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SUPPLENESS OF SOME QUOTIENT SPACES OF MICROFUNCTIONS

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ABSTRACT. The theory of microfunctions which Sato introduced contributed very much to that of linear partial differential equations. He constructed a useful transformation of these equations by expanding the analytic singularities of a hyperfunction onto the cosphere bundle. We stick not only to the analytic singularities but also to some other singularities of some subclasses of hyperfunctions which Komatsu [**7**], Hörmander [**4**], Eida [**2**] and others have introduced. For this reason, in this paper we develop the theory of some quotient spaces of microfunctions. We state the suppleness of the sheaves of these functions which is important for the theory of microdifferential equations. Our investigations enable us to introduce the notions SS^* and $SS^{1/*}$ for hyperfunctions.

1. Notations and definitions. We consider some classes of hyperfunctions, that is, ultradifferentiable functions and ultradistributions. We will recall the basic definitions [**2, 6**]. Let M_p be a sequence of positive numbers satisfying the following conditions.

$$\text{(M.0)} \qquad M_0 = M_1 = 1;$$

$$\text{(M.1)} \qquad M_p^2 \le M_{p-1}M_{p+1}, \quad p = 1, 2, \dots ;$$

$$\text{(M.2)} \qquad M_p/M_qM_{p-q} \le AB^p, \quad 0 \le q \le p;$$

$$\text{(M.3)}' \qquad \sum_{p=1}^{\infty} M_{p-1}/M_p < \infty.$$

We note that the Gevrey sequence $M_p = (p!)^s$ or p^{ps} or $\Gamma(1+ps)$, for $s > 1$, satisfies the above conditions.

A function $f(x)$ on an open set U in $\mathbf{R}^n$ is called an ultradifferentiable function of class (M_p) (respectively $\{M_p\}$), if for any compact set K

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in U and for any $h > 0$ there is $C > 0$ (respectively if for any compact set K in U there are $h > 0$ and $C > 0$) such that

$$\sup_{x \in K} |D^\alpha f(x)| \leq Ch^{|\alpha|} M_{|\alpha|},$$

where α is any n-tuple of nonnegative integers. We denote by $Df^{(M_p)}(U)$ (respectively $Df^{\{M_p\}}(U)$) the space of all ultradifferentiable functions of class (M_p) (respectively $\{M_p\}$) on U. From now on, $*$ will stand for both (M_p) and $\{M_p\}$.

We denote by $Df_c^*(U)$ the space of ultradifferentiable functions of class $*$ with compact support in U. Refer to Komatsu [**6**] and others for its topology. The space $Db^*(U)$ of ultradistributions of class $*$ is defined as the strong dual space of $Df_c^*(U)$.

Note that the presheaves $U \to Df^*(U)$, $U \to Db^*(U)$, $U \subset \mathbf{R}^n$, form sheaves. These sheaves are known to be soft, that is, if K is a closed set in an open set U in $\mathbf{R}^n$ and if f is a section on a neighborhood of K in U, then there is an extension $\tilde{f}$ on U.

$\mathcal{B}(U)$ denotes the space of Sato's hyperfunctions on an open set U in $\mathbf{R}^n$. We do not give a formal definition by using relative colomology groups. It is known that $U \to \mathcal{B}(U)$ forms a sheaf and that this sheaf is flabby, that is, the restriction mappings $S : \mathcal{B}(U) \to \mathcal{B}(U_1)$, $U_1 \subset U$, are always surjective. Refer to [**11**] for more details.

So far, all the above sheaves are defined on $\mathbf{R}^n$. However, they can be defined also on a real analytic manifold M.

There exist injections

$$\mathcal{D}f_M^* \hookrightarrow \mathcal{B}_M, \qquad \mathcal{D}b_M^* \hookrightarrow \mathcal{B}_M.$$

According to Kaneko [**5**], a hyperfunction f on U can be written as a formal sum of boundary values of holomorphic functions $F_j(z)$ defined on infinitesimal wedges $U + \sqrt{-1}\,\Gamma_j 0$, where Γ_j are open cones on $\mathbf{R}^n$:

$$f(x) = \sum_{j=1}^{N} F_j(x + \sqrt{-1}\,\Gamma_j 0).$$

For an open set U in a real analytic manifold M, we denote by S^*U the cosphere bundle of U. Let V be the complexification of U. Then

S_U^*V can be identified as $\sqrt{-1}\,S^*U$. From now on we are interested in the microlocal calculus on this space.

Let $f(x) \in \mathcal{B}(U)$. It is said that f is microanalytic at the point $(x, \sqrt{-1}\,\xi\infty) \in \sqrt{-1}\,S^*U$ if, for a suitable representation as above, of f on a neighborhood of x, $\Gamma_j \cap \{y \in \mathbf{R}^n; \langle \xi, y\rangle < 0\} \neq \varnothing$ holds for all $j \in \{1, \dots, N\}$.

The set of all points at which f is not microanalytic is called the singular spectrum of f, which we denote by $SS(f)$.

Let $\pi : \sqrt{-1}\,S^*M \to M$ be the canonical projection. The sheaf of microfunctions on $\sqrt{-1}\,S^*M$ is the associated sheaf of the presheaf:

$$\sqrt{-1}\,S^*M \supset \Omega \to \Gamma(\pi(\Omega); \mathcal{B}_M)/\{u \in \Gamma(\pi(\Omega); \mathcal{B}_M) \mid SS(u) \cap \Omega = \varnothing\}.$$

This sheaf is denoted by $\mathcal{C}_M$. It enjoys the exact sequence

$$0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{B}_M \longrightarrow \pi_*\mathcal{C}_M \longrightarrow 0,$$

which is due to Sato et al. [**11**]. Here $\mathcal{A}_M$ denotes the sheaf of real analytic functions on M. Moreover, there exists a canonical surjective spectrum map

$$Sp_M : \pi^{-1}\mathcal{B}_M \longrightarrow \mathcal{C}_M.$$

Then, for $u \in \mathcal{B}_M$, $SS(u) = \operatorname{supp}(Sp_M(u))$. The injection

$$\mathcal{D}b_M^* \hookrightarrow \mathcal{B}_M, \quad (\text{respectively } \mathcal{D}f_M^* \hookrightarrow \mathcal{B}_M),$$

induces a sheaf homomorphism

$$\pi^{-1}\mathcal{D}b_M^* \longrightarrow \mathcal{C}_M, \quad (\text{respectively } \pi^{-1}\mathcal{D}f_M^* \to \mathcal{C}_M).$$

We define a subsheaf $\mathcal{C}_M^*$ (respectively $\mathcal{C}_M^{d,*}$) of $\mathcal{C}_M$ as the image of the above morphism and call it the sheaf of microfunctions of class $*$ (respectively $d, *$). Furthermore, we have a canonical exact sequence

$$0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{D}b_M^* \longrightarrow \pi_*\mathcal{C}_M^* \longrightarrow 0$$
$$(\text{respectively } 0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{D}f_M^* \longrightarrow \pi_*\mathcal{C}_M^{d,*} \longrightarrow 0).$$

2. Quotient sheaves of microfunctions. We introduce an order to the set of sequences satisfying conditions (M.0), (M.1), (M.2) and

(M.3)$'$. This order will imply the corresponding inclusions of test function spaces.

(i) $(M_p) \leq (N_p)$, $\{M_p\} \leq \{N_p\}$ if there are constants L and C such that

$$M_p \leq CL^p N_p, \quad p = 0, 1, 2, \ldots$$

(ii) $\{M_p\} \leq (N_p)$ if for any $\varepsilon > 0$ there is a constant C_ε such that

$$M_p \leq C_\varepsilon \varepsilon^p N_p, \quad p = 0, 1, 2, \ldots$$

(iii) $(M_p) \leq \{M_p\}$.

We let $\dagger$ and $*$ denote $(N_p), \{N_p\}$ or $(M_p), \{M_p\}$ in the sequel and use the above orderings on such sequences. If $\dagger \leq *$, we have canonical injections

$$\mathcal{C}_M^{d,\dagger} \hookrightarrow \mathcal{C}_M^{d,*} \hookrightarrow \mathcal{C}_M^{*} \hookrightarrow \mathcal{C}_M^{\dagger} \hookrightarrow \mathcal{C}_M.$$

From now on $\mathcal{C}_M^1 = \mathcal{C}_M$ and, by definition, 1 corresponds to the sequence $\{p!\}$; (M.3)$'$ implies $1 \leq *$ for any $*$.

Definition 2.1. We define sheaves $\mathcal{C}_M^{\dagger,*}, \mathcal{C}_M^{\dagger/*}, \mathcal{C}_M^{d,\dagger,*}$ on $\sqrt{-1}\, S^*M$ by the following exact sequences:

(i) $0 \to \mathcal{C}_M^{d,*} \to \mathcal{C}_M^{\dagger} \to \mathcal{C}_M^{\dagger,*} \to 0$, when $\dagger \leq *$,

(ii) $0 \to \mathcal{C}_M^{*} \to \mathcal{C}_M^{\dagger} \to \mathcal{C}_M^{\dagger/*} \to 0$, when $\dagger \leq *$,

(iii) $0 \to \mathcal{C}_M^{d,*} \to \mathcal{C}_M^{d,\dagger} \to \mathcal{C}_M^{d,\dagger,*} \to 0$, when $* \leq \dagger$.

The canonical surjective spectrum map Sp_M induces the following surjective spectrum maps:

$$Sp_M^{1,*} : \pi^{-1}\mathcal{B}_M \longrightarrow \mathcal{C}_M^{1,*},$$
$$Sp_M^{1/*} : \pi^{-1}\mathcal{B}_M \longrightarrow \mathcal{C}_M^{1/*}.$$

Definition 2.2. Let $u \in \mathcal{B}_M$. We define the singular spectrum of class $*$, $SS^*(u)$ and that of class $1/*, SS^{1/*}(u)$, in the following way.

$$SS^*(u) = \operatorname{supp}(Sp_M^{1,*}(u)),$$
$$SS^{1/*}(u) = \operatorname{supp}(Sp_M^{1/*}(u)).$$

It is known that if $u \in \mathcal{D}b^*$, then $SS^*(u)$ is identical with $WF_*(u)$ defined by Hörmander [**4**].

3. Suppleness. Let $\mathcal{F}$ be a sheaf of Abelian groups on a topological space X. From [**1**], recall that $\mathcal{F}$ is supple if, for any open set Ω of X, any closed Z, Z_1, Z_2 of Ω which satisfy $Z = Z_1 \cup Z_2$ and any section $u \in \Gamma_Z(\Omega, \mathcal{F})$, there exists $u_i \in \Gamma_{Z_i}(\Omega, \mathcal{F})$, $i = 1, 2$, with $u = u_1 + u_2$.

We will use the following decomposition of δ, [**8, 9**],

$$\delta(x - x') = \frac{1}{(2\pi\sqrt{\pi}\,)^n} \int_{\mathbf{R}^n \times \mathbf{R}^n} (2|\xi|)^{n/2} \left\{ 1 + \frac{1}{2} \frac{\xi}{|\xi|}(x - x') \right\} \cdot e^{\sqrt{-1}\,\xi(x-x') + |\xi|\{(x-u)^2 + (x-x')^2\}}\, du\, d\xi.$$

First we prove a lemma which is stated in [**9**] with a sketched proof. In the sequel we identify $S^*\mathbf{R}^n$ with $\mathbf{R}^n \times (\sqrt{-1}\, S^{n-1})$.

Lemma 3.1. *Let Z be a closed subset of* $\mathbf{R}^n \times \sqrt{-1}\, S^{n-1}$,

$$\tilde{Z} = \left\{ (x, \xi) \in \mathbf{R}^n \times (\mathbf{R}^n \backslash \{0\});\ \left(x, \sqrt{-1}\, \frac{\xi}{|\xi|} \right) \in Z \right\}$$

and let

$$W_{\tilde{Z}}(x, x') = \frac{1}{(2\pi\sqrt{\pi}\,)^n} \int_{\tilde{Z}} (2|\xi|)^{n/2} \left\{ 1 + \frac{1}{2} \frac{\xi}{|\xi|}(x - x') \right\} \cdot e^{\sqrt{-1}\,\xi(x-x') + |\xi|\{(x-u)^2 + (x'-u)^2\}}\, du\, d\xi.$$

Then $W_{\tilde{Z}}$ is a distribution and

$$SS(W_{\tilde{Z}}) \subset \{(x, x', \sqrt{-1}\,(\xi, \xi'));\ x = x', \xi = -\xi', (x, \sqrt{-1}\,\xi) \in Z\}.$$

Proof. Denote by $\mathcal{Z}$ the set on the righthand side of the above inclusion. We will prove that the complement, in $\mathbf{R}^n \times \mathbf{R}^n \backslash \{0\}$, of $\mathcal{Z}$, $C\mathcal{Z}$, is a subset of the complement of $SS(W_{\tilde{Z}})$. This implies the assertion. Let $x_1 \neq x$ or $\xi_1 \neq -\xi$ which implies that $(x, x_1, \sqrt{-1}\,(\xi, \xi_1)) \in C\mathcal{Z}$. Then it is clear that $(x, x_1, \sqrt{-1}\,(\xi, \xi_1)) \notin SS(W_{\tilde{Z}})$. Let $(x_0, \sqrt{-1}\,\xi_0) \notin Z$. This is the last possible case with $(x_0, x_0, \sqrt{-1}\,(\xi_0, \xi_0)) \in C\mathcal{Z}$. We have to prove that

$$(x_0, x_0, \sqrt{-1}\,(\xi_0, -\xi_0)) \notin SS(W_{\tilde{Z}}).$$

There exists an open set $O_{x_0} \ni x_0$ and an open cone $\Gamma_{\xi_0} \ni \xi_0$ such that

$$(x_0, \xi_0) \in (O_{x_0} \times \Gamma_{\xi_0}) \cap \tilde{Z} = \varnothing.$$

Let Γ_j, $j = 1, \dots, r$, be open convex proper cones such that

$$\mathbf{R}^n \backslash \Gamma_{\xi_0} = \bigcup_{j=1}^{r} \bar{\Gamma}_j,$$

and $\operatorname{mes}(\bar{\Gamma}_j \backslash \Gamma_j) = 0$, $j = 1, \dots, r$. We put

$$\tilde{Z} \cap (\mathbf{R}^n \times \Gamma_j) = \tilde{Z}_j, \quad j = 1, \dots, r.$$

Note, if $y \in \Gamma_j^*$, $y \in -\Gamma_j^*$, $t \in \Gamma_j$, then

$$\langle t, y - y' \rangle > \varepsilon_j |t|\,|y - y'| \quad \text{for some } \varepsilon_j > 0.$$

Let $j \in \{1, \dots, r\}$. Put

$$B_j = bv_{z,z'} \int_{\tilde{Z} \cap (\mathbf{R}^n \times \Gamma_j)} (2|t|)^{n/2} \left\{ 1 + \frac{1}{2} \frac{t}{|t|}(z - z') \right\} \cdot e^{\sqrt{-1}\,t(z-z') - |t|\{(z-u)^2 + (z'-u)^2\}}\, du\, dt, \quad (x, x') \in O_{x_0} \times O_{x_0},$$

where the boundary value is taken over $y \in \Gamma_j^*$, $y' \in -\Gamma_j^*$ and $|y| + |y'| < C_j$ where C_j is a suitable constant. We have that each B_j, $j = 1, \dots, r$, is a distribution which has singular spectrum contained in

$$(O_{x_0} \times O_{x_0}) \times (\sqrt{-1}\,(\Gamma_j \cap S^{n-1}) \times \sqrt{-1}\,(-\Gamma_j \cap S^{n-1})).$$

Since

$$W_{\tilde{Z}}(x, x') = \sum_j B_j, \quad (x, x') \in O_{x_0},$$

it follows that $(x_0, x_0, \sqrt{-1}\,(\xi_0, -\xi_0)) \notin SS(W_{\tilde{Z}})$.

Theorem 3.1. a) *The quotient sheaf* $\mathcal{D}b^*_M/\mathcal{D}f^*_M$ *is supple.*

b) *The sheaves* $\mathcal{C}^{1,*}_M$ *and* $\mathcal{C}^{1/*}_M$ *are flabby and the following sequences are exact*

$$0 \longrightarrow \mathcal{D}f^*_M \longrightarrow \mathcal{B}_M \longrightarrow \pi_*\mathcal{C}^{1,*}_M \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{D}b^*_M \longrightarrow \mathcal{B}_M \longrightarrow \pi_*\mathcal{C}^{1/*}_M \longrightarrow 0.$$

c) *Let* $1 \leq \dagger \leq *$. *The sheaves* $\mathcal{C}^{\dagger,*}_M$ *and* $C^{\dagger/*}_M$ *are supple and the sequences*

$$0 \longrightarrow \mathcal{D}f^*_M \longrightarrow \mathcal{D}^{\dagger}_M \longrightarrow \pi_*\mathcal{C}^{\dagger,*}_M \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{D}b^*_M \longrightarrow \mathcal{D}b^{\dagger}_M \longrightarrow \pi_*\mathcal{C}^{\dagger/*}_M \longrightarrow 0,$$

are exact.

d) *Let* $1 \leq * \leq \dagger$. *The sheaf* $\mathcal{C}^{d,\dagger,*}_M$ *is supple and the sequence*

$$0 \longrightarrow \mathcal{D}f^*_M \longrightarrow \mathcal{D}f^{\dagger}_M \longrightarrow \pi_*\mathcal{C}^{d,\dagger,*}_M \longrightarrow 0$$

is exact.

Proof. a) It is enough to prove the assertion for $M = \mathbf{R}^n$. Let $G \in (\mathcal{D}b^*/\mathcal{D}f^*)(\mathbf{R}^n)$. By partition of unity we may suppose that $\operatorname{supp} G$ is a compact set K. The use of the partition of unity also implies that we may suppose that there is a representative $\tilde{G} \in \mathcal{D}b^*$ of G such that $SS^*\tilde{G} = K$ and that $\tilde{G}$ is compactly supported.

Now by the same arguments as in [**3**], we have that for closed sets K_1 and K_2 with the union K there exist $\tilde{G}_1$ and $\tilde{G}_2$ of $\mathcal{D}b^*_M$ such that they are compactly supported,

$$\tilde{G} = \tilde{G}_1 + \tilde{G}_2 \quad \text{and} \quad SS^*\tilde{G}_i \subset K_i, \quad i = 1, 2. \tag{3.1}$$

Let Γ_j, $j = 1, \ldots, r$, be convex closed proper cones such that measure$(\Gamma_j \cap \Gamma_i) = 0$, $i \neq j$, $\cup_{j=1}^r \Gamma_j = \mathbf{R}^n$ and that for every j there exists a convex closed cone $\tilde{\Gamma}_j$ such that $\Gamma_j \subset\subset \mathring{\tilde{\Gamma}}_j$. By using [**10**, Corollary 1], we obtain that $\tilde{G} = \sum_{j=1}^r \tilde{G}^j$ such that

$$SS^*\tilde{G}_j \subset \mathbf{R}^n \times \sqrt{-1}\,(S^{n-1} \cap \Gamma_j) \cap K, \quad j = 1, \ldots, r.$$

Put

$$\tilde{K} = \left\{ (x,\xi) \in \mathbf{R}^n \times (\mathbf{R}^n \backslash \{0\});\ \left(x, \sqrt{-1}\frac{\xi}{|\xi|} \right) \in K \right\}$$

and define $\tilde{K}_1$ and $\tilde{K}_2$ in the same way. Put

$$\tilde{K}_{ij} = (\mathbf{R}^n \times \Gamma_j) \cap \tilde{K}_i, \quad i = 1,2, \quad j = 1, \dots, r.$$

The corresponding sets in $\mathbf{R}^n \times \sqrt{-1}\, S^{n-1}$ are denoted by K_{ij}. Define

$$\tilde{G}^j_{\tilde{\Gamma}_j} = \int_{\mathbf{R}^n} W_{\mathbf{R}^n \times \tilde{\Gamma}_j}(x - x')\tilde{G}^j(x')\, dx', \quad x \in \mathbf{R}^n.$$

Then, by Theorem 1.9 in [**2**] we have $SS^*\tilde{G}^j_{\tilde{\Gamma}_j} \subset K_{1j} \cup K_{2j}$. Note $SS^*(G - \tilde{G}^j_{\tilde{\Gamma}_j}) = 0$. We put

$$\begin{aligned} A_{1j} &= \{(x,\xi) \in \mathbf{R}^n \times \tilde{\Gamma}_j;\ d((x,\xi), K_{1j}) \le d((x,\xi), \tilde{K}_{2j})\}, \\ A_{2j} &= \{(x,\xi) \in \mathbf{R}^n \times \tilde{\Gamma}_j;\ d((x,\xi), K_{1j}) \ge d((x,\xi), \tilde{K}_{2j})\}, \\ \tilde{G}_{A_{ij}}(x) &= \int_{\mathbf{R}^n} W_{A_{ij}}(x,x')\tilde{G}^j(x')\, dx', \quad i = 1,2. \end{aligned}$$

By Theorem 1.9 in [**2**] these ultradistributions are well defined and

$$SS^*\tilde{G}^j_{A_{ij}} \subset A_{ij} \cap SS^*\tilde{G}^j \subset K_{ij}, \quad i = 1,2, \quad j = 1,2,\dots, r.$$

Note that

$$\begin{aligned} \tilde{G}^j - \tilde{G}_{A_{1j}} - \tilde{G}_{A_{2j}} = \tilde{G}^j - \tilde{G}^j_{\tilde{\Gamma}_j} \\ + \int_{\mathbf{R}^n} (W_{A_{1j} \cup A_{2j}}(x,x') - W_{A_{1j}}(x,x') - W_{A_{2j}}(x,x')) \\ \cdot \tilde{G}^j(x')\, dx'. \end{aligned}$$

We will prove that

$$SS^*(\tilde{G}^j - \tilde{G}_{A_{1j}} - \tilde{G}_{A_{2j}}) \subset K_{1j} \cap K_{2j}, \quad j = 1, \dots, r.$$

It is enough to prove

$$SS(W_{A_{1j} \cup A_{2j}} - W_{A_{1j}} - W_{A_{2j}}) \subset \{(x,x,\xi,-\xi);\ (x,\xi) \in A_{1j} \cap A_{2j}\}.$$

Let $(x_0, \xi_0) \in (A_{1j} \cup A_{2j}) \backslash (A_{1j} \cap A_{2j})$ and assume $(x_0, \xi_0) \in A_{1j} \backslash A_{2j}$. There exists an open ball $B((x_0, \xi_0), \varepsilon)$ such that $B((x_0, \xi_0), \varepsilon) \cap A_{2j} = \phi$. We have

$$\begin{aligned} &W_{(A_{1j} \cup A_{2j})} - W_{A_{1j}} - W_{A_{2j}} \\ &\qquad = W_{(A_{1j} \cup A_{2j}) \backslash B((x_0, \xi_0), \varepsilon)} - W_{A_{1j} \backslash B((x_0, \xi_0), \varepsilon)} - W_{A_{2j}} \end{aligned}$$

and this implies that

$$(x_0, x_0, \xi_0, -\xi_0) \notin SS(W_{(A_{1j} \cup A_{2j})} - W_{A_{1j}} - W_{A_{2j}}).$$

Then

$$SS^*(\tilde{G}^j - \tilde{G}_{A_{1j}} - \tilde{G}_{A_{2j}}) \subset K_1 \cap K_2,$$
$$SS^* \tilde{G}_i \subset K_i \quad \text{and} \quad \tilde{G}_i = \sum_{j=1}^{r} \tilde{G}_{A_{ij}}, \quad i = 1, 2,$$

holds. Since

$$\sum_{j=1}^{r} (\tilde{G}^j - \tilde{G}_{A_{1j}} - \tilde{G}_{A_{2j}}) + \tilde{G}_1 + \tilde{G}_2 = \tilde{G},$$

we obtain the desired decomposition (3.1). This implies

$$G = G_1 + G_2, \operatorname{supp} G_i \subset K_i,$$

where G_i are elements of $(\mathcal{D}b^* / \mathcal{D}f^*)(\mathbf{R}^n)$ determined by $\tilde{G}_i$, $i = 1, 2$.

b)–d). We will prove the first diagram in b), the flabbiness of $\mathcal{C}_M^{1,*}$ in b) and the suppleness of $\mathcal{C}_M^{\dagger,*}$ in c). The other parts of the theorem may be proved in a similar way. The diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{A}_M \arrow[r] \arrow[d] & \mathcal{A}_M \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{D}f_M^* \arrow[r] \arrow[d] & \mathcal{D}b_M^* \arrow[r] \arrow[d] & \pi_*\mathcal{C}_M^{\dagger,*} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_*\mathcal{C}_M^{d,*} \arrow[r] \arrow[d] & \pi_*\mathcal{C}_M^{\dagger} \arrow[r] \arrow[d] & \pi_*\mathcal{C}_M^{\dagger,*} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

implies the exactness is of the first sequence in b).

Suppose that U is an open set of M. It is well known that $H^n(U,\mathcal{A}_M)=0$, $n>0$ and since $\mathcal{D}f_M^*$ and $\mathcal{D}b_M^*$ are soft, we have

$$H^n(U,\mathcal{D}f^*)=0, \qquad H^n(U,\mathcal{D}b^*)=0, \quad n>0.$$

Since

$$0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{D}f_M^* \longrightarrow \pi_*\mathcal{C}_M^{d,*} \longrightarrow 0$$

is exact, it follows that $H^n(\Omega,\mathcal{C}^{d,*})=0$, $n>0$, for any open set Ω of $\sqrt{-1}\,S^*M$.

This implies that

$$0 \longrightarrow \Gamma(\Omega,\mathcal{C}_M^{d,*}) \longrightarrow \Gamma(\Omega,\mathcal{C}_M^{1}) \longrightarrow \Gamma(\Omega,\mathcal{C}_M^{1,*}) \longrightarrow 0$$

is exact and the flabbiness of $\mathcal{C}_M^{1,*}$ follows from the diagram

$$\begin{array}{ccccccc}
\Gamma(X,\mathcal{C}^1) & \longrightarrow & \Gamma(X,\mathcal{C}^{1,*}) & \longrightarrow & 0 & & \text{exact} \\
\downarrow & & \downarrow & & & & \\
\Gamma(\Omega,\mathcal{C}^1) & \longrightarrow & \Gamma(\Omega,\mathcal{C}^{1,*}) & \longrightarrow & 0 & & \text{exact} \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

exact.

Let $F \in \Gamma(U, \mathcal{C}_M^{\dagger,*})$, $\operatorname{supp} F = K = K_1 \cup K_2$ where K_i, $i = 1, 2$, are closed sets. Then for every x there is an open neighborhood of x, W_x, such that $F|_{W_x}$ is determined by a section $\tilde{F} \in \Gamma(W_x, \mathcal{C}_M^{\dagger})$.

Let $\{U_\alpha;\, \alpha \in \mathbf{N}\}$ be a locally finite covering of U which refines the cover $\{W_x;\, x \in U\}$, and let κ_α, $\alpha \in \mathbf{N}$, be a partition of unity of $\mathcal{D}f_M^*(U)$ which is subordinated to $\{U_\alpha;\, \alpha \in \mathbf{N}\}$. Thus, if $\alpha \in \mathbf{N}$, we have

$$\operatorname{supp} \kappa_\alpha \subset U_\alpha \subset W_{x_0} \quad \text{for some } x_0 \in U.$$

Put $F = \sum_{\alpha \in \mathbf{N}} F\kappa_\alpha$. Then $\operatorname{supp} F\kappa_\alpha \subset K$ and if

$$\tilde{F}_\alpha \in \Gamma(U_\alpha, \mathcal{C}_M^{\dagger}), \quad \alpha \in \mathbf{N},$$

corresponds to $F\kappa_\alpha$, then

$$K_\alpha = SS^* \tilde{F}_\alpha = \operatorname{supp} F\kappa_\alpha, \quad \alpha \in \mathbf{N}.$$

Now, by part (a), there exist $\tilde{F}_{\alpha 1}$ and $\tilde{F}_{\alpha 2}$ in $\tilde{\Gamma}(U_\alpha, \mathcal{C}_M^{\dagger})$ such that

$$SS^* \tilde{F}_{\alpha i} \subset K_i \cap K_\alpha, \quad i = 1, 2, \quad \alpha \in \mathbf{N}.$$

Then, F_1 and F_2 of $\Gamma(U, \mathcal{C}_M^{\dagger,*})$, which correspond to

$$\sum_{\alpha \in \mathbf{N}} \tilde{F}_{\alpha 1} \quad \text{and} \quad \sum_{\alpha \in \mathbf{N}} \tilde{F}_{\alpha 2} \quad \text{respectively},$$

have the properties

$$F = F_1 + F_2,\ \operatorname{supp} F_i \subset K_i, \quad i = 1, 2.$$

Corollary 3.1. *Let* Ω *be an open subset of* $\sqrt{-1}\,S^*M$, *and let* $u \in \mathcal{D}b^*_M(\pi(\Omega))$.

(i) $(SS^*u) \cap \Omega = \varnothing \Leftrightarrow Sp(u)|_\Omega \in \mathcal{C}^{d,*}_M(\Omega)$.

(ii) *For* $u \in \mathcal{D}b^\dagger_M(\pi(\Omega))$, $\dagger \le *$, *we have*

$$(SS^{1/*}u) \cap \Omega = \varnothing \Longleftrightarrow Sp(u)|_\Omega \in \mathcal{C}^*_M(\Omega).$$

The quoted assertions imply that we can generalize the definitions of SS_* and SS^* of $u \in \mathcal{B}_M$ in $\sqrt{-1}\,S^*M$ as follows.

Let $q = (\overset{\circ}{x}, \sqrt{-1}, \overset{\circ}{\xi}\infty) \in \sqrt{-1}\,S^*M$. Then

$$\overset{\circ}{q} \notin SS^*u \quad \text{if} \quad Sp(u)_{\overset{\circ}{q}} \in \mathcal{C}^{d,*}_M,$$
$$\overset{\circ}{q} \notin SS^{1/*}u \quad \text{if} \quad Sp(u)_{\overset{\circ}{q}} \in \mathcal{C}^*_M.$$

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