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LOCAL COHOMOLOGY FOR COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. The purpose of this paper is to introduce a local cohomology theory in the unital commutative Banach algebras context and to describe a connection between the local cohomology functor and direct limit of hom functors.

1. Introduction. The local cohomology theory in the context of commutative ring theory was introduced by Grothendieck [3] and developed by Brodmann, Mcdonald and Sharp and some other mathematicians [2]. In this paper we introduce a version of local cohomology for unital commutative Banach algebras and Banach modules. After introduction, we specialize in Section 2 to local cohomology functor, torsion and torsion-free modules with respect to a given ideal and also some related examples. Section 3 provides the connected right sequence of local cohomology functors. In the last section the local cohomology functor is described as a direct limit of some hom functors.

Throughout the paper, A is a fixed unital commutative Banach algebra with unit e, ||e|| = 1, and I is a fixed closed ideal of A. We follow the notation and terminology of [5] or [6], but with some exceptions as the following:

Definition 1.1. A Banach *A*-module is a Banach space with an algebraic unital symmetric *A*-bimodule structure satisfying

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A module morphism between Banach A-modules X and Y is a linear mapping $f: X \to Y$ such that

$$f(ax) = af(x); \quad x \in X, a \in A.$$

Notation 1.2. We denote the category of all Banach A-modules and module morphisms between them by \mathcal{C} , the corresponding positive (cochain) complexes by $\overline{\mathcal{C}}$, and the category of algebraic A-modules with underlying complete semi-normed spaces satisfying (\diamondsuit) and their module morphisms by $\langle \mathcal{C} \rangle$.

Let S be a subset of A and $X \in C$; then the set of all $x \in X$ such that Sx = 0 is a Banach A-submodule of X which is denoted by $(0:_X S)$.

2. Local cohomology functor with respect to an ideal.

Definition 2.1. Let $X \in C$; then the local cohomology of X with respect to I, denoted by $\Gamma_I(X)$, is the closure of $\{x \in X; I^n x = 0 \text{ for some } n \in \mathbb{N}\}$, where I^n is the closed ideal generated by $a_1 a_2 \cdots a_n, 1 \leq i \leq n, a_i \in I$. $\Gamma_I(X)$ is a Banach A-submodule of X and so it belongs to C. It follows therefore that

$$\Gamma_I(X) = \overline{\bigcup_{n=1}^{\infty} (0:_X I^n)}.$$

An A-module X is said to be I-torsion, respectively, I-torsion-free, whenever $\Gamma_I(X) = X$, respectively $\Gamma_I(X) = 0$.

Examples 2.2. (i) If I has a bounded approximate identity, then clearly $I^n = I$. Hence $\Gamma_I(X) = (0:_X I)$ for each Banach A-module X; in particular, $\Gamma_I(A) = \operatorname{Ann}(I)$, where $\operatorname{Ann}(I)$ is the annihilator of I. Moreover, if I has a bounded approximate identity for X, i.e., there is a net $\{e_\lambda\}_{\lambda \in \Lambda}$ with $e_\lambda \in I$ such that $\lim_\lambda e_\lambda x = x$ for all $x \in X$ and $\sup\{\|e_\lambda\|; \lambda \in \Lambda\} < \infty$, e.g., X = I, then $\Gamma_I(X) = 0$. Thus, in this case, X is I-torsion-free.

(ii) If X is a Banach A-module and $\tilde{\Gamma}_I(X) = \bigcup_{n=1}^{\infty} (0:_X I^n)$, then

$$\Gamma_I(X) = \overline{\tilde{\Gamma}_I(X)} = \overline{\tilde{\Gamma}_I(\tilde{\Gamma}_I(X))} \subseteq \overline{\tilde{\Gamma}_I(\Gamma_I(X))} = \Gamma_I(\Gamma_I(X)) \subseteq \Gamma_I(X).$$

Hence, $\Gamma_I(\Gamma_I(X)) = \Gamma_I(X)$. Thus $\Gamma_I(X)$ is *I*-torsion.

(iii) Let A be an abelian von Neumann algebra and I a weak-operator closed ideal of A. Then $A \simeq C(\Omega)$ for some extremely disconnected compact Hausdorff space Ω [7, Theorem 5.2.1] and I is of the form Ac for a projection $c \in A$ [8, Theorem 6.8.8]. Since I has the identity c, Example 2.2 (i) shows that $\Gamma_I(X) = \{x \in X; cx = 0\}$. If X = A and I is nontrivial, A is clearly neither I-torsion nor I-torsion-free.

If $f: X \to Y$ is a module morphism in \mathcal{C} and x is annihilated by I^n for some n, then for each $a \in I^n$, af(x) = f(ax) = 0, so I^n annihilates f(x). Hence, $f(\Gamma_I(X)) \subseteq \Gamma_I(Y)$, by the continuity of f. Let $\Gamma_I(f)$ be the restriction and corestriction of f to $\Gamma_I(X)$ and $\Gamma_I(Y)$, respectively. Then it can be checked that $\Gamma_I(\cdot)$ is a functor from \mathcal{C} to \mathcal{C} . We call this the local cohomology functor with respect to I. It is additive, \mathbb{C} linear and A-linear, i.e., for objects X and Y in \mathcal{C} , module morphisms $f: X \to Y$ and $g: X \to Y \lambda \in \mathbb{C}$ and $a \in A$:

$$\Gamma_I(f+g) = \Gamma_I(f) + \Gamma_I(g), \Gamma_I(\lambda f) = \lambda \Gamma_I(f) \text{ and } \Gamma_I(af) = a \Gamma_I(f).$$

3. The *i*th local cohomology functors with respect to an ideal, $i \ge 0$. We recall that if E is a Banach space, then $\mathcal{B}(A, E)$ is a Banach A-module together with the following action:

$$(a\phi)(b) = \phi(ba); \phi \in \mathcal{B}(A, E), \quad a, b \in A.$$

It is an injective Banach A-module [6, Chapter III, Section 1.4].

Consider the normalized injective resolution for $X \in \mathcal{C}$ [6, Chapter III, Section 2]:

$$0 \longrightarrow J^0(X) \xrightarrow{d^0} J^1(X) \longrightarrow \cdots \longrightarrow J^i(X) \xrightarrow{d^i} J^i(X) \longrightarrow \cdots (\mathcal{J}(X)).$$

This complex has the property that the following complex is admissible:

$$0 \longrightarrow X \xrightarrow{\tilde{\pi}} J^0(X) \xrightarrow{d^0} J^1(X) \longrightarrow \cdots \longrightarrow J^i(X) \xrightarrow{d^i} J^{i+1}(X) \longrightarrow \cdots$$

In the latter complex, $\tilde{\pi}_X : X \to \mathcal{B}(A, X)$ is given by $(\tilde{\pi}(x))(a) = ax$; also if $C(X) = \mathcal{B}(A, X)/\operatorname{Im} \tilde{\pi}_X, C^{-1}(X) = X$ and $C^i(X) = ax$

 $C(C^{i-1}(X)), i \geq 0$, then for each $i \geq 0, J^i(X) = \mathcal{B}(A, C^{i-1}(X))$ and d^i is the composition of $J^i(X) \xrightarrow{\text{nat.}} C^i(X) \xrightarrow{\tilde{\pi}_{C^i(X)}} J^{i+1}(X)$. In fact \mathcal{J} is a functor from \mathcal{C} to $\overline{\mathcal{C}}$.

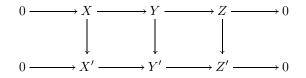
Definition 3.1. The *n*th injective derived functor of $\Gamma_I(\cdot)$, i.e., $H^i \circ \overline{\Gamma}_I \circ \mathcal{J} = H_I^i$ is called the *i*th local cohomology functor with respect to I, [6, Chapter III, Section 3]. $H_I^i(X)$ is called the *i*th local cohomology module of X with respect to I. The functors H_I^i , $i \geq 0$, are additive, **C**-linear, A-linear and covariant functors from \mathcal{C} to $\langle \mathcal{C} \rangle$. $H_I^i(X)$ is independent of the choice of injective resolution for X up to an isomorphism in $\langle \mathcal{C} \rangle$ [6, Theorem 3.3.10].

Remark 3.2. If Q is an injective Banach A-module, then $H_I^i(Q) = 0$ for all i > 0. In fact, the exact complex $0 \to Q \xrightarrow{1_Q} Q \to 0 \to \cdots$ shows that $0 \to Q \to 0 \to \cdots (Q)$ is an injective resolution for Q so, for all $i > 0, H_I^i(Q) = H^i(\Gamma_I(Q)) = 0.$

Definition 3.3. A sequence $(T^i)_{i\geq 0}$ of covariant functors from C to $\langle C \rangle$ is called a connected right sequence of covariant functors if the following conditions are satisfied:

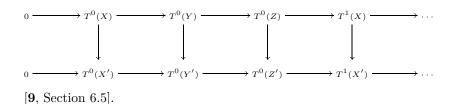
(i) If $0 \to X \to Y \to Z \to 0$ is an admissible short complex in \mathcal{C} , there are defined continuous connecting morphisms $T^n(Z) \to T^{n+1}(X)$, $n \ge 0$, in $\langle \mathcal{C} \rangle$ such that $0 \to T^0(X) \to T^0(Y) \to T^0(Z) \to T^1(X) \to \cdots$ is a complex.

(ii) Whenever



is a commutative diagram in C with admissible rows, there is defined a morphism between the corresponding complexes:

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Theorem 3.4. Let $0 \to X \to Y \to Z \to 0$ (S) be an admissible short complex in C. Then there exists a long exact complex in $\langle C \rangle$ as

$$0 \longrightarrow H^0_I(X) \longrightarrow H^0_I(Y) \longrightarrow H^0_I(Z) \xrightarrow{\zeta_0} H^1_I(X) \longrightarrow \cdots$$

with continuous connecting morphisms $\zeta_n : H^n_I(Z) \to H^{n+1}_I(X)$. Moreover, $(H^i_I)_{i\geq 0}$ is a connected right sequence of covariant functors.

Proof. Suppose that $0 \to X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \to 0$ (S) is an admissible short complex in C so that there exist continuous operators $\rho: Y \to X$ and $\sigma: Z \to Y$ such that $\rho \circ \phi = 1_X$, $\psi \circ \sigma = 1_Z$, $\phi \circ \rho + \sigma \circ \psi = 1_Y$ [6, Proposition 3.1.8]. Then the following short sequence of complexes is admissible:

$$0 \longrightarrow C(X) \longrightarrow C(Y) \longrightarrow C(Z) \longrightarrow 0.$$

Applying Proposition III.1.5 and Theorem III.1.9 of [6] to the functor $\mathcal{B}(A,?)$, we conclude that, for any $i \geq 0$, $J^i(\mathcal{S})$ splits. Thus $0 \to \Gamma_I(\mathcal{J}(X)) \to \Gamma_I(\mathcal{J}(Y)) \to \Gamma_I(\mathcal{J}(Z)) \to 0$ is exact. Now we may use the fundamental lemma of homological algebra [6, Theorem 0.5.7] in order to get

$$o \longrightarrow H^0_I(X) \longrightarrow H^0_I(Y) \longrightarrow H^0_I(Z) \xrightarrow{\zeta_0} H^1_I(X) \longrightarrow \cdots$$

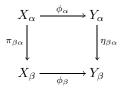
with continuous connecting morphisms $\zeta_n : H_I^n(Z) \to H_I^{n+1}(X)$. The rest is a well-known technique in homological algebra, [9, Section 6.3] and [6, Chapter 0, Section 5]. \Box

4. Direct limit and local cohomology functors. Let $\{X_{\alpha}\}_{\alpha \in K}$ be a family of Banach A-modules. Recall that the topological direct

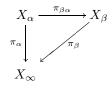
sum of $\{X_{\alpha}\}_{\alpha\in K}$ is defined as the completion of algebraic direct sum $\bigoplus_{\alpha\in K} X_{\alpha}$ with respect to the norm $||(x_{\alpha})_{\alpha\in K}|| = \sum_{\alpha\in K} ||x_{\alpha}||$ which will also be denoted by $\bigoplus_{\alpha\in K} X_{\alpha}$. This is in \mathcal{C} and consists of all elements $(x_{\alpha})_{\alpha\in K}$ of the algebraic direct product $\prod_{\alpha\in K} X_{\alpha}$ such that $\sum_{\alpha\in K} ||x_{\alpha}|| < \infty$, see [10, Section 2.1] and [1, Section 9].

Definition 4.1. (i) Let (D, \leq) be a directed set. A direct system over D in \mathcal{C} consists of families $\{X_{\alpha}\}_{\alpha\in D}$ of Banach A-modules and $\{\pi_{\beta\alpha}\}_{(\alpha,\beta)\in D\times D,\alpha\leq\beta}$ of module morphisms such that for all $\alpha,\beta,\gamma\in$ $D, \ \pi_{\alpha\alpha} = 1_{X_{\alpha}}$ and $\pi_{\gamma\beta} \circ \pi_{\beta\alpha} = \pi_{\gamma\alpha}$ whenever $\alpha \leq \beta \leq \gamma$ and $\sup_{\alpha\leq\beta} \|\pi_{\beta\alpha}\| < \infty$. It is denoted by (X, π, D) .

(ii) Let (X, π, D) and (Y, η, D) be direct systems. A map ϕ from (X, π, D) to (Y, η, D) is a family $\{\phi_{\alpha}\}_{\alpha \in D}$ of module morphisms $\phi_{\alpha} : X_{\alpha} \to Y_{\alpha}$ such that it is uniformly bounded, i.e., $\sup_{\alpha \in D} \|\phi_{\alpha}\| < \infty$, and for $\alpha \leq \beta$ the following diagram commutes:

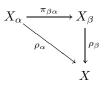


(iii) A direct limit of a direct system (X, π, D) is a Banach Amodule X_{∞} together with a family $\{\pi_{\alpha}\}_{\alpha \in D}$ of module morphisms $\pi_{\alpha} : X_{\alpha} \to X_{\infty}$ such that $\sup_{\alpha \in D} ||\pi_{\alpha}|| < \infty$ and for all $\alpha, \beta \in D$ if $\alpha \leq \beta$ then the diagram

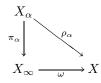


commutes. In addition, it has the following "universal property":

For each Banach A-module X and each family $\{\rho_{\alpha}\}_{\alpha\in D}$ of module morphisms $\rho_{\alpha}: X_{\alpha} \to X$ such that $\sup_{\alpha\in D} \|\rho_{\alpha}\| < \infty$ and for $\alpha \leq \beta$, the diagram



commutes, we have a unique module morphism $\omega : X_{\infty} \to X$ such that, for all $\alpha \in D$ we have the following commutative diagram:



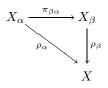
It is obvious that X_{∞} and $\{\pi_{\alpha}\}_{\alpha\in D}$ are unique up to a module isomorphism in \mathcal{C} . We denote the direct limit of (X, π, D) by $(X_{\infty}, \{\pi_{\alpha}\}_{\alpha\in D})$ or $\varinjlim X_{\alpha}$.

α

Proposition 4.2. Any direct system in C has a direct limit.

Proof. Suppose that (X, π, D) is a direct system in \mathcal{C} . Let $i_{\alpha} : X_{\alpha} \to \bigoplus_{\alpha \in D} X_{\alpha}$ be the natural injection and R the Banach A-submodule of $\bigoplus_{\alpha \in D} X_{\alpha}$ generated by $i_{\beta}(\pi_{\beta\alpha}(x_{\alpha})) - i_{\alpha}(x_{\alpha}), (\alpha, \beta) \in D, \alpha \leq \beta, x_{\alpha} \in X_{\alpha}$. Set $X_{\infty} = (\bigoplus_{\alpha \in D} X_{\alpha})/R$ and suppose that, for each $\alpha \in D, \pi_{\alpha}$ is the composition $X_{\alpha} \xrightarrow{i_{\alpha}} \bigoplus_{\alpha \in D} X_{\alpha} \xrightarrow{\text{nat}} X_{\infty}$. Obviously $\sup_{\alpha \in D} \|\pi_{\alpha}\| \leq 1$. We shall show that $(X_{\infty}, \{\pi_{\alpha}\}_{\alpha \in D})$ is the direct limit of (X, π, D) .

Let $X \in \mathcal{C}$, $\rho_{\alpha} : X_{\alpha} \to X$, $\alpha \in D$ be module morphisms such that for each $\alpha \leq \beta$ the diagram



commutes and $M = \sup_{\alpha \in D} \|\rho_{\alpha}\| < \infty$. Suppose $(x_{\alpha})_{\alpha \in D}$ is an element of the algebraic direct sum of $\{X_{\alpha}\}_{\alpha \in D}$. Then $x_{\alpha} = 0$ for

all except finitely many α . If $\Theta((x_{\alpha})_{\alpha \in D}) = \sum_{\alpha \in D} \rho_{\alpha}(x_{\alpha})$, then $\|\Theta((x_{\alpha})_{\alpha \in D})\| \leq \sum_{\alpha \in D} \|\rho_{\alpha}(x_{\alpha})\| \leq M \sum_{\alpha \in D} \|x_{\alpha}\| = M \|(x_{\alpha})_{\alpha \in D}\|$. So we can extend Θ by the continuity to $\bigoplus_{\alpha \in D} X_{\alpha}$, denoted by the same Θ . For $(\alpha, \beta) \in D$, $\alpha \leq \beta$ and $x_{\alpha} \in X_{\alpha}$, $\Theta(i_{\beta}(\pi_{\beta\alpha}(x_{\alpha})) - i_{\alpha}(x_{\alpha})) = \rho_{\beta}(\pi_{\beta\alpha}(x_{\alpha})) - \rho_{\alpha}(x_{\alpha}) = \rho_{\alpha}(x_{\alpha}) - \rho_{\alpha}(x_{\alpha}) = 0$. Hence $R \subseteq \ker \Theta$. Thus we have module morphism $\omega : X_{\infty} \to X$ defined by $\omega(u + R) = \Theta(u)$, $u \in \bigoplus_{\alpha \in D} X_{\alpha}$. We next have $(\omega \circ \pi_{\alpha})(x_{\alpha}) = \omega(i_{\alpha}(x_{\alpha}) + R) = \Theta(i_{\alpha}(x_{\alpha})) = \rho_{\alpha}(x_{\alpha}), x_{\alpha} \in X_{\alpha}$; hence, $\omega \circ \pi_{\alpha} = \rho_{\alpha}$.

If $\{\phi_{\alpha}\}_{\alpha\in D}$ is a mapping from (X, π, D) to (Y, η, D) , then it is easy to verify that there exists a unique, in a certain meaning, module morphism $\phi_{\infty}: X_{\infty} \to Y_{\infty}$. Moreover, it is possible to consider "direct limit" as a functor [4].

Example 4.3. Suppose that D is a directed set, $\{X_{\alpha}\}_{a\in D}$ a family of Banach A-submodules of a given $X \in C$ and whenever $\alpha \leq \beta$, $X_{\alpha} \subseteq X_{\beta}$ and $\pi_{\beta\alpha} : X_{\alpha} \to X_{\beta}$ is the inclusion map, then it is clear that $\lim_{\alpha \to \infty} X_{\alpha} = \bigcup_{\alpha \in D} X_{\alpha}$ and $\pi_{\alpha} : X_{\alpha} \to \lim_{\alpha \to \infty} X_{\alpha}$ is also the inclusion map.

Now let, for $X \in \mathcal{C}$ and $n \in \mathbf{N}$, $X_n = {}_Ah((A/I^n), X)$; for $m \leq n$, $\pi_{nm} : X_m \to X_n$ given by $\pi_{nm}(\alpha) = \alpha \circ \delta_{nm}$ where $\delta_{nm} : A/I^n \to A/I^m$ is defined naturally by $\delta_{nm}(a + I^n) = a + I^m$ (note that $I^n \subseteq I^m$ whenever $m \leq n$). It follows that $\{X_n\}_{n \in \mathbf{N}}$ together with $\{\pi_{nm}\}_{(m,n)\in\mathbf{N}\times\mathbf{N},m\leq n}$ is a direct system. Also if $f : X \to Y$ is a module morphism and $\phi_n : X_n \to Y_n$ is given by $\phi_n(\alpha) = f \circ \alpha$, then $\sup_n \|\phi_n\| \leq \|f\|$ and the commutativity of

$$\begin{array}{c|c} X_m & \xrightarrow{\phi_m} & Y_m \\ & & & \downarrow_A h(\delta_{nm}, X) \\ & & & \downarrow_A h(\delta_{nm}, Y) \\ & & X_n & \xrightarrow{\phi_n} & Y_n \end{array}$$

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The next theorem is the Banach theory version of an important purely algebraic theorem, [2, Theorem 1.2.11].

Theorem 4.4. The functors $\Gamma_I(\cdot)$ and $\varinjlim_n h((A/I^n), \cdot)$ are naturally acquired ent

rally equivalent.

Proof. For each $n, \psi_n : {}_Ah((A/I^n), X) \to (0 :_X I^n)$ given by $\psi_n(f) = f(e + I^n)$ is obviously a module isomorphism and $\sup_n \|\psi_n\| \le 1$. In addition the following diagram commutes:

For the direct limit is a functor, we have a module isomorphism $\psi(X) : \varinjlim_n h((A/I^n), X) \simeq \to \varinjlim_n (0 :_X I^n).$

By Example 4.3 and $(0:_X I^n) \subseteq (0:_X I^{n+1})$, $\lim_{n \to \infty} (0:_X I^n) = \Gamma_I(X)$ and so $\varinjlim_A h((A/I^n), X) \simeq \Gamma_I(X)$. Also it is easy to check that if $f: X \to Y$ is a morphism in \mathcal{C} , the following diagram is well-defined and commutative.

Thus $\varinjlim_n Ah((A/I^n), \cdot)$ is naturally equivalent to $\Gamma_I(\cdot)$. \Box

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