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A GENERALIZATION OF A CONJECTURE OF HARDY AND LITTLEWOOD TO ALGEBRAIC NUMBER FIELDS

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ABSTRACT. We generalize conjectures of Hardy and Littlewood concerning the density of twin primes and k-tuples of primes to arbitrary algebraic number fields.

In one of their great *Partitio Numerorum* papers [**7**], Hardy and Littlewood advance a number of conjectures involving the density of pairs and k-tuples of primes separated by fixed gaps. For example, if d is even, we define

$$P_d(x) = |\{0 < n < x : n, n+d \text{ are both prime}\}|.$$

They conjecture both that

$$\lim_{x\to\infty} \frac{P_d(x)}{P_2(x)} = \prod_{\text{odd } p|d} \frac{p-1}{p-2}$$

and that $P_2(x)$ is asymptotic to

$$2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\int_2^x \frac{dy}{(\log y)^2}.$$

We will refer to the first equation as the "relative conjecture" and the second as the "absolute conjecture."

There has been much numerical verification of these conjectures and many attempts at proofs. Balog [**1**] proves a result that implies that the conjectures are true "on average," where the average is taken over the possible shapes of the k-tuples. Golubev [**6**] compares these conjectures

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with provable analogous limit results for patterns of numbers prime to n. Turàn [**18**] relates such theorems to zeros of the ζ-function, using the large sieve rather than Hardy and Littlewood's circle method.

There are also many generalizations to specific fields. Most of those generalizations use "Conjecture H" of Sierpiński and Schinzel [**14, 15**]. For example, Sierpiński [**17**] shows that Conjecture H implies the existence of infinitely many prime Gaussian integers differing by two. Bateman and Horn [**2, 3**] quote a quantitative form of Conjecture H which allows them to estimate the density of rational twin primes. Shanks [**16**] numerically verifies that the density of prime pairs of the form $a+i$, $a+2+i$ in the Gaussian integers matches that of the quantitative form of Conjecture H. Rieger [**13**] proves an upper bound for the density of twin primes of certain forms, using sieve methods. Holben and Jordan [**10**] venture weaker analogs of these conjectures for Gaussian integers. Hensley [**9**] considers the distribution of primes in regions of quadratic number fields.

Jordan and Rabung [**11**] conjecture that certain patterns of Gaussian primes exist and in fact occur infinitely often. Our conjectures would imply this and give asymptotic estimates (difficult to compute in some cases) for how often. We defer a detailed discussion of how our conjectures would imply theirs and others until the end of this paper.

Our goal in this paper is to give a very strong analog of these conjectures for all number fields. We will first present the numerical verification for various types of number fields (imaginary quadratic class number one, real quadratic class number one, Galois cubic, non-Galois cubic, imaginary quadratic with larger class number) and then state a conjecture which is numerically plausible in all of these cases.

The heuristic motivating the first conjecture is relatively simple to understand. For those odd primes p which divide d, there are $p-1$ possible equivalence classes for $n \pmod p$ for which n and $n+d$ are both prime to p; in contrast, there are $p-2$ congruence classes for which n and $n+2$ are both prime to p. On the other hand, those primes which do not divide d should not play a different role in the two densities. If we assume that the events involved are independent, then the conjecture follows.

The heuristic naturally carries over to algebraic number fields. We consider first the imaginary quadratic number fields of class number one and define

$$P_d^{(D)}(x) = \Big|\Big\{a+b\sqrt{-D}\in\mathcal{O}_{\mathbf{Q}[\sqrt{-D}]} :\ a>0,\ b\geq 0,\ N(a+b\sqrt{-D})<x,\\ (a+b\sqrt{-D}),\ (a+b\sqrt{-D}+d) \text{ prime ideals}\Big\}\Big|$$

for $D \neq 3$. For that exceptional case, we instead consider $a+b\omega$, where $\omega = (1+\sqrt{-3})/2$. Note that for $D \neq 1, 3$, the conditions on a and b mean that the region contains half of the prime ideals of norm $< x$, while when $D = 1$ or 3, the region contains all such prime ideals.

By analogy with the case of $\mathbf{Z}$, for those fields in which there is at least one prime ideal of norm 2, we conjecture that

$$\lim_{x\to\infty} \frac{P_d^{(D)}(x)}{P_2^{(D)}(x)} = \prod_{\substack{\mathfrak{p}|d \\ N(\mathfrak{p})>2}} \frac{N(\mathfrak{p})-1}{N(\mathfrak{p})-2},$$

where the product is 1 if $N(d)$ is a power of 2. If there are no prime ideals of norm 2, then we conjecture that

$$\lim_{x\to\infty} \frac{P_d^{(D)}(x)}{P_1^{(D)}(x)} = \prod_{\mathfrak{p}|d} \frac{N(\mathfrak{p})-1}{N(\mathfrak{p})-2},$$

where the product is 1 if d is a unit.

We compute the value of the lefthand side of the relative conjecture for $x = 10^6$ for these number fields for various small values of d and compare the result with the product on the right in Tables 1–3.

To find the proper absolute conjecture, we must review the heuristic that leads to the conjecture for rational primes. Perhaps the simplest route is to follow the suggestion of Polya [**12**]. (See also Hardy and Wright [**8**] and Cherwell and Wright [**5**] for other forms of this heuristic.)

Fix some large integer x, and we consider the fraction of pairs of integers n and $n+2$ which are less than x and not divisible by a prime

TABLE 1. Relative conjecture for $\mathbf{Q}[i]$, $\mathbf{Q}[\sqrt{-2}]$, $\mathbf{Q}[\sqrt{-7}]$.

D	1		2		7	
d	Obs.	Theor.	Obs.	Theor.	Obs.	Theor.
4	1.00	1.00	1.02	1.00	1.01	1.00
6	1.15	1.14	4.07	4.00	1.16	1.14
8	1.01	1.00	1.03	1.00	1.02	1.00
10	1.78	1.78	1.07	1.04	1.06	1.04
12	1.14	1.14	4.08	4.00	1.15	1.14
14	1.02	1.02	1.04	1.02	1.22	1.20
16	1.00	1.00	1.03	1.00	1.01	1.00
18	1.14	1.14	4.07	4.00	1.16	1.14
20	1.77	1.78	1.07	1.04	1.05	1.04
22	1.00	1.01	1.25	1.23	1.25	1.23
24	1.14	1.14	4.09	4.00	1.15	1.14
26	1.19	1.19	1.01	1.01	1.01	1.01
28	1.02	1.02	1.04	1.02	1.21	1.20
30	2.02	2.03	4.25	4.17	1.20	1.19
40	1.77	1.78	1.07	1.04	1.05	1.04
50	1.77	1.78	1.03	1.04	1.05	1.04
60	2.01	2.03	4.21	4.17	1.18	1.19
70	1.79	1.82	1.08	1.07	1.24	1.25

p. If $p = 2$, then the fraction of such pairs is approximately $1/2$. If p is an odd prime, then the fraction of such pairs is approximately $(p-2)/p$. Thus, the density of pairs with no prime dividing n or $n+2$ is

$$\frac{1}{2} \prod_{\text{odd } p} \frac{p-2}{p}$$

where we deliberately are not specifying the range of the product, other than to take it over a set of odd primes all smaller than x. We now simplify that product:

$$\begin{aligned}\frac{1}{2}\prod_{\text{odd } p}\frac{p-2}{p} &= \frac{1}{2}\prod_{\text{odd } p}\frac{p(p-2)}{(p-1)^2}\prod_{\text{odd } p}\left(\frac{p-1}{p}\right)^2 \\ &= 2\prod_{\text{odd } p}\frac{p(p-2)}{(p-1)^2}\prod\left(\frac{p-1}{p}\right)^2.\end{aligned}$$

The prime number theorem can be interpreted as saying that the density of primes less than x is approximately $(1/\log x)$. Thus, one might hope to find a factor of $1/(\log x)^2$ as part of this computation, and, indeed, the second product is asymptotic to $(\log x)^{-2}$ when taken over an appropriate range of primes (Polya points out that the appropriate range is neither $p < x$ nor $p < \sqrt{x}$, but rather $p < x^{e^{-\gamma}} \approx x^{0.5615}$, where $\gamma = 0.577\ldots$ is Euler's constant; this is a consequence of Merten's theorem), while the first product is the convergent one

$$\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

Thus, the number of prime pairs less than x can be approximated by

$$2\prod_{p\geq 2}\left(1-\frac{1}{(p-1)^2}\right)\int_2^x \frac{dy}{(\log y)^2}.$$

Let us attempt to generalize this to algebraic number fields, beginning with $\mathbf{Q}[i]$. The probability that both $a+bi$ and $a+bi+2$ are divisible by $1+i$, the prime above the rational prime 2, is again $1/2$. We consider a similar product and simplify similarly:

$$\begin{aligned}\frac{1}{2}\prod_{\mathfrak{p}\neq(1+i)}\frac{N(\mathfrak{p})-2}{N(\mathfrak{p})} &= \frac{1}{2}\prod_{\mathfrak{p}\neq(1+i)}\frac{N(\mathfrak{p})(N(\mathfrak{p})-2)}{(N(\mathfrak{p})-1)^2}\prod_{\mathfrak{p}\neq(1+i)}\left(\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})}\right)^2 \\ &= 2\prod_{\mathfrak{p}\neq(1+i)}\frac{N(\mathfrak{p})(N(\mathfrak{p})-2)}{(N(\mathfrak{p})-1)^2}\prod_{\mathfrak{p}}\left(\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})}\right)^2.\end{aligned}$$

TABLE 2. Relative conjecture for $\mathbf{Q}[\sqrt{-3}]$.

d	Obs.	Theor.	12	3.01	3.00	24	2.97	3.00
1	–	–	13	1.18	1.19	25	1.03	1.04
2	1.49	1.50	14	2.15	2.16	26	1.77	1.79
3	1.99	2.00	15	2.08	2.09	27	1.98	2.00
4	1.50	1.50	16	1.49	1.50	28	2.13	2.16
5	1.04	1.04	17	0.99	1.00	29	0.98	1.00
6	3.01	3.00	18	2.99	3.00	30	3.11	3.13
7	1.45	1.44	19	1.11	1.12	31	1.06	1.07
8	1.50	1.50	20	1.55	1.57	32	1.47	1.50
9	2.00	2.00	21	2.85	2.88	33	2.00	2.03
10	1.56	1.57	22	1.50	1.51	34	1.49	1.51
11	1.00	1.01	23	1.00	1.00	35	1.49	1.50

TABLE 3. Relative conjecture for $\mathbf{Q}[\sqrt{-11}]$, $\mathbf{Q}[\sqrt{-19}]$, $\mathbf{Q}[\sqrt{-43}]$, $\mathbf{Q}[\sqrt{-67}]$, $\mathbf{Q}[\sqrt{-163}]$.

D	11		19		43		67		163	
d	Ob.	Th.	Ob.	Th.	Ob.	Th.	Ob.	Th.	Ob.	Th.
2	1.49	1.50	1.52	1.50	1.50	1.50	1.48	1.50	1.50	1.50
3	3.96	4.00	1.16	1.14	1.14	1.14	1.13	1.14	1.14	1.14
4	1.49	1.50	1.53	1.50	1.50	1.50	1.49	1.50	1.51	1.50
5	1.80	1.78	1.83	1.78	1.05	1.04	1.04	1.04	1.06	1.04
6	5.98	6.00	1.75	1.71	1.69	1.71	1.69	1.71	1.71	1.71
7	1.03	1.02	1.43	1.44	1.04	1.02	1.00	1.02	1.02	1.02
8	1.46	1.50	1.51	1.50	1.51	1.50	1.47	1.50	1.50	1.50
9	3.97	4.00	1.16	1.14	1.14	1.14	1.12	1.14	1.15	1.14
10	2.66	2.67	2.68	2.67	1.56	1.57	1.55	1.57	1.56	1.57

If we set $\chi(p) = -1$ if p is inert in $\mathbf{Q}[i]$, and $\chi(p) = 1$ if p factors in $\mathbf{Q}[i]$, and $\chi(2) = 0$, then we can factor this last product:

$$\prod_{\mathfrak{p}} \left(\frac{N(\mathfrak{p}) - 1}{N(\mathfrak{p})} \right)^2 = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})} \right)^2 = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(p)}{p} \right)^2 \prod_{p} \left(1 - \frac{1}{p} \right)^2.$$

The final product, over an appropriate range of primes p as above, again can be approximated by $(\log N(a+bi))^{-2}$, while the first product converges to $L(1,\chi)^{-2}$. If we now set

$$\mathcal{R} = \{a + bi : a > 0,\ b \geq 0,\ a, b \in \mathbf{Z},\ 2 \leq N(a + bi) \leq x\},$$

then we expect that

$$P_2^{(1)}(x) \approx 2 \prod_{\mathfrak{p} \neq (1+i)} \left(1 - \frac{1}{(N(\mathfrak{p}) - 1)^2} \right) L(1,\chi)^{-2} \cdot \sum_{a+bi \in \mathcal{R}} \frac{1}{(\log(N(a + bi)))^2}.$$

Though the above approximation is numerically verifiable, we can speed computation by simplifying the final term. Since the same simplification will work for all of the imaginary quadratic fields we considered above, we do the computation in general.

We begin by approximating

$$\sum_{a+b\sqrt{-D} \in \mathcal{R}} \frac{1}{(\log N(a + b\sqrt{-D}))^2} \approx \int_{\mathcal{S}} \frac{dA}{(\log N(a + b\sqrt{-D}))^2} = \int_{\mathcal{S}} \frac{da\, db}{(\log(a^2 + Db^2))^2}$$

where

$$\mathcal{S} = \{a + b\sqrt{-D} : a \geq 0,\ b \geq 0,\ 2 \leq a^2 + Db^2 \leq x\}.$$

Now an appropriate change of variables transforms the last integral into

$$\frac{\pi}{4\sqrt{D}} \int_2^x \frac{dy}{(\log y)^2}.$$

Miraculously, the residue formula for the ζ-function of a number field means that the constant in front of the integral exactly cancels one factor of $L(1,\chi)$, sometimes with an extra factor of $1/2$ when the region $\mathcal{R}$ contains only half of the primes of the region $\{a+b\sqrt{-D} \mid N(a+b\sqrt{-D} \leq x\}$.

Therefore, our final approximation is

$$P_2^{(D)}(x) \approx C_D \prod_{\mathfrak{p}\neq(1+i)} \left(1-\frac{1}{N(\mathfrak{p})-1}\right)^2 L(1,\chi)^{-1} \int_2^x \frac{dy}{(\log y)^2},$$

where C_D is either 1 or 2 depending on whether $\mathcal{R}$ contains all of the primes $\mathfrak{p}$ with $N(\mathfrak{p}) \leq x$ or only half.

If we define

$$R^{(D)}(x) = P_d^{(D)}(x) \Big/ \prod_{\mathfrak{p}\neq(1+i)} \left(1-\frac{1}{N(\mathfrak{p})-1}\right)^2 L(1,\chi)^{-1} \int_2^x \frac{dy}{(\log y)^2}$$

where $d=2$ if $D=1,2$ or 7 and $d=1$ otherwise, then Tables 4 and 5 show values of $R^{(D)}$ for various values of x. In these computations, we have used the approximation

$$\int_2^x \frac{dy}{(\log y)^2} \approx \frac{x}{(\log x)^2}\left(1+\frac{2}{\log x}+\frac{6}{(\log x)^2}+\frac{24}{(\log x)^3}\right),$$

which is accurate to within one part in a hundred. The expected value of $R^{(D)}$ can vary from 2 to 0.5 depending on the behavior of the rational prime 2 in the field and on the fraction of prime ideals considered in the definition of the function $P_d^{(D)}$.

TABLE 4. Absolute conjecture for $\mathbf{Q}[i]$, $\mathbf{Q}[\sqrt{-2}]$, $\mathbf{Q}[\sqrt{-3}]$, $\mathbf{Q}[\sqrt{-7}]$.

x	$R^{(1)}(x)$	$R^{(2)}(x)$	$R^{(3)}(x)$	$R^{(7)}(x)$
200000	1.99	0.94	0.99	1.98
400000	2.01	0.97	1.00	1.99
600000	2.00	0.97	1.00	2.00
800000	1.99	0.98	1.00	1.99
1000000	1.99	0.98	1.00	1.98
1200000	2.00	0.98	1.00	1.98

TABLE 5. Absolute conjecture for $\mathbf{Q}[\sqrt{-11}]$, $\mathbf{Q}[\sqrt{-19}]$, $\mathbf{Q}[\sqrt{-43}]$, $\mathbf{Q}[\sqrt{-67}]$, $\mathbf{Q}[\sqrt{-163}]$.

x	$R^{(11)}(x)$	$R^{(19)}(x)$	$R^{(43)}(x)$	$R^{(67)}(x)$	$R^{(163)}(x)$
200000	0.49	0.50	0.50	0.50	0.51
400000	0.48	0.50	0.50	0.50	0.51
600000	0.50	0.50	0.50	0.50	0.51
800000	0.50	0.50	0.50	0.50	0.51
1000000	0.50	0.49	0.50	0.50	0.50
1200000	0.50	0.50	0.50	0.50	0.50

Hardy and Littlewood advance similar conjectures for triples and more generally, k-tuples, of primes. For example, if we let

$$P_{a,b}(x) = |\{0 < n < x : n,\ n+a,\ n+a+b \text{ all prime}\}|,$$

then they conjecture

$$\lim_{x\to\infty} \frac{P_{a,b}(x)}{P_{b,a}(x)} = 1.$$

They also advance a conjecture for the relative density of $P_{a,b}$ compared to $P_{2,4}$:

$$\lim_{x\to\infty} \frac{P_{a,b}(x)}{P_{2,4}(x)} = \varepsilon(a,b) \prod_{p||ab(a+b)} \frac{p-2}{p-3} \prod_{p|a,p|b} \frac{p-1}{p-3}.$$

The first two products are over primes larger than 3, the notation $p||x$ means that $p|x$ but $p^2 \nmid x$, and the factor $\varepsilon(a,b)$ is one unless both a and b are multiples of 3, in which case it is 2. We also must have $3|ab(a+b)$ and a and b both even, or else $P_{a,b}$ will be trivial.

We can easily try to generalize these conjectures to $\mathbf{Q}[i]$, where 3 no longer needs special treatment (because the rational prime 3 is inert in this field):

$$\lim_{x\to\infty} \frac{P^{(1)}_{a,b}(x)}{P^{(1)}_{2,2}(x)} = \prod_{\mathfrak{p}||ab(a+b)} \frac{N(\mathfrak{p})-2}{N(\mathfrak{p})-3} \prod_{\mathfrak{p}|a,\mathfrak{p}|b} \frac{N(\mathfrak{p})-1}{N(\mathfrak{p})-3}$$

where both products are over primes $\mathfrak{p}$ with odd norms.

We can compute these ratios for various small values of a and b with $x = 10^6$ and compare the results with what the conjecture predicts. In each pair of columns of Table 6, the first number is the observed ratio and the second number is the conjectured one.

Similarly, we can generalize the conjecture with no difficulty to $\mathbf{Q}[\sqrt{-19}]$, say, because neither 2 nor 3 splits in that field; we can conjecture that

$$\lim_{x\to\infty} \frac{P^{(1)}_{a,b}(x)}{P^{(1)}_{1,1}(x)} = \varepsilon(a,b) \prod_{\mathfrak{p}||ab(a+b)} \frac{N(\mathfrak{p})-2}{N(\mathfrak{p})-3} \prod_{\mathfrak{p}|a,\mathfrak{p}|b} \frac{N(\mathfrak{p})-1}{N(\mathfrak{p})-3},$$

where the products are over all prime ideals $\mathfrak{p}$ with odd norm and $\varepsilon(a,b)$ is 1 unless both a and b are multiples of 2, in which case it is $3/2$.

TABLE 6. Relative prime triple conjecture for $\mathbf{Q}[i]$.

$a\backslash b$	2		4		6		8		10		12	
2	1.00	1.00	1.14	1.17	1.17	1.17	2.23	2.25	2.59	2.63	1.16	1.19
4	1.14	1.17	0.99	1.00	2.60	2.63	1.16	1.17	2.29	2.30	1.17	1.17
6	1.14	1.17	2.60	2.63	1.30	1.36	1.18	1.19	2.60	2.63	1.32	1.33
8	2.26	2.25	1.12	1.17	1.19	1.19	0.99	1.00	2.59	2.63	2.58	2.63
10	2.56	2.63	2.25	2.30	2.58	2.63	2.61	2.63	3.90	4.00	2.59	2.65
12	1.17	1.19	1.14	1.17	1.31	1.33	2.59	2.63	2.61	2.65	1.29	1.33

The results of the computations for small values of a and b when $x = 10^6$ are in Table 7.

TABLE 7. Relative prime triple conjecture for $\mathbf{Q}[\sqrt{-19}]$.

$a\backslash b$	1		2		3		4		5		6	
1	1.00	1.00	1.22	1.17	1.26	1.17	2.47	2.25	2.63	2.63	1.76	1.82
2	1.22	1.17	1.50	1.50	2.72	2.63	1.88	1.75	3.54	3.52	1.68	1.75
3	1.25	1.17	2.75	2.63	1.33	1.36	1.70	1.82	2.80	2.63	1.35	1.33
4	2.27	2.25	1.79	1.75	1.84	1.82	1.54	1.50	2.77	2.63	4.07	3.94
5	2.65	2.63	3.50	3.52	2.74	2.63	2.67	2.63	4.17	4.00	3.44	3.32
6	1.82	1.82	1.75	1.75	1.35	1.33	4.00	3.94	3.36	3.32	2.08	2.00

Hardy and Littlewood also offer a conjecture about the size of $P_{2,4}(x)$:

$$P_{2,4}(x) \approx \frac{9}{2} \prod_{p>3} \left(\frac{p}{p-1}\right)^2 \left(\frac{p-3}{p-1}\right) \int_2^x \frac{dy}{(\log y)^3}.$$

We can compute an analogous estimate for $\mathbf{Q}[i]$ for example. Because 3 is inert in this field, we estimate $P_{2,2}^{(1)}(x)$. The probability that $a+bi$, $a+bi+2$ and $a+bi+4$ are all not divisible by $1+i$ is $1/2$; for the remaining primes, we simply have $(N(\mathfrak{p})-3)/N(\mathfrak{p})$. We therefore consider

$$\begin{aligned}
\frac{1}{2} \prod_{N(\mathfrak{p})\neq 2} \frac{N(\mathfrak{p})-3}{N(\mathfrak{p})} &= \frac{1}{2} \prod_{N(\mathfrak{p})\neq 2} \left(\frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\right)^2 \left(\frac{N(\mathfrak{p})-3}{N(\mathfrak{p})-1}\right) \\
&\quad \cdot \prod_{N(\mathfrak{p})\neq 2} \left(\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})}\right)^3 \\
&= 4 \prod_{N(\mathfrak{p})\neq 2} \left(\frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\right)^2 \left(\frac{N(\mathfrak{p})-3}{N(\mathfrak{p})-1}\right) \\
&\quad \cdot \prod \left(\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})}\right)^3.
\end{aligned}$$

As before, we can break the last product up into a product of $L(1,\chi)^{-1}$ and a product which is asymptotic to $1/\log N(a+bi)$, so we conjecture that

$$P_{2,2}^{(1)}(x) \approx 4 \prod_{N(\mathfrak{p})\neq 2} \left(\frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\right)^2 \left(\frac{N(\mathfrak{p})-3}{N(\mathfrak{p})-1}\right) L(1,\chi)^{-2} \int_2^x \frac{dy}{(\log y)^3},$$

after we perform the same change of variables in the integral as before.

If we define

$$S^{(1)}(x) = P_{2,2}^{(1)}(x) \Big/ \prod_{N(\mathfrak{p})\neq 2} \left(\frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\right)^2 \left(\frac{N(\mathfrak{p})-3}{N(\mathfrak{p})-1}\right) L(1,\chi)^{-2} \int_2^x \frac{dy}{(\log y)^3},$$

then we can compute $S^{(1)}(x)$ for various values of x, making the approximation

$$\int_2^x \frac{dy}{(\log y)^3} \approx \frac{x}{(\log x)^3}\left(1 + \frac{3}{\log x} + \frac{12}{(\log x)^2} + \frac{60}{(\log x)^3} + \frac{360}{(\log x)^4}\right),$$

which is accurate to one part in 50. The results are in Table 8.

We can repeat the computation for any of the fields. The expected answer differs depending on the properties of the rational primes 2 and 3 in the fields. For simplicity, we present only the case in which 2 and 3 are both inert; we can then approximate $P_{1,1}^{(D)}(x)$. The prime 2 contributes a factor of $1/2$, while every other prime contributes a factor of $(N(\mathfrak{p})-3)/N(\mathfrak{p})$. We have

$$\frac{1}{2}\prod_{N(\mathfrak{p})>4}\frac{N(\mathfrak{p})-3}{N(\mathfrak{p})} = 2\prod\left(\frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\right)^2\left(\frac{N(\mathfrak{p})-3}{N(\mathfrak{p})-1}\right)\prod\left(\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})}\right)^3.$$

Therefore, for $D = 19$, 43, 67 and 163, we expect

$$S^{(D)}(x) = P_{1,1}^{(D)}(x)\Big/\prod\left(\frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\right)^2\left(\frac{N(\mathfrak{p})-3}{N(\mathfrak{p})-1}\right)L(1,\chi)^{-2} \cdot \int_2^x \frac{dy}{(\log y)^3} \approx 1,$$

because we have defined $P_{1,1}^{(D)}$ to include only half of the primes in the field.

The computed data are in Table 8.

TABLE 8. Absolute conjecture for prime triples.

x	$S^{(1)}(x)$	$S^{(19)}(x)$	$S^{(43)}(x)$	$S^{(67)}(x)$	$S^{(163)}(x)$
200000	3.89	1.09	0.99	0.99	1.06
400000	3.92	1.06	1.00	0.98	1.04
600000	4.05	0.98	0.97	0.97	1.03
800000	4.05	0.95	0.98	1.01	1.03
1000000	4.00	0.98	0.97	1.01	1.02
1200000	4.03	0.98	0.97	1.01	1.02

We turn next to other fields. If K is a real quadratic field with class number 1 and $\mathcal{R}$ is a set of algebraic integers in K, we can define

$$P_d(K,\mathcal{R}) = \{r \in \mathcal{R} : (r), (r+d) \text{ both prime ideals}\}.$$

If $\mathcal{R}$ is "large" in some appropriate sense and there is a prime ideal $\mathfrak{p} \subset K$ with $N(\mathfrak{p}) = 2$, then we can conjecture that

$$\frac{P_d(K,\mathcal{R})}{P_2(K,\mathcal{R})} \approx \prod_{\substack{\mathfrak{p}|d \\ N(\mathfrak{p})\neq 2}} \frac{N(\mathfrak{p})-1}{N(\mathfrak{p})-2}.$$

If there is no prime ideal in K with norm 2, then

$$\frac{P_d(K,\mathcal{R})}{P_1(K,\mathcal{R})} \approx \prod_{\mathfrak{p}|d} \frac{N(\mathfrak{p})-1}{N(\mathfrak{p})-2}.$$

The numerical evidence supporting this relative conjecture is very strong in the fields $\mathbf{Q}[\sqrt{2}]$, $\mathbf{Q}[\sqrt{3}]$, $\mathbf{Q}[\sqrt{5}]$ and $\mathbf{Q}[\sqrt{6}]$.

In addition, when the rational prime 2 ramifies or splits, we conjecture that

$$P_2(K,\mathcal{R}) \approx 2 \prod_{N(\mathfrak{p})\neq 2} \left(1 - \frac{1}{(N(\mathfrak{p})-1)^2}\right) L(1,\chi)^{-2} \sum_{\substack{\alpha\in\mathcal{R} \\ |N(\alpha)|\neq 0,1}} \frac{1}{(\log |N(\alpha)|)^2}.$$

If 2 is inert, we conjecture that

$$P_1(K,\mathcal{R}) \approx \prod_{\mathfrak{p}} \left(1 - \frac{1}{(N(\mathfrak{p})-1)^2}\right) L(1,\chi)^{-2} \sum_{\substack{\alpha\in\mathcal{R} \\ |N(\alpha)|\neq 0,1}} \frac{1}{(\log |N(\alpha)|)^2}.$$

The numerical evidence for this absolute conjecture is also strong. For example, if we define

$$\mathcal{R}(x) = \{a + b\sqrt{d} : 0 \le a \le x,\ 0 \le b \le x\}$$

for $d = 2, 3, 6$ and

$$\begin{aligned} &R(\mathbf{Q}[\sqrt{d}], x) \\ &\quad = P_2(\mathbf{Q}[\sqrt{d}], \mathcal{R}(x)) \Big/ \prod_{N(\mathfrak{p})\neq 2} \left(1 - \frac{1}{(N(\mathfrak{p})-1)^2}\right) L(1,\chi)^{-2} \\ &\qquad\qquad \cdot \sum_{\substack{\alpha\in\mathcal{R}(x) \\ |N(\alpha)|\neq 0,1}} \frac{1}{(\log |N(\alpha)|)^2}, \end{aligned}$$

we summarize the supporting data in Table 9.

TABLE 9. Absolute conjecture for $\mathbf{Q}[\sqrt{2}]$, $\mathbf{Q}[\sqrt{3}]$, $\mathbf{Q}[\sqrt{6}]$.

x	$R(\mathbf{Q}[\sqrt{2}],x)$	$R(\mathbf{Q}[\sqrt{3}],x)$	$R(\mathbf{Q}[\sqrt{6}],x)$
100	1.72	1.66	1.69
200	1.84	1.86	1.89
300	1.91	1.90	1.96
400	1.94	1.90	1.92
500	1.94	1.95	1.93
600	1.95	1.95	1.97

Next we consider a cubic Galois field. If ρ is a root of $x^3-7x+7=0$, the field $\mathbf{Q}[\rho]$ is a Galois of degree 3. There is a congruence condition for which rational primes remain prime and which split, simplifying computations. Again, we conjecture

$$\frac{P_d(\mathbf{Q}[\rho],\mathcal{R})}{P_1(\mathbf{Q}[\rho],\mathcal{R})} \approx \prod_{\mathfrak{p}|d} \frac{N(\mathfrak{p})-1}{N(\mathfrak{p})-2}.$$

We let

$$\mathcal{R} = \{a\rho^2+b\rho+c : 0 \le a,b,c \le 30\},$$

and present numerical evidence for this relative conjecture in Table 10. These results are not as strong as the ones shown earlier, presumably because the region $\mathcal{R}$ is quite small, because of computational complexity, compared to some of the regions considered earlier.

If χ_1 and χ_2 are the two nontrivial cubic characters, then the approximation

$$P_1(\mathbf{Q}[\rho],\mathcal{R}) \approx \prod_{\mathfrak{p}} \left(1-\frac{1}{(N(\mathfrak{p})-1)^2}\right) L(1,\chi_1)^{-2} L(1,\chi_2)^{-2} \cdot \sum_{\substack{\alpha\in\mathcal{R} \\ |N(\alpha)|\ne 0,1}} \frac{1}{(\log|N(\alpha)|)^2}$$

TABLE 10. Relative conjecture for $\mathbf{Q}[\rho]$.

d	Obs.	Theor.	d	Obs.	Theor.	d	Obs.	Theor.
			11	1.00	1.00	21	1.13	1.25
2	1.18	1.17	12	1.17	1.21	22	1.10	1.17
3	1.05	1.04	13	1.21	1.30	23	0.96	1.00
4	1.19	1.17	14	1.34	1.40	24	1.16	1.21
5	0.97	1.01	15	1.03	1.05	25	0.95	1.01
6	1.16	1.21	16	1.13	1.17	26	1.37	1.51
7	1.15	1.20	17	0.96	1.00	27	0.93	1.04
8	1.16	1.17	18	1.18	1.21	28	1.28	1.40
9	1.02	1.04	19	0.91	1.00	29	1.02	1.12
10	1.16	1.18	20	1.09	1.18	30	1.14	1.22

can also be checked. Using the region $\mathcal{R}$ as above, the ratio

$$P_1(\mathbf{Q}[\rho],\mathcal{R}) \Big/ \prod_{\mathfrak{p}} \left(1 - \frac{1}{(N(\mathfrak{p})-1)^2}\right) L(1,\chi_1)^{-2} L(1,\chi_2)^{-2} \cdot \sum_{\substack{\alpha\in\mathcal{R} \\ |N(\alpha)|\neq 0,1}} \frac{1}{(\log|N(\alpha)|)^2}$$

is approximately 0.91. Again, this evidence is not as strong as for the quadratic fields, presumably because the region $\mathcal{R}$ is relatively small.

The field $\mathbf{Q}[\sqrt[3]{2}]$ is not Galois, but we still can use the heuristic given above, factoring the product as before. We can always factor the Euler product for $\zeta_K(s)$ as a product over rational primes $\prod(1-(1/p^s))^{-1}$, which of course is the ordinary ζ-function, and another product, which is not an L-series, but which nevertheless converges at $s=1$. Letting s tend to 1 and computing residues, we have that the residue of $\zeta_K(s)$ at $s=1$ precisely equals the remaining term (because the residue of the ordinary ζ-function is 1). In our case we are actually manipulating the reciprocal of $\zeta_K(s)$, so we end up with the reciprocal of the residue in our estimate.

The numerical evidence for both the relative and the absolute conjectures is very convincing. Set

$$\mathcal{R} = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : 0 \le a, b, c \le 75\},$$

and we can compute $P_d(\mathbf{Q}[\sqrt[3]{2}], \mathcal{R})/P_2(\mathbf{Q}[\sqrt[3]{2}], \mathcal{R})$ for various values of d. The results are in Table 11.

TABLE 11. Relative conjecture for $\mathbf{Q}[\sqrt[3]{2}]$.

d	Obs.	Theor.	6	2.00	2.00	14	1.00	1.00
$\sqrt[3]{2}$	1.01	1.00	8	0.99	1.00	16	0.99	1.00
$\sqrt[3]{4}$	1.00	1.00	10	1.38	1.39	18	1.96	2.00
4	1.00	1.00	12	1.97	2.00	20	1.37	1.39

The absolute conjecture can also be checked for this region $\mathcal{R}$. The ratio

$$P_2(\mathbf{Q}[\sqrt[3]{2}], \mathcal{R}) \Big/ \prod_{\mathfrak{p}} \left(1 - \frac{1}{(N(\mathfrak{p})-1)^2}\right) \left(\operatorname*{Res}_{s=1} \zeta_{\mathbf{Q}[\sqrt[3]{2}]}(s)\right)^{-2} \cdot \sum_{\substack{\alpha \in \mathcal{R} \\ |N(\alpha)| \neq 0,1}} \frac{1}{(\log |N(\alpha)|)^2}$$

is 1.97, while conjecturally it should be 2.

Finally we consider fields with class number larger than 1. The fields $\mathbf{Q}[\sqrt{-d}]$, when $d = 5$, 6, 14, 15, 21, 23 and 30, have class numbers 2, 2, 4, 3, 4, 3 and 4, respectively. In each of these fields, 2 ramifies or splits. We can consider the regions

$$\mathcal{R}(x, d) = \{\alpha \in \mathbf{Q}[\sqrt{-d}] : \alpha \text{ is an algebraic integer in the first quadrant, } 1 < N(\alpha) < x\},$$

and compute

$$R(x, d) = P_2(\mathbf{Q}[\sqrt{-d}], \mathcal{R}(x, d)) \Big/ \prod_{N(\mathfrak{p}) \neq 2} \left(1 - \frac{1}{(N(\mathfrak{p})-1)^2}\right) L(1, \chi)^{-2} \cdot \sum_{\alpha \in \mathcal{R}(x)} \frac{1}{(\log |N(\alpha)|)^2},$$

for the above values of d and a range of x. The results are in Table 12.

TABLE 12. Absolute conjecture for $\mathbf{Q}[\sqrt{5}]$, $\mathbf{Q}[\sqrt{6}]$, $\mathbf{Q}[\sqrt{14}]$, $\mathbf{Q}[\sqrt{15}]$, $\mathbf{Q}[\sqrt{21}]$, $\mathbf{Q}[\sqrt{23}]$, $\mathbf{Q}[\sqrt{30}]$.

x	$R(5,x)$	$R(6,x)$	$R(14,x)$	$R(15,x)$	$R(21,x)$	$R(23,x)$	$R(30,x)$
200000	1.87	1.81	2.08	3.90	1.99	4.06	2.01
400000	1.93	1.82	2.08	3.92	2.07	4.06	2.03
600000	1.95	1.89	2.01	3.93	2.08	4.04	2.00
800000	1.96	1.91	2.02	3.93	2.10	4.03	2.01
1000000	1.99	1.93	2.04	3.92	2.09	3.99	1.97
1200000	1.99	1.96	2.06	3.93	2.07	4.02	1.97

The evidence is as convincing as it was for class number 1. We do not present any data supporting the relative conjecture, but they are equally strong.

In order to make the broadest possible conjecture, we need to define the type of sets of algebraic integers over which we would like to take limits. One possibility is the following:

Definition. Let $\mathcal{O}_K$ be represented as $\mathbf{Z}^n$ with respect to some basis. Suppose that $\mathcal{S} \subset \mathbf{R}^n$ is bounded, with nonempty interior and is a finite union or difference of convex sets (these properties are independent of the choice of basis). Let c_i be a set of real numbers tending to ∞. We then call the sequence of sets $\mathcal{R}_i = c_i\mathcal{S} \cap \mathcal{O}_K$ *reasonable.*

The above examples then suggest the following generalizations of Hardy and Littlewood's conjectures. We first state a conjecture for pairs of primes.

Conjecture. *Let K be a number field. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the complete list of prime ideals of K with norm 2, and let α be an element of $\mathcal{O}_K$ divisible by each $\mathfrak{p}_i$ and only by those ideals. (If there are no prime ideals of K with norm 2, let $\alpha = 1$.) Let $\mathcal{R}$ be a reasonable large subset* *of $\mathcal{O}_K$. Then*

$$\frac{P_d(K,\mathcal{R})}{P_\alpha(K,\mathcal{R})} \approx \prod_{\substack{\mathfrak{p}|d \\ N(\mathfrak{p})\neq 2}} \frac{N(\mathfrak{p})-1}{N(\mathfrak{p})-2}$$

and

$$P_\alpha(K,\mathcal{R}) \approx 2^r \prod_{\substack{\mathfrak{p} \\ N(\mathfrak{p})\neq 2}} \left(1-\frac{1}{(N(\mathfrak{p})-1)^2}\right)\left(\operatorname*{Res}_{s=1}\zeta_K(s)\right)^{-2} \cdot \sum_{\substack{\beta\in\mathcal{R} \\ |N(\beta)|\neq 0,1}} \frac{1}{(\log|N(\beta)|)^2}$$

where the interpretation of the symbol $\approx$ is that for a "reasonable" sequence of increasing regions $\mathcal{R}_i$, the ratio of the two sides tends to 1.

We can similarly extend the conjectures to arbitrary k-tuples. The following statement includes the previous conjecture as a special case and also combines both elements of the Hardy-Littlewood conjectures into one statement.

Conjecture. *Let K be a number field. Let $d_1, d_2, \dots, d_k \in \mathcal{O}_K$. For each prime ideal $\mathfrak{p} \subset \mathcal{O}_K$, let $k_\mathfrak{p}$ be the number of distinct residue classes of $d_1, \dots, d_k$ $\pmod{\mathfrak{p}}$. (Note that for all but finitely many ideals $\mathfrak{p}$, $k_\mathfrak{p} = k$.) Let R be reasonable, as above. Let $\pi(\mathcal{R}, d_1, d_2, \dots, d_k)$ be the number of elements $x \in \mathcal{R}$ so that $(x+d_1), (x+d_2), \dots, (x+d_k)$ are all prime ideals. Then*

$$\begin{aligned}\pi(\mathcal{R}, d_1, d_2, \dots, d_k) \approx & \prod_{N(\mathfrak{p})>k}\left(\frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\right)^{k-1}\left(\frac{N(\mathfrak{p})-k}{N(\mathfrak{p})-1}\right) \\ & \prod_{N(\mathfrak{p})\leq k}\left(\frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\right)^{k-1}\left(\frac{N(\mathfrak{p})-k_\mathfrak{p}}{N(\mathfrak{p})-1}\right) \\ & \prod_{\substack{N(\mathfrak{p})>k \\ k_\mathfrak{p}\neq k}}\left(\frac{N(\mathfrak{p})-k_\mathfrak{p}}{N(\mathfrak{p})-k}\right)\left(\operatorname*{Res}_{s=1}\zeta_K(s)\right)^{-k} \\ & \sum_{\substack{\alpha\in\mathcal{R} \\ |N(\alpha)|\neq 0,1}} \frac{1}{\log|N(\alpha)|^k}.\end{aligned}$$

Note that the first conjecture is an easy consequence of the second. For example, the factor of 2^r in the first conjecture results from simplification of

$$\prod_{N(\mathfrak{p})\le k}\left(\frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\right)^{k-1}\left(\frac{N(\mathfrak{p})-k_{\mathfrak{p}}}{N(\mathfrak{p})-1}\right)$$

after setting $k=2$ and $k_p=1$.

Remarks. There are many quantitative and qualitative conjectures regarding the distribution of primes in the literature. Many of these follow from our conjectures above.

In [**11**], Jordan and Rabung consider certain patterns of primes in $\mathbf{Z}[i]$. Any two positions differ by multiples of $1+i$, otherwise the positions would fill up a class $\pmod{1}+i$ forcing one of the positions to contain $1+i$ or an associate. Of the five connected patterns of four positions, where each pattern is taken to include its image under horizontal and/or vertical reflections, they observe from their data, "prime formations of types b, c, e seem significantly more numerous than those of forms a and d." If we break a type x up into reflection classes, e.g., $a=\{a_1,a_2\}$, then our relative conjecture would imply that there are $5/3$ as many of type b_i or c_i as a_i and $5/6$ as many of types d_i or e_i, where $d=\{d_1\}$, as a_i. Hence, b would be $10/3$ as dense as a, c and e $5/3$ as dense and d $5/12$ as dense.

In [**4**] Bergum makes similar conjectures concerning the field of cube roots of 1, in particular, that there are infinitely many n-tuples of "consecutive" primes for $2\le n\le 5$ (consecutive meaning "can be ordered so that each differs from its successor by a unit") and only finitely many such n-tuples for $n\ge 6$. Of the possible connected configurations of six positions, only the hexagon fails to cover some set of classes $\pmod{2}$ or $\pmod{2+\omega}$, where $\omega=e^{2\pi i/3}$, hence any other 6-tuple and any larger tuple occur only finitely often (must in fact contain one of these integers or an associate). On the other hand, the absolute conjecture would imply that there are infinitely many hexagons of primes and infinitely many k-tuples for $k<6$, though the conjectured density makes hexagons fairly sparse.

The computations in this paper were done using the programming language C, with 32-bit integers. The computations presented no major computational problems and used facts about algebraic number theory which can be found in any standard text.

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