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EXACT SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS OF LAMÉ'S TYPE AND ITS APPLICATIONS TO CONTACT GEOMETRY

BANG-YEN CHEN

ABSTRACT. The study of linear differential equations with one or more of their coefficients involving Jacobi's elliptic functions was initiated by Picard. Among such linear differential equations perhaps the most famous one is the equation of Lamé. The methods of finding the exact solutions of the Lamé equation have been investigated by many mathematicians. In this note we investigate a class of differential equations of Lamé's type which arise naturally in the study of Legendre curves in contact geometry. We present the exact solutions of this class of differential equations and apply them to determine explicitly the Legendre curves associated with the exact solutions of this class of differential equations.

1. Introduction. The study of linear differential equations with coefficients involving uniform doubly periodic functions of the independent variable was initiated by Picard. For instance, Picard had shown that every linear differential equation with uniform doubly periodic coefficients and possessing only uniform solutions has always at least one solution which is a doubly periodic function of the second kind. Among linear differential equations with uniform doubly periodic coefficients perhaps the most famous one is the equation of Lamé:

$$\frac{d^2y}{dx^2} = [n(n+1)k^2\mathrm{sn}^2(x,k) + c]y.$$

The methods of finding the exact solutions of the Lamé equation have been studied by many mathematicians. For a recent study on Lamé's equation and its applications to physics, see, for instance, [**4**].

Legendre curves are known to play an important role in the study of contact manifolds, e.g., a diffeomorphism of a contact manifold

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is a contact transformation if and only if it maps Legendre curves to Legendre curves. The investigation of Legendre curves from the Riemannian point of view has been investigated in [**1, 2**] among others.

Legendre curves in a 3-sphere or in a three-dimensional anti-de Sitter space-time H_1^3 arise from the solutions of the differential equation:

$$z''(x) = i\lambda(x)z'(x) - cz(x) \tag{1.1}$$

where $\lambda(x)$ is a real-valued function and c is a nonzero constant. It is well known in the theory of differential equations that exact solutions of second order differential equations are usually difficult to obtain. In this note we show that the exact solutions of equation (1.1) can actually be derived for every real-valued solution λ of the second order nonlinear differential equation:

$$\lambda'' = -c\lambda - \frac{2}{9}\lambda^3. \tag{1.2}$$

Since the solutions of (1.2) are "generically" given by functions involving Jacobi's elliptic functions, this leads to another interesting class of differential equations whose coefficients also involve Jacobi's elliptic functions. In this note, we also show the precise way to construct the Legendre curves in $S^3(c)$ or in $H_1^3(c)$ using the exact solutions of such differential equations.

2. Exact solutions and Legendre curves. By a contact manifold we mean a smooth manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. A curve $\gamma = \gamma(t)$ in a contact manifold is called a *Legendre curve* if $\eta(\gamma'(t)) = 0$ along γ.

Let $\mathbf{C}^{n+1}$ and $\mathbf{C}_1^{n+1}$ denote respectively the complex Euclidean $(n+1)$-space and the complex pseudo-Euclidean $(n+1)$-space with metric

$$g = -dz_1 d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j d\bar{z}_j.$$

We put $S^{2n+1}(c) = \{z = (z_1, \dots, z_{n+1}) \in \mathbf{C}^{n+1} : \langle z, z\rangle = (1/c) > 0\}$ and $H_1^{2n+1}(c) = \{z = (z_1, z_2, \dots, z_{n+1}) \in \mathbf{C}_1^{n+1} : \langle z, z\rangle = (1/c) < 0\}$, where $\langle\ ,\ \rangle$ denotes the inner product induced from the metrics. $H_1^{2n+1}(c)$ is known as an anti-de Sitter space-time. It is well known that both $S^{2n+1}(c)$ and $H_1^{2n+1}(c)$ admit canonical contact structures induced from the complex structures on $\mathbf{C}^{n+1}$ and on $\mathbf{C}_1^{n+1}$, respectively.

The following lemmas from [**2**] provide a simple relationship between Legendre curves and differential equation (1.1).

Lemma 1. *Let c be a positive number and $z = (z_1, z_2) : I \to S^3(c) \subset \mathbf{C}^2$ a unit speed curve where I is either an open interval or a circle. If $z : I \to \mathbf{C}^2$ satisfies*

$$z''(x) = i\lambda(x)z'(x) - cz(x) \tag{A}$$

for some nonzero real-valued function λ on I, it defines a Legendre curve in $S^3(c)$.

Conversely, if z defines a Legendre curve in $S^3(c)$, it satisfies differential equation (A) *for some real-valued function λ.*

Lemma 2. *Let c be a negative number and $z = (z_1, z_2) : I \to H_1^3(c) \subset \mathbf{C}_1^2$ a unit speed curve where I is an open interval. If $z : I \to \mathbf{C}_1^2$ satisfies*

$$z''(x) = i\lambda(x)z'(x) - cz(x) \tag{B}$$

for some nonzero real-valued function λ on I, then it defines a Legendre curve in $H_1^3(C)$.

Conversely, if z defines a Legendre curve in $H_1^3(c)$, then it satisfies differential equation (B) *for some real-valued function λ.*

The main purpose of this note is to prove the following.

Theorem. *For any constant c and any nontrivial real-valued solution $\lambda = \lambda(x)$ of the differential equation*

$$\lambda'' = -c\lambda - \frac{2}{9}\lambda^3, \tag{2.1}$$

we have

(i) *the differential equation*

$$z''(x) = i\lambda(x)z'(x) - cz(x) \tag{2.2}$$

has the following two independent complex-valued solutions

$$z_1(x) = \lambda(x)\exp\left(\frac{i}{3}\int \lambda(x)\,dx\right), \tag{2.3}$$

$$z_2(x) = z_1(x)\int\left\{\frac{9}{\lambda^2(x)}\exp\left(\frac{i}{3}\int^x \lambda(t)\,dt\right)\right\}dx. \tag{2.4}$$

(ii) *If* $c > 0$, *then for any nontrivial solution* $\lambda(x)$ *of* (2.1), *there exist two real numbers* α, γ *and two solutions of* (2.2) *in the forms of* (2.3) *and* (2.4) *such that* $z(x) = (\alpha z_2(x), \alpha\gamma z_1(x))$ *defines a unit speed Legendre curve in* $S^3(c) \subset \mathbf{C}^2$, *and*

(iii) *if* $c < 0$, *then for any nontrivial solution* $\lambda(x)$ *of* (2.1), *there exist two real numbers* α, γ *and two solutions of* (2.2) *in the forms of* (2.3) *and* (2.4) *such that one of the following two maps*

$$z(x) = (\alpha\gamma z_1(x), \alpha z_2(x)), \qquad z(x) = (\alpha z_2(x), \alpha\gamma z_1(x))$$

defines a unit speed Legendre curve in $H_1^3(c) \subset \mathbf{C}_1^2$.

Proof. Let c be a given constant and λ a nontrivial solution of the differential equation (2.1). We put

$$z_1(x) = \psi(x)w(x), \qquad w(x) = \exp\left(\frac{i}{3}\int \lambda(x)\,dx\right). \tag{2.5}$$

Then we have

$$\begin{aligned} z_1' &= \left(\psi' + \frac{i}{3}\lambda\psi\right)w, \\ z_1'' &= \left(\psi'' + \frac{2}{3}i\lambda\psi' + \frac{i}{3}\lambda'\psi - \frac{\lambda^2}{9}\psi\right)w. \end{aligned} \tag{2.6}$$

Thus $z_1(x) = \psi(x)w(x)$ is a solution of (2.2) for some real-valued function $\psi(x)$ if and only if ψ satisfies the following system of differential equations

$$\psi'' = -\left(c + \frac{2}{9}\lambda^2\right)\psi \tag{2.7}$$

$$\lambda\psi' = \lambda'\psi. \tag{2.8}$$

Since λ satisfies (2.1), $\psi = \lambda$ is a solution of (2.7)–(2.8). Therefore, $z_1(x) = \lambda(x)\exp((i/3)\int \lambda(x)\,dx)$ is a solution of (2.2). The second independent solution $z_2 = z_1(x)\int\{(9/\lambda^2(x))\exp((i/3)\int^x \lambda(t)\,dt)\}\,dx$ can be obtained by applying the method of reduction of order.

Now let $F(x)$ and $v(x)$ be anti-derivatives of $\lambda(x)/3$ and of $(9/\lambda^2)e^{iF(x)}$, respectively. We put

$$u(x) = e^{iF(x)}, \qquad w(x) = \langle u(x), v(x)\rangle. \tag{2.9}$$

Then we have

$$u' = \frac{i}{3}\lambda u, \qquad v' = \frac{9}{\lambda^2}u, \qquad w' = \frac{1}{3}\langle iu, v\rangle\lambda + \frac{9}{\lambda^2}. \tag{2.10}$$

From (2.10) we find

$$w'' + \frac{\lambda^2}{9}w = \frac{\lambda'}{\lambda} - \frac{81\lambda'}{\lambda^4}. \tag{2.11}$$

Also, from (2.1) we find

$$\lambda'^2 = 9b - c\lambda^2 - \frac{\lambda^4}{9}, \tag{2.12}$$

where b is a constant. By using (2.1), (2.3) and a direct computation, we may prove that

$$w = c_1\cos F(x) + c_2\sin F(x) - \frac{\lambda'}{b\lambda}, \tag{2.13}$$

is the general solution of the differential equation (2.11) where c_1, c_2 are constants.

Since $z_2 = \lambda uv$ and $\langle v, v\rangle' = (18/\lambda^2)w$, (2.4) and (2.13) yield

$$\begin{aligned}\langle z_2, z_2\rangle = \frac{9}{b} + c_3\lambda^2 + 18c_1\lambda^2 \int \left(\frac{1}{\lambda^2}\cos F(x)\right)dx \\ + 18c_2\lambda^2 \int \left(\frac{1}{\lambda^2}\sin F(x)\right)dx,\end{aligned} \tag{2.14}$$

where c_3 is a constant.

On the other hand, by (2.3) and (2.4), we have

$$\langle z_2, (c_1 + ic_2)z_1\rangle = 9c_1\lambda^2 \int \left(\frac{1}{\lambda^2}\cos F(x)\right) dx + 9c_2\lambda^2 \int \left(\frac{1}{\lambda^2}\sin F(x)\right) dx. \tag{2.15}$$

Therefore, we obtain

$$\langle z_2 - (c_1 + ic_2)z_1, z_2 - (c_1 + ic_2)z_1\rangle = \frac{9}{b} + (c_3 + c_1^2 + c_2^2)\lambda^2. \tag{2.16}$$

On the other hand, we have

$$z_2 - (c_1 + ic_2)z_1 = z_1\left(\int \left(\frac{9}{\lambda^2}\cos F(x)\right) dx - c_1\right) + iz_1\left(\int \left(\frac{9}{\lambda^2}\sin F(x)\right) dx - c_2\right).$$

Therefore, by choosing a suitable anti-derivative, say $G(x)$, of $(9/\lambda^2)e^{iF(x)}$ for z_2 in (2.4), we have

$$\langle z_2, z_2\rangle = \frac{9}{b} + c_3\lambda^2 = \frac{9}{b} + c_3\langle z_1, z_1\rangle \tag{2.17}$$

$$\langle v, v\rangle = \frac{9}{b\lambda^2} + c_3. \tag{2.18}$$

For simplicity, we put $G(x) = \int_{x_0}^{x}(9/\lambda^2(t))e^{iF(t)}\,dt$. Combining (2.9) and (2.18), we obtain

$$\langle G(x), G(x)\rangle = \frac{9}{b\lambda^2(x)} + c_3. \tag{2.19}$$

Now, by taking the first and second derivatives of (2.18) and applying (2.1) and (2.12), we find

$$\langle u, v\rangle = -\frac{\lambda'}{b\lambda}, \qquad \langle iu, v\rangle = \frac{\lambda}{3b}. \tag{2.20}$$

Since

$$z_1' = \lambda' u + \frac{i}{3}\lambda^2 u, \qquad z_2' = \left(\lambda' + \frac{i}{3}\lambda^2\right) uv + \frac{9u^2}{\lambda}, \tag{2.21}$$

(2.9), (2.12) and (2.19) imply

$$\langle z_1', z_1' \rangle = 9b - c\lambda^2, \qquad \langle z_2', z_2' \rangle = \frac{9c}{b} + c_3(9b - c\lambda^2). \tag{2.22}$$

Case a). $c > 0$. In this case (2.22) implies $b > 0$. On the other hand, (2.19) yields $0 = (9/(b\lambda^2(x_0))) + c_3$. Thus, $c_3 < 0$. We put

$$z(x) = (\alpha z_2(x), \alpha\gamma z_1(x)), \tag{2.23}$$

where $c_3 = -\gamma^2$, $\alpha = (1/3)\sqrt{b/c}$. By applying (2.17), (2.22) and (2.23), we obtain $\langle z, z \rangle = (1/c)$, $\langle z', z' \rangle = 1$. Since the map $z = z(x)$ given by (2.23) satisfies the differential equation (2.1). Thus, according to Lemma 2.1, $z = z(x)$ defines a unit speed Legendre curve in $S^3(c) \subset \mathbf{C}^2$. This proves statement (ii).

Case b). $c < 0$. In this case we have $c_3 > 0$. This can be seen as follows. If $c_3 < 0$, then (2.22) implies $b > 0$. On the other hand, (2.17) implies $b > 0$. This is a contradiction. Therefore, $c_3 > 0$ and there is a positive number γ such that $c_3 = \gamma^2$.

Case b-1). $b > 0$. In this case we put

$$z(x) = (\alpha z_2(x), \alpha\gamma z_1(x)), \tag{2.24}$$

where $\alpha = (1/3)\sqrt{b/(-c)}$. By applying (2.17), (2.22) and (2.24), we may obtain $\langle z, z \rangle = (1/c)$, $\langle z', z' \rangle = 1$. Since $z = z(x)$ satisfies differential equation (1.1), Lemma 2.2 implies that $z = z(x)$ defines a unit speed Legendre curve in $H_1^3(c) \subset \mathbf{C}_1^2$.

Case b-2). $b < 0$. In this case we put

$$z(x) = (\alpha\gamma z_1(x), \alpha z_2(x)), \tag{2.25}$$

where $\alpha = (1/3)\sqrt{b/c}$. By applying an argument similar to that of Case b-2), we know that $z = z(x)$ defines a unit speed Legendre curve in $H_1^3(c) \subset \mathbf{C}_1^2$. This proves statement (iii). $\square$

3. Some examples. Let $\mathrm{cn}(u,k)$, $\mathrm{dn}(u,k)$, $\mathrm{sn}(u,k)$ denote the three main Jacobi's elliptic functions with modulus k and complementary modulus k_1. The other Jacobi's elliptic functions are defined by taking reciprocals and quotients (see [**3**] for details). For example, we have

$$\mathrm{cd}\,(u) = \frac{\mathrm{cn}(u)}{\mathrm{dn}(u)}, \qquad \mathrm{sd}(u) = \frac{\mathrm{sn}(u)}{\mathrm{dn}(u)}, \qquad \mathrm{nd}(u) = \frac{1}{\mathrm{dn}(u)}. \tag{3.1}$$

Example 1. The following Jacobi's elliptic functions

$$\begin{gathered} \lambda(x) = 3akk_1\mathrm{sd}(ax,k), \quad 3a\mathrm{dn}(ax,k), \quad 3ak\mathrm{cn}(ax,k), \\ 3ak_1\mathrm{nd}(ax,k), \quad 3a\mathrm{dn}(ax,1) \end{gathered} \tag{3.2}$$

are respectively solutions of the differential equation $\lambda'' = -c\lambda - (2/9)\lambda^3$ for

$$\begin{gathered} c = (1-2k^2)a^2, \quad (k^2-2)a^2, \quad (1-2k^2)a^2, \\ (k^2-2)a^2, \quad -a^2. \end{gathered} \tag{3.3}$$

Example 2. According to Theorem (i), for each $a > 0$ and each k with $0 < k < 1$, the following differential equations of Lamé type:

$$z''(x) = 3akk_1 i\,\mathrm{sd}(ax,k)z' + (2k^2-1)a^2 z, \tag{3.4}$$

$$z''(x) = 3aki\,\mathrm{cn}(ax,k)z' + (2k^2-1)a^2 z, \tag{3.5}$$

$$z''(x) = 3ak_1 i\,\mathrm{nd}(ax,k)z' + (2-k^2)a^2 z, \tag{3.6}$$

$$z''(x) = 3ai\,\mathrm{dn}(ax,k)z' + (2-k^2)a^2 z, \tag{3.7}$$

$$z''(x) = 3ai\,\mathrm{dn}(ax,1)z' + a^2 z \tag{3.8}$$

have independent solutions given respectively by

$$\begin{cases} z_1 = \mathrm{sd}(ax,k)(k\,\mathrm{cd}(ax,k) + ik_1\mathrm{nd}(ax,k)), \\ z_2 = (k\,\mathrm{cd}(ax,k) + ik_1\mathrm{nd}(ax,k))(k_1\mathrm{cd}(ax,k) - ik\,\mathrm{nd}(ax,k)), \end{cases} \tag{3.9}$$

(3.10)
$$\begin{cases} z_1 = \mathrm{cn}(ax,k)(\mathrm{dn}(ax,k) + ik\,\mathrm{sn}(ax,k)), \\ z_2 = (\mathrm{dn}(ax,k) + ik\,\mathrm{sn}(ax,k))(k_1\mathrm{sn}(ax,k) + ik\,\mathrm{dn}(ax,k)), \end{cases}$$

(3.11)
$$\begin{cases} z_1 = \mathrm{nd}(ax,k)(k_1\mathrm{sd}(ax,k) - i\,\mathrm{cd}(ax,k)), \\ z_2 = (k_1\mathrm{cd}(ax,k) + i\,\mathrm{sd}(ax,k))(k_1\mathrm{sd}(ax,k) - i\,\mathrm{cd}\,(ax,k)), \end{cases}$$

(3.12)
$$\begin{cases} z_1 = \mathrm{dn}(ax,k)(\mathrm{cn}(ax,k) + i\,\mathrm{sn}(ax,k)), \\ z_2 = (k_1^2\mathrm{sn}(ax,k) - i\,\mathrm{cn}(ax,k))(\mathrm{cn}(ax,k) + i\,\mathrm{sn}(ax,k)), \end{cases}$$

(3.13)
$$\begin{cases} z_1 = \mathrm{dn}(ax,1)(\mathrm{dn}(ax,1) + i\,\mathrm{sn}(ax,1)), \\ z_2 = (\mathrm{dn}(ax,1) + i\,\mathrm{sn}(ax,1))(\mathrm{sn}(ax,1) + \frac{i}{2}\mathrm{nd}(ax,1)). \end{cases}$$

Applying Theorem (ii) and (iii) we have the following.

Example 3. For each $a > 0$ and each k with $0 < k < \sqrt{1/2}$,

$$\begin{aligned} \psi_a(x) = \frac{1}{a\sqrt{1-2k^2}} &\left(k\ \mathrm{cd}(ax,k) + i\sqrt{1-k^2}\ \mathrm{nd}(ax,k)\right) \\ &\left(\sqrt{1-k^2}\ \mathrm{cd}(ax,k) - ik\ \mathrm{nd}(ax,k),\ \sqrt{1-2k^2}\ \mathrm{sd}(ax,k)\right) \end{aligned} \tag{3.14}$$

is a Legendre curve in $S^3(c)$ with $c = (1-2k^2)a^2$.

Example 4. For each $a > 0$ and each k with $1/\sqrt{2} < k < 1$,

$$\begin{aligned} \psi_a(x) = \frac{1}{a\sqrt{2k^2-1}} &\left(k\ \mathrm{cd}(ax,k) + i\sqrt{i-k^2}\ \mathrm{nd}(ax,k)\right) \\ &\left(\left(\sqrt{1-k^2}\ \mathrm{cd}(ax,k) - ik\ \mathrm{nd}(ax,k)\right),\ \sqrt{2k^2-1}\ \mathrm{sd}(ax,k)\right), \end{aligned} \tag{3.15}$$

is a Legendre curve in $H_1^3(c)$ with $c = (1-2k^2)a^2 < 0$.

REFERENCES

1. C. Baikoussis and D.E. Blair, *On Legendre curves in contact* 3-*manifolds*, Geom. Dedicata **49** (1994), 135–142.

2. B.Y. Chen, *Interaction of Legendre curves and Lagrangian submanifolds*, Israel J. Math. **99** (1998), 69–108.

3. D.F. Lawden, *Elliptic functions and applications*, Springer-Verlag, New York, 1989.

4. R.S. Ward, *The Nahn equations, finite-gap potentials and Lamé functions*, J. Phys. A **20** (1987), 2679–2683.

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824-1027
E-mail address: bychen@math.msu.edu