begin by showing that there is a spherical geodesic arc γ such that γ has spherical length strictly less than $\pi/2$, the endpoints of γ lie in Ω , γ is contained in $\overline{\Omega}$, $\gamma \cap \partial\Omega \neq \emptyset$ and there is a circular arc δ in Ω connecting the endpoints of γ so that $\gamma \cup \delta$ is a Jordan curve whose interior is contained in Ω . Since Ω is not spherically convex,

there exist $a, b \in \Omega$ and a spherical geodesic arc σ between them such that σ is not entirely in Ω . Trivially, σ has spherical length at most $\pi/2$. If σ has spherical length $\pi/2$ (equivalently, a and b are antipodal), then we may move b slightly along σ towards a and still remain in Ω . Thus, there is no harm in assuming σ has spherical length strictly less than $\pi/2$. Also, we may suppose $a = 0$; if not, rotate the sphere to achieve this. Because Ω is a region and $0, b \in \Omega \cap \mathbf{C}$, the points 0 and b can be joined by a polygonal arc in $\Omega \cap \mathbf{C}$. This means that a finite sequence $0 = w_0, w_1, \dots, w_n = b$ exists in Ω such that the straight line segment $[w_j, w_{j+1}]$ is contained in Ω for $j = 0, 1, \dots, n-1$. Recall that straight line segments through the origin are spherical geodesics. Note that $[0, w_1] \subset \Omega$ but $\sigma = [0, b] = [0, w_n]$ is not entirely contained in Ω . This implies that there exists an integer j , $1 \leq j \leq n-1$, such that $[0, w_j] \subset \Omega$ while $[0, w_{j+1}] \not\subset \Omega$. Let $w(t) = (1-t)w_j + tw_{j+1}$ for $0 \leq t \leq 1$. Because Ω is open, $[0, w(t)] \subset \Omega$ for all t sufficiently small. Set

$$T = \sup\{t \in [0, 1] : [0, w(s)] \subset \Omega \text{ for } 0 \leq s \leq t\}.$$

Then $0 < T \leq 1$, $\gamma = [0, w(T)] \subset \overline{\Omega}$, 0 and $w(T)$ belong to Ω and γ meets $\partial\Omega$. Moreover, as 0 and $w(T)$ are not antipodal, γ has spherical length strictly less than $\pi/2$. Note that the interior of the triangle with vertices $0, w_j$ and $w(T)$ is contained in Ω . Then we can find a circular arc δ from 0 to $w(T)$ in Ω in the interior of this triangle, so $-\gamma \cup \delta$ is a Jordan curve whose interior lies in Ω .

Now we complete the proof by making use of γ and δ . By rotating \mathbf{P} if necessary, we may suppose that γ is an interval of the real axis \mathcal{R} centered at the origin, say $\gamma = [-x, x]$, and δ lies in the closed upper half-plane $\overline{\mathbf{H}}$, where $\mathbf{H} = \{w : \operatorname{Im}(w) > 0\}$. Since γ has spherical length strictly less than $\pi/2$, we conclude that $0 < x < 1$. Because Ω is open, we can determine $\eta > 0$ so that the vertical line segments $[x, x - i\eta]$ and $[-x, -x - i\eta]$ lie in Ω . Note that the great circle $\{w : d_{\mathbf{P}}(-i, w) = \pi/4\}$ is $\mathcal{R} \cup \{\infty\}$. By selecting $\rho \in (0, 1)$ sufficiently close to 1 , we see that the spherical circle $\{w : d_{\mathbf{P}}(w, -i\rho) = \arctan(\rho) = d_{\mathbf{P}}(0, -i\rho)\}$, which is tangent to \mathcal{R} at the origin, will meet both $[x, x - i\eta]$ and $[-x, -x - i\eta]$. Let β be the arc of this circle which joins these segments and contains 0 . By making η smaller if necessary, we may assume that β joins $-x - i\eta$ to $x - i\eta$. Let K be the compact set that is bounded by the closed curve $[-x, -x - i\eta] \cup \beta \cup [x, x - i\eta] \cup [-x, x]$. Since γ meets $\partial\Omega$, K

intersects $\partial\Omega$. Set

$$R = \max\{d_{\mathbf{P}}(-i\rho, w) : w \in K \cap (\mathbf{P} \setminus \Omega)\}.$$

Note that

$$R \leq d_{\mathbf{P}}(-i\rho, \pm x) = \arctan \sqrt{\frac{x^2 + \rho^2}{1 + \rho^2 x^2}} < \pi/4$$

since

$$\frac{x^2 + \rho^2}{1 + \rho^2 x^2} < 1$$

follows from $0 < x < 1$ and $0 < \rho < 1$. Select $c \in K \cap (\mathbf{P} \setminus \Omega)$ with $d_{\mathbf{P}}(c, -i\rho) = R$. Note that c is at a positive distance from both segments $[x, x - i\eta]$ and $[-x, -x - i\eta]$. We want to show $c \in \partial\Omega$. If $c \in (-x, x)$, then $c \in \partial\Omega$ is certainly true. Suppose $c \in K \setminus [-x, x]$ and $c \notin \partial\Omega$. But then c would be an interior point of $\mathbf{P} \setminus \Omega$ and this would violate the definition of R and the choice of c . Thus, $c \in \partial\Omega$. Finally, we can select $r \in (0, R)$ so that $D_{\mathbf{P}}(c, r) \setminus \overline{D_{\mathbf{P}}(-i\rho, R)} \subset \Omega$. If $c \in (-x, x)$, the existence of such an r makes use of the fact that the interior of $-\gamma \cup \delta$ is contained in Ω . For $c \in K \setminus [-x, x]$, the existence of r follows from the definition of R and the choice of c .

Theorem 1. *Suppose Ω is a hyperbolic region on \mathbf{P} . Then Ω is spherically convex if and only if*

$$(1) \quad \mu_{\Omega}(w) \geq \frac{1 + E_{\Omega}^2(w)}{2E_{\Omega}(w)}$$

for all $w \in \Omega$.

Proof. The inequality (1) for spherically convex regions was established in [13, Theorem 1] (see [11] for a generalization).

Next we prove that if Ω is not spherically convex, then (1) is not valid for some $w \in \Omega$. If Ω is not spherically convex, then the preceding lemma applies. Since μ_{Ω} and E_{Ω} are both invariant under $\text{Rot}(\mathbf{P})$, we can assume $c = 0$. More precisely, we may assume there is a complex number b with $\text{Re}\{b\} > 0$, $\text{Im}\{b\} < 0$ and a crescent-shaped region

$\Delta \subset \Omega$ bounded below by the circular arc through $-\bar{b}$, 0 and b and above by a circular arc from $-\bar{b}$ to b that meets \mathbf{H} . Then

$$h(w) = \left[\frac{b(\bar{b} + w)}{\bar{b}(b - w)} \right]^{\pi/\varphi},$$

where φ is the angle at each vertex of Δ , is a conformal map of Δ onto \mathbf{H} with $h(0) = 1$. The hyperbolic metric on \mathbf{H} is $\lambda_{\mathbf{H}}(z)|dz| = |dz|/(2\operatorname{Im}\{z\})$, so

$$\lambda_{\Delta}(w) = \frac{|h'(w)|}{2\operatorname{Im}\{h(w)\}}.$$

Since

$$h'(w) = \frac{2\pi\operatorname{Re}\{b\}h(w)}{\varphi(\bar{b} + w)(b - w)},$$

we find that

$$\lambda_{\Delta}(w) = \frac{\pi\operatorname{Re}\{b\}}{\varphi|\bar{b} + w||b - w|\sin\{(\pi/\varphi)\arg(b(\bar{b} + w)/(\bar{b}(b - w)))\}}.$$

For $w = i\varepsilon$ this becomes

$$\begin{aligned} \lambda_{\Delta}(i\varepsilon) &= \frac{\pi\operatorname{Re}\{b\}}{\varphi|b - i\varepsilon|^2\sin((2\pi/\varphi)\arg(b(\bar{b} + i\varepsilon)))} \\ &= \frac{\pi\operatorname{Re}\{b\}}{\varphi|b - i\varepsilon|^2\sin((2\pi/\varphi)\arctan(\varepsilon\operatorname{Re}\{b\}/(|b|^2 - \varepsilon\operatorname{Im}\{b\})))}. \end{aligned}$$

This gives

$$\varepsilon\lambda_{\Delta}(i\varepsilon) = \frac{1}{2} + \frac{\varepsilon\operatorname{Im}\{b\}}{2|b|^2} + \mathcal{O}(\varepsilon^2).$$

Because $\Delta \subset \Omega$, $\lambda_{\Delta}(w) \geq \lambda_{\Omega}(w)$ for $w \in \Delta$. For $w = i\varepsilon$, $\varepsilon > 0$ small, $E_{\Omega}(i\varepsilon) = \varepsilon$. Thus, for $\varepsilon > 0$ sufficiently small

$$\begin{aligned} \frac{2E_{\Omega}(i\varepsilon)}{1 + E_{\Omega}^2(i\varepsilon)}\mu_{\Omega}(i\varepsilon) &= \frac{2\varepsilon}{1 + \varepsilon^2}(1 + \varepsilon^2)\lambda_{\Omega}(i\varepsilon) \\ &= 2\varepsilon\lambda_{\Omega}(i\varepsilon) \leq 2\varepsilon\lambda_{\Delta}(i\varepsilon) \\ &= 1 + \frac{\varepsilon\operatorname{Im}\{b\}}{|b|^2} + \mathcal{O}(\varepsilon^2) < 1 \end{aligned}$$

since $\operatorname{Im}\{b\} < 0$. Thus, inequality (4) cannot hold for all $w \in \Omega$.

5. Main results. We start by establishing certain differential identities that are needed later. Suppose f is a locally univalent meromorphic function defined on \mathbf{D} . Let $\gamma : z = z(s)$, $-L \leq s \leq L$, be a smooth path in \mathbf{D} parametrized by hyperbolic arclength. This means that $z'(s) = (1 - |z(s)|^2)e^{i\theta(s)}$, where $\theta = \arg z'(s)$ so $e^{i\theta(s)}$ is a unit tangent vector to γ at $z(s)$, and $2L$ is the hyperbolic length of γ .

It is straightforward to show that

$$\frac{d}{ds}|D_1f(z(s))| = |D_1f(z(s))|\operatorname{Re}\{Q_f(z(s))e^{i\theta(s)}\}$$

and

$$\begin{aligned} (2) \quad & \frac{d}{ds} \left(\frac{|D_1f(z(s))|}{1 + \sqrt{1 - |D_1f(z(s))|^2}} \right)^p \\ &= p \left(\frac{|D_1f(z(s))|}{1 + \sqrt{1 - |D_1f(z(s))|^2}} \right)^p \frac{\operatorname{Re}\{Q_f(z(s))e^{i\theta(s)}\}}{\sqrt{1 - |D_1f(z(s))|^2}}. \end{aligned}$$

Here and below we are assuming that $|D_1f(z(s))| < 1$ for $s \in [-L, L]$. Since

$$\frac{d}{ds} \frac{1}{\sqrt{1 - |D_1f(z(s))|^2}} = \frac{|D_1f(z(s))|^2 \operatorname{Re}\{Q_f(z(s))e^{i\theta(s)}\}}{(1 - |D_1f(z(s))|^2)^{3/2}},$$

we have

$$\begin{aligned} & \frac{d^2}{ds^2} \left(\frac{|D_1f(z(s))|}{1 + \sqrt{1 - |D_1f(z(s))|^2}} \right)^p \\ &= p \left(\frac{|D_1f(z(s))|}{1 + \sqrt{1 - |D_1f(z(s))|^2}} \right)^p \frac{1}{\sqrt{1 - |D_1f(z(s))|^2}} \\ & \quad \cdot \left[\frac{p \operatorname{Re}^2\{Q_f(z(s))e^{i\theta(s)}\}}{\sqrt{1 - |D_1f(z(s))|^2}} \right. \\ & \quad + \frac{|D_1f(z(s))|^2 \operatorname{Re}^2\{Q_f(z(s))e^{i\theta(s)}\}}{1 - |D_1f(z(s))|^2} \\ & \quad \left. + \operatorname{Re} \left\{ e^{i\theta(s)} \frac{d}{ds} Q_f(z(s)) + Q_f(z(s)) \frac{d}{ds} e^{i\theta(s)} \right\} \right]. \end{aligned}$$

In [9] we proved that

$$\frac{d}{ds}e^{i\theta(s)} = i\kappa_h(z(s), \gamma)e^{i\theta(s)} + (z(s) - \bar{z}(s))e^{2i\theta(s)},$$

where $\kappa_h(z(s), \gamma)$ denotes the hyperbolic curvature of γ at $z(s)$. A tedious calculation shows that

$$\begin{aligned} & \operatorname{Re} \left\{ e^{i\theta(s)} \frac{d}{ds} Q_f(z(s)) + Q_f(z(s)) \frac{d}{ds} e^{i\theta(s)} \right\} \\ &= \operatorname{Re} \left\{ \frac{1}{2} e^{2i\theta(s)} Q_f^2(z(s)) + e^{2i\theta(s)} (1 - |z(s)|^2)^2 S_f(z(s)) \right. \\ & \quad \left. + i\kappa_h(z(s), \gamma) e^{i\theta(s)} Q_f(z(s)) - 2|D_1 f(z(s))|^2 - 2 \right\}. \end{aligned}$$

This implies that

$$\begin{aligned} & \frac{\sqrt{1 - |D_1 f(z(s))|^2}}{p} \left(\frac{|D_1 f(z(s))|}{1 + \sqrt{1 - |D_1 f(z(s))|^2}} \right)^{-p} \\ & \cdot \frac{d^2}{ds^2} \left(\frac{|D_1 f(z(s))|}{1 + \sqrt{1 - |D_1 f(z(s))|^2}} \right)^p \\ &= \frac{p \operatorname{Re} \{ Q_f(z(s)) e^{i\theta(s)} \}}{\sqrt{1 - |D_1 f(z(s))|^2}} + \frac{|D_1 f(z(s))|^2 \operatorname{Re} \{ Q_f(z(s)) e^{i\theta(s)} \}}{1 - |D_1 f(z(s))|^2} \\ &+ \operatorname{Re} \left\{ \frac{1}{2} e^{2i\theta(s)} Q_f^2(z(s)) + e^{2i\theta(s)} (1 - |z(s)|^2)^2 S_f(z(s)) \right. \\ & \quad \left. + i\kappa_h(z(s), \gamma) e^{i\theta(s)} Q_f(z(s)) - 2|D_1 f(z(s))|^2 - 2 \right\}. \end{aligned}$$

The connection between the hyperbolic curvature of a path γ in \mathbf{D} and the spherical curvature of the image $f \circ \gamma$ under a locally univalent meromorphic function f is [7]

$$\kappa_s(f(z(s)), f \circ \gamma) |D_1 f(z(s))| = \kappa_h(z(s), \gamma) + \operatorname{Im} \{ Q_f(z(s)) e^{i\theta(s)} \}.$$

Hence, if $f \circ \gamma$ is a spherical geodesic, then $\kappa_s(f(z(s)), f \circ \gamma) = 0$ and so

$$\kappa_h(z(s), \gamma) = -\operatorname{Im} \{ Q_f(z(s)) e^{i\theta(s)} \}.$$

If we make use of this identity, then we obtain

$$\begin{aligned}
 (3) \quad & \frac{\sqrt{1 - |D_1 f(z(s))|^2}}{p} \left(\frac{|D_1 f(z(s))|}{1 + \sqrt{1 - |D_1 f(z(s))|^2}} \right)^{-p} \\
 & \cdot \frac{d^2}{ds^2} \left(\frac{|D_1 f(z(s))|}{1 + \sqrt{1 - |D_1 f(z(s))|^2}} \right)^p \\
 & = |Q_f(z(s))|^2 - 2(1 + |D_1 f(z(s))|^2) \\
 & + \frac{p\sqrt{1 - |D_1 f(z(s))|^2} + 2|D_1 f(z(s))|^2 - 1}{1 - |D_1 f(z(s))|^2} \operatorname{Re}^2 \{Q_f(z(s))e^{i\theta(s)}\} \\
 & + \operatorname{Re} \left\{ \frac{1}{2} e^{2i\theta(s)} Q_f^2(z(s)) + e^{2i\theta(s)} (1 - |z(s)|^2)^2 S_f(z(s)) \right\}.
 \end{aligned}$$

The results in the following example are needed in the proof of Theorem 2.

Example. Suppose Ω is a hemisphere, A is the spherical center of Ω , $B \in \Omega$ and Γ is the spherical geodesic from A to B . Note that Γ is also a hyperbolic geodesic. Let $f : \mathbf{D} \rightarrow \Omega$ be a conformal mapping, $\gamma = f^{-1} \circ \Gamma$, $a = f^{-1}(A)$ and $b = f^{-1}(B)$. Assume $\gamma : z = z(s)$, $-L \leq s \leq L$, is a hyperbolic arclength parametrization of γ . Since $|D_1 f(a)| = 1$, the preceding identities (2) and (3) do not apply. Nevertheless, we prove that for $p > 0$ the function

$$v(s) = \left(\frac{|D_1 f(z(s))|}{1 + \sqrt{1 - |D_1 f(z(s))|^2}} \right)^p$$

is of class \mathcal{C}^2 on $[-L, L]$ and $|v'| = 2pv$, $v'' = 4p^2v$. By replacing f with $R \circ f \circ T$ for appropriate $R \in \operatorname{Rot}(\mathbf{P})$ and $T \in \operatorname{Aut}(\mathbf{D})$, we see that there is no harm in assuming $\Omega = \mathbf{D}$, $A = a = 0$, $B = b = r \in (0, 1)$ and $f(z) \equiv z$. In this situation $\Gamma = \gamma = [0, r]$ and $z(s) = \tanh(s + L)$, where $2L = \tanh^{-1}(r)$. Then

$$D_1 f(z) = \frac{1 - |z|^2}{1 + |z|^2}$$

and so

$$D_1 f(z(s)) = \frac{1}{\cosh(2(s + L))}.$$

This gives

$$v(s) = \left(\frac{1}{\cosh(2(s+L)) + \sinh(2(s+L))} \right)^p = e^{-2p(s+L)},$$

so $v' = -2pv$ and $v'' = 4p^2v$. The same type of result holds if B is the spherical center.

We also wish to note that if $-1 < A < 0 < B < 1$ and $\gamma = \Gamma = [A, B]$, then v is not of class \mathcal{C}^2 on $[-L, L]$. For simplicity we assume $B = r \in (0, 1)$ and $A = -r$. Then $z(s) = \tanh(s)$, $s \in [-L, L]$ ($L = \tanh(r)$), is a hyperbolic arclength parametrization of $\gamma = \Gamma$ and

$$v(s) = \left(\frac{1}{\cosh(s) + |\sinh(s)|} \right)^p = e^{-p|s|}$$

which is not differentiable at $s = 0$.

Theorem 2. *Suppose Ω is a spherically convex hyperbolic region on \mathbf{P} . Then for any $p \geq 1$ and all $A, B \in \Omega$,*

(4)

$$\begin{aligned} d_{\mathbf{P}}(A, B) \\ \geq \arctan \frac{2 \sinh(d_{\Omega}(A, B))}{\left[\frac{2 \cosh(pd_{\Omega}(A, B))}{H^p(\mu_{\Omega}(A)) + H^p(\mu_{\Omega}(B))} \right]^{1/p} + \left[\frac{H^p(\mu_{\Omega}(A)) + H^p(\mu_{\Omega}(B))}{2 \cosh(pd_{\Omega}(A, B))} \right]^{1/p}}, \end{aligned}$$

where $H(t) = 1/(t + \sqrt{t^2 - 1})$. Equality holds for distinct $A, B \in \Omega$ if and only if Ω is a hemisphere, the spherical geodesic Γ between A and B is the arc of a great circle through the spherical center of Ω , but the spherical center is not an interior point of Γ . Conversely, if Ω is a hyperbolic region on \mathbf{P} and (4) holds for some $p \geq 1$ and all $A, B \in \Omega$, then Ω is spherically convex.

Proof. We first show that a hyperbolic region which satisfies the inequality (4) for some $p \geq 1$ must be spherically convex. The assumption that the inequality holds implicitly means that $\mu_{\Omega} \geq 1$. Now, fix $A \in \Omega$ and choose $\alpha \in \partial\Omega$ with $d_{\mathbf{P}}(A, \alpha) = \varepsilon_{\Omega}(A)$. Then let $B \in \Omega$ tend to α along the spherical geodesic arc between A and

α . Then $d_{\Omega}(A, B) \rightarrow \infty$ since the hyperbolic metric is complete and $\mu_{\Omega}(B) \rightarrow \infty$ [15], so the inequality yields

$$\varepsilon_{\Omega}(A) \geq \arctan H(\mu_{\Omega}(A)).$$

This is equivalent to

$$\mu_{\Omega}(A) \geq \frac{1 + E_{\Omega}^2(A)}{2E_{\Omega}(A)}.$$

Theorem 1 implies that Ω is spherically convex.

Next, we turn to the proof that inequality (4) holds for spherically convex regions. Suppose Ω is spherically convex and $f : \mathbf{D} \rightarrow \Omega$ is a conformal mapping. Fix $A, B \in \Omega$. Since Ω is spherically convex, the spherical geodesic Γ joining A to B lies in Ω . Then $\gamma = f^{-1} \circ \Gamma$ is a smooth path in \mathbf{D} from $a = f^{-1}(A)$ to $b = f^{-1}(B)$. Suppose $\gamma : z = z(s)$, $-L \leq s \leq L$, is a parametrization of γ by hyperbolic arclength. Then $2L \geq d_{\mathbf{D}}(a, b) = d_{\Omega}(A, B)$ with equality if and only if γ is the hyperbolic geodesic from a to b . We assume $|D_1 f(z)| < 1$ for $z \in \gamma$ except possibly at a or b . This only rules out the possibility that Ω is a hemisphere and the spherical center of Ω is on Γ strictly between A and B . We refer to this as the exceptional case.

For $p \geq 1$ define

$$v(s) = \left(\frac{|D_1 f(z(s))|}{1 + \sqrt{1 - |D_1 f(z(s))|^2}} \right)^p.$$

We will show that $|v'| \leq 2pv$ and $v'' \leq 4p^2v$. Since the preceding example proves that $|v'| = 2pv$ and $v'' = 4p^2v$ when $|D_1 f(a)| = 1$ or $|D_1 f(b)| = 1$, we need only consider the possibility that $0 < |D_1 f(z(s))| < 1$ for $s \in [-L, L]$. This inequality implies $0 < v(s) < 1$ on $[-L, L]$. Then identity (2) gives

$$v'(s) = pv(s) \frac{\operatorname{Re}\{Q_f(z(s))e^{i\theta(s)}\}}{\sqrt{1 - |D_1 f(z(s))|^2}}.$$

Property (ii) of spherically convex functions yields

$$|v'(s)| \leq 2pv(s).$$

Because $\Gamma = f \circ \gamma$ is a spherical geodesic, formula (3) produces

$$\begin{aligned} v''(s) = & \frac{pv(s)}{\sqrt{1 - |D_1 f(z(s))|^2}} \left[|Q_f(z(s))|^2 - 2(1 + |D_1 f(z(s))|^2) \right. \\ & + \frac{p\sqrt{1 - |D_1 f(z(s))|^2} + 2|D_1 f(z(s))|^2 - 1}{1 - |D_1 f(z(s))|^2} \\ & \cdot \operatorname{Re}^2 \{Q_f(z(s))e^{i\theta(s)}\} \\ & \left. + \operatorname{Re} \left\{ \frac{1}{2} e^{2i\theta(s)} Q_f^2(z(s)) + e^{2i\theta(s)} (1 - |z(s)|^2)^2 S_f(z(s)) \right\} \right]. \end{aligned}$$

Because $p \geq 1$ and $0 < |D_1 f(z(s))| < 1$,

$$p\sqrt{1 - |D_1 f(z(s))|^2} + 2|D_1 f(z(s))|^2 - 1 \geq 0$$

for $s \in [-L, L]$. Consequently,

$$\begin{aligned} v''(s) \leq & \frac{pv(s)}{\sqrt{1 - |D_1 f(z(s))|^2}} \\ & \cdot \left[\left(\frac{3}{2} + \frac{p\sqrt{1 - |D_1 f(z(s))|^2} + 2|D_1 f(z(s))|^2 - 1}{1 - |D_1 f(z(s))|^2} \right) |Q_f(z(s))|^2 \right. \\ & \left. - 2(1 + |D_1 f(z(s))|^2) + (1 - |z(s)|^2)^2 |S_f(z(s))| \right]. \end{aligned}$$

By making use of property (iii) of spherically convex functions we obtain

$$\begin{aligned} v''(s) \leq & \frac{pv(s)}{\sqrt{1 - |D_1 f(z(s))|^2}} \\ & \cdot \left[\frac{p\sqrt{1 - |D_1 f(z(s))|^2} + |D_1 f(z(s))|^2}{1 - |D_1 f(z(s))|^2} |Q_f(z(s))|^2 \right. \\ & \left. - 4|D_1 f(z(s))|^2 \right]. \end{aligned}$$

Application of property (ii) of spherically convex functions results in

$$v''(s) \leq 4p^2 v(s)$$

for $s \in [-L, L]$. This demonstrates that $v(s)$ satisfies the hypotheses of Proposition 2 for $k = 2$ and all $p \geq 1$. Note that if $v'' = 4p^2v(s)$, then equality holds in property (ii) of spherically convex functions and so $\Omega = f(\mathbf{D})$ is a hemisphere.

Now, we establish (4). Since $\Gamma = f \circ \gamma$ is a spherical geodesic,

$$\begin{aligned} d_{\mathbf{P}}(A, B) &= \int_{\Gamma=f \circ \gamma} \lambda_{\mathbf{P}}(w) |dw| = \int_{\gamma} f^{\#}(z) |dz| \\ &= \int_{-L}^L f^{\#}(z(s)) |z'(s)| ds = \int_{-L}^L |D_1 f(z(s))| ds \\ &= 2 \int_{-L}^L \frac{v(s)^{1/p}}{1 + v(s)^{2/p}} ds. \end{aligned}$$

Because $v(s)$ satisfies the hypotheses of Proposition 2, we conclude

$$d_{\mathbf{P}}(A, B) \geq \arctan \frac{2 \sinh(2L)((v(L) + v(-L))/(2 \cosh(2pL)))^{1/p}}{1 + ((v(L) + v(-L))/(2 \cosh(2pL)))^{2/p}}$$

and equality implies $v''(s) = 4p^2v(s)$ so Ω is a hemisphere. For $p \geq 1$ the function

$$g(t) = \frac{2 \sinh(t)(c/\cosh(pt))^{1/p}}{1 + (c/\cosh(pt))^{2/p}}$$

is a strictly increasing function of t . Therefore,

$$\begin{aligned} d_{\mathbf{P}}(A, B) &\geq \arctan \frac{2 \sinh(d_{\Omega}(A, B))((v(L) + v(-L))/(2 \cosh(pd_{\Omega}(A, B))))^{1/p}}{1 + ((v(L) + v(-L))/(2 \cosh(pd_{\Omega}(A, B))))^{2/p}} \end{aligned}$$

and equality implies both that Ω is a hemisphere and that Γ is a hyperbolic geodesic joining A and B . The latter follows from equality implying $2L = d_{\Omega}(A, B)$. As

$$v(-L) = \left(\frac{|D_1 f(a)|}{1 + \sqrt{1 - |D_1 f(z)|^2}} \right)^p = H^p(\mu_{\Omega}(A))$$

and

$$v(L) = \left(\frac{|D_1 f(b)|}{1 + \sqrt{1 - |D_1 f(b)|^2}} \right)^p = H^p(\mu_{\Omega}(B)),$$

we have established inequality (4) except in the exceptional case.

Since $|v'(s)| = 2pv(s)$ and $v''(s) = 4p^2v(s)$ when Ω is a hemisphere and either A or B is the spherical center of Ω , and Γ is also a hyperbolic geodesic in this case, the preceding work shows that equality holds in (4) in this situation.

The exceptional case follows easily from the nonexceptional case. Suppose Ω is a hemisphere, $A, B \in \Omega$ and the spherical geodesic Γ joining A to B contains the spherical center of Ω . Choose a sequence $\{B_n\}_{n=1}^\infty$ in Ω so that $B_n \rightarrow B$ and the spherical geodesic Γ_n from A to B_n does not pass through the spherical center of Ω . Then (4) holds for A and B_n . By letting $n \rightarrow \infty$ we obtain (4) for A and B .

Finally, we deal with the sharpness of inequality (4). Suppose equality holds in (4) in the nonexceptional case. Then we know that Ω must be a hemisphere and the spherical geodesic Γ must also be a hyperbolic geodesic. By making use of the rotational invariance of the quantities involved, we may assume that $\Omega = \mathbf{D}$. We show that Γ must lie on a line through the origin. Since Γ is a hyperbolic geodesic, it is an arc of a circle C which is orthogonal to $\partial\mathbf{D}$. Let α be one of the two points in which C meets $\partial\mathbf{D}$. Since Γ is also a spherical geodesic, C must be a great circle on \mathbf{P} . This implies that for each point $w \in C$ the antipodal point $-1/\bar{w}$ must be on C . Thus $-1/\bar{\alpha} = -\alpha \in C$. Since the diametrically opposite points $\alpha, -\alpha$ of $\partial\mathbf{D}$ belong to C and C is orthogonal to $\partial\mathbf{D}$, C is the straight line through $\alpha, -\alpha$. In particular, $0 \in C$. Thus, equality in (4) in a nonexceptional case implies that Ω is a hemisphere and A, B are located as in the equality statement of the theorem.

All that remains is to prove that strict inequality holds in the exceptional case. We begin by noting that the righthand side of inequality (4) is a decreasing function of p . We omit the details except to indicate that one only needs to prove that

$$L(p) = \left[\frac{H^p(\mu_\Omega(A)) + H^p(\mu_\Omega(B))}{2 \cosh(pd_\Omega(A, B))} \right]^{-1/p} + \left[\frac{H^p(\mu_\Omega(A)) + H^p(\mu_\Omega(B))}{2 \cosh(pd_\Omega(A, B))} \right]^{1/p}$$

is an increasing function of p . This is accomplished by showing $L'(p) > 0$. Thus, if we can prove that strict inequality holds in (4)

in the exceptional case when $p = 1$, then strict inequality also holds for $p \geq 1$. It suffices to consider the case in which $\Omega = \mathbf{D}$ and $-1 < A < 0 < B < 1$. We need to verify that

$$(5) \quad \frac{B-A}{1+AB} > \frac{2 \sinh(d_{\mathbf{D}}(A, B))}{\frac{2 \cosh(d_{\mathbf{D}}(A, B))}{H(\mu_{\mathbf{D}}(A)) + H(\mu_{\mathbf{D}}(B))} + \frac{H(\mu_{\mathbf{D}}(A)) + H(\mu_{\mathbf{D}}(B))}{2 \cosh(d_{\mathbf{D}}(A, B))}}.$$

By making use of the identities

$$d_{\mathbf{D}}(A, B) = \frac{1}{2} \log \frac{(1-A)(1+B)}{(1+A)(1-B)},$$

$$H(\mu_{\mathbf{D}}(A)) = \frac{1+A}{1-A}$$

and

$$H(\mu_{\mathbf{D}}(B)) = \frac{1-B}{1+B},$$

we find that (5) becomes

$$(6) \quad \frac{B-A}{1+AB} > \frac{(1+AB)(1-AB)(B-A)}{1-AB-2AB^2+2A^2B+A^2B^2-A^3B^3}.$$

Because $-1 < A < 0 < B < 1$, the denominator on the righthand side is positive, so (6) is equivalent to

$$1-AB-2AB^2+2A^2B+A^2B^2-A^3B^3 > (1+AB)^2(1-AB),$$

or

$$-2AB(1-A)(1+B) > 0.$$

Because $-1 < A < 0 < B < 1$, this is trivially valid. This demonstrates that strict inequality holds in the exceptional case for all $p \geq 1$, so the proof is complete. \square

Corollary 1. *Suppose f is spherically convex in \mathbf{D} . Then for $p \geq 1$ and $a, b \in \mathbf{D}$*

(7)

$$d_{\mathbf{P}}(f(a), f(b)) \geq \arctan \frac{2 \sinh(d_{\mathbf{D}}(a, b))}{\left(\frac{2 \cosh(pd_{\mathbf{D}}(a, b))}{K^p(|D_1 f(a)|) + K^p(|D_1 f(b)|)} \right)^{1/p} + \left(\frac{K^p(|D_1 f(a)|) + K^p(|D_1 f(b)|)}{2 \cosh(pd_{\mathbf{D}}(a, b))} \right)^{1/p}}$$

where $K(t) = t/(1 + \sqrt{1 - t^2})$. Equality holds for distinct $a, b \in \mathbf{D}$ if and only if $f = R \circ T$, where $R \in \text{Rot}(\mathbf{P})$, $T \in \text{Aut}(\mathbf{D})$ and $a, b \in T^{-1}(0, 1)$. Conversely, if a nonconstant meromorphic function f defined on \mathbf{D} satisfies (7) for some $p \geq 1$ and all $a, b \in \mathbf{D}$, then f is spherically convex.

Proof. Suppose f is spherically convex. Then Theorem 2 applies to the region $\Omega = f(\mathbf{D})$. If we apply Theorem 2 to the points $A = f(a)$, $B = f(b)$ and use $d_{\mathbf{D}}(a, b) = d_{\Omega}(f(a), f(b))$, $\mu_{\Omega}(f(z)) = 1/|D_1 f(z)|$, then we obtain inequality (7). If equality holds, then Ω must be a hemisphere and so we can find $R \in \text{Rot}(\mathbf{P})$ with $R^{-1}(\Omega) = \mathbf{D}$ and $R^{-1}(A), R^{-1}(B) \in (0, 1)$. Then $T = R^{-1} \circ f \in \text{Aut}(\mathbf{D})$ and $T(a), T(b) \in (0, 1)$. This proves that $f = R \circ T$ for some $R \in \text{Rot}(\mathbf{P})$, $T \in \text{Aut}(\mathbf{D})$ and $a, b \in T^{-1}(0, 1)$. On the other hand, if f has this form, then it is straightforward to show that equality holds for all $a, b \in T^{-1}(0, 1)$.

Conversely, suppose a nonconstant holomorphic function f defined on \mathbf{D} satisfies (7) for some $p \geq 1$ and all $a, b \in \mathbf{D}$. As in the proof of the invariant form of the Koebe distortion theorem for holomorphic univalent functions in [4, p. 144], we can conclude that f is univalent on \mathbf{D} . Set $\Omega = f(\mathbf{D})$. Since f is a conformal map, $d_{\Omega}(f(a), f(b)) = d_{\mathbf{D}}(a, b)$ for all $a, b \in \mathbf{D}$ and $\mu_{\Omega}(f(z)) = 1/|D_1 f(z)|$. Thus, inequality (7) for f implies inequality (4) for Ω . This implies that Ω is a spherically convex region, so f is spherically convex. \square

As we noted in the proof of Theorem 2, the righthand side of inequality (4) is a decreasing function of p . Similarly, the righthand side of inequality (7) is a decreasing function of p . Thus, in both instances the largest lower bound is obtained when $p = 1$ while the limiting case $p \rightarrow \infty$ produces the smallest lower bound. The limiting case of inequality (7) is an invariant version of a known growth theorem for

spherically convex functions as we now show. Suppose f is spherically convex on \mathbf{D} . If we let $p \rightarrow \infty$ in inequality (7) we obtain

$$d_{\mathbf{P}}(f(a), f(b)) \geq \arctan \frac{d_{\mathbf{D}}(a, b)}{\frac{\exp(d_{\mathbf{D}}(a, b))}{\max\{K(|D_1 f(a)|), K(|D_1 f(b)|)\}} + \frac{\max\{K(|D_1 f(a)|), K(|D_1 f(b)|)\}}{\exp(d_{\mathbf{D}}(a, b))}}.$$

Since $t \mapsto t + (1/t)$ is decreasing on $(0, 1)$, we get

$$d_{\mathbf{P}}(f(a), f(b)) \geq \arctan \frac{d_{\mathbf{D}}(a, b)}{\frac{\exp(d_{\mathbf{D}}(a, b))}{K(|D_1 f(a)|)} + \frac{K(|D_1 f(a)|)}{\exp(d_{\mathbf{D}}(a, b))}}.$$

If f is normalized by $f(0) = 0$, $f'(0) = \alpha \in (0, 1]$ and we choose $a = 0$, $b = z$, then the preceding inequality simplifies to

$$d_{\mathbf{P}}(0, f(z)) \geq \arctan \frac{\alpha|z|}{1 + \sqrt{1 - \alpha^2}|z|},$$

or

$$|f(z)| \geq \frac{\alpha|z|}{1 + \sqrt{1 - \alpha^2}|z|}.$$

This is the known sharp lower bound on $|f(z)|$ for normalized spherically convex functions [6].

6. Concluding remarks. So far we have not been able to establish sharp upper bounds in either the two-point distortion theorem for spherically convex functions or the two-point comparison theorem for spherically convex regions. This is curious since in [8] and [9] the corresponding upper bounds were easier to obtain than the lower bounds in the sense that the proofs of the upper bounds used weaker coefficient bounds than the proofs of the lower bounds. On the other hand, in dealing with other spherical situations sharp upper bounds are sometimes harder to obtain. For example, the sharp upper bound on $|f'(z)|$ for normalized meromorphic univalent functions on \mathbf{D} has not been given for $|z| > p$, where p is the simple pole of f in \mathbf{D} , while the sharp lower bound was completely determined in [5]. A similar

situation holds for spherically convex functions. The sharp lower bound on $|f'(z)|$ is known for all $z \in \mathbf{D}$ while the sharp upper bound on $|f'(z)|$ has been established only for z near the origin; see [6] for the explicit results.

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SCHOOL OF INTEGRATED STUDIES, PENNSYLVANIA COLLEGE OF TECHNOLOGY,
WILLIAMSPORT, PA 17701-5799
E-mail address: `wma@pct.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF CINCINNATI, CINCINNATI,
OH 45221-0025
E-mail address: `david.minda@math.uc.edu`