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WEIGHTED VERSION OF MULTIVARIATE OSTROWSKI TYPE INEQUALITIES

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ABSTRACT. We establish two weighted integral identities and use them to prove a number of inequalities of Ostrowski type for functions of several variables. The results in the paper extend some known results of Pečarić and Savić as well as some recent results of Anastassiou.

1. Introduction and preliminary results. The results in this paper are motivated by the following integral inequality that was proved in 1938 by Ostrowski [**12**].

Theorem A. *Let f be a differentiable function on $[a,b]$, and let $|f'(x)| \le M$ on $[a,b]$. Then, for every $x \in [a,b]$,*

$$(1.1)\qquad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\big(x-((a+b)/2)\big)^2}{(b-a)^2}\right](b-a)M.$$

Some generalizations of this inequality, obtained by Milovanović [**8, 9**], Milovanović and Pečarić [**10**] and Fink [**4**] were noted in [**11**, pp. 468–471]. Recently, Anastassiou [**1**], [**2**] proved some more general inequalities of this type.

We are interested in generalization of (1.1) for functions of several variables. In 1984 Pečarić and Savić [**14**, pp. 263–264] proved the following result.

Theorem B. *Consider a real linear space X of real valued functions $f : Q \to \mathbf{R}$, where Q is a subset of $\mathbf{R}^m$, $m \in \mathbf{N}$, and assume that $\mathbf{1} \in X$ (here $\mathbf{1}$ denotes the constant function $\mathbf{x} \mapsto \mathbf{1}$, $\mathbf{x} \in Q$). Let $A : X \to \mathbf{R}$*

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be a positive linear functional on X such that $A(\mathbf{1}) = \mathbf{1}$ and let $f, g \in X$ be such that $fg \in X$. Suppose that f satisfies the condition

$$|f(\mathbf{x}) - f(\mathbf{y})| \le \sum_{i=1}^{m} N_i |x_i - y_i|^{r_i} \quad \text{for all } \mathbf{x}, \mathbf{y} \in Q, \tag{1.2}$$

where $N_i \ge 0$ and $0 < r_i \le 1$, $i = \mathbf{1}, \dots, m$, are some constants. For any fixed $\mathbf{x} \in Q$, define $f_i : Q \to \mathbf{R}$, $i = 1, \dots, m$, as

$$f_i(\mathbf{y}) = |y_i - x_i|^{r_i}, \quad \mathbf{y} \in Q, \quad i = 1, \dots, m.$$

(i) *If $g(\mathbf{y}) \ge 0$ for all $\mathbf{y} \in Q$ and $A(g) > 0$, then*

$$\left| f(\mathbf{x}) - \frac{A(fg)}{A(g)} \right| \le \sum_{i=1}^{m} N_i \frac{A(f_i g)}{A(g)}. \tag{1.3}$$

(ii) *If*

$$0 < c \le g(\mathbf{y}) \le \lambda c \quad \text{for all } \mathbf{y} \in Q \tag{1.4}$$

with some constants c and λ, then

$$\left| f(\mathbf{x}) - \frac{A(fg)}{A(g)} \right| \le \sum_{i=1}^{m} N_i \frac{A(f_i g)}{A(g)} \le \sum_{i=1}^{m} N_i \frac{\lambda T_i A(f_i)}{T_i + (\lambda - 1) A(f_i)}, \tag{1.5}$$

where

$$T_i = \sup_{\mathbf{y} \in Q} f_i(\mathbf{y}), \quad i = \mathbf{1}, \dots, m.$$

A special case of Theorem B can be obtained in the following way, [**14**, p. 264]. Take

$$Q = D := \prod_{i=1}^{m} [a_i, b_i], \quad a_i, b_i \in \mathbf{R},\ a_i < b_i, \quad i = \mathbf{1}, \dots, m,$$

and let X be a linear space which contains all functions $f : D \to \mathbf{R}$ integrable on D. Define $A : X \to \mathbf{R}$ as

$$A(f) := \frac{1}{\prod_{i=1}^{m} (b_i - a_i)} \int_D f(\mathbf{y})\, d\mathbf{y}, \quad f \in X.$$

Then (1.3) becomes

$$
\left| f(\mathbf{x}) - \frac{\int_D f(\mathbf{y}) g(\mathbf{y})\, d\mathbf{y}}{\int_D g(\mathbf{y})\, d\mathbf{y}} \right| \le \sum_{i=1}^{m} N_i \frac{\int_D |y_i - x_i|^{r_i} g(\mathbf{y})\, d\mathbf{y}}{\int_D g(\mathbf{y})\, d\mathbf{y}}, \quad \mathbf{x} \in D \tag{1.6}
$$

while (1.5) becomes

$$
\begin{aligned}
\left| f(\mathbf{x}) - \frac{\int_D f(\mathbf{y}) g(\mathbf{y})\, d\mathbf{y}}{\int_D g(\mathbf{y})\, d\mathbf{y}} \right| &\le \sum_{i=1}^{m} N_i \frac{\int_D |y_i - x_i|^{r_i} g(\mathbf{y})\, d\mathbf{y}}{\int_D g(\mathbf{y})\, d\mathbf{y}} \\
&\le \sum_{i=1}^{m} N_i \frac{\lambda T_i S_i}{T_i + (\lambda - 1) S_i}, \quad \mathbf{x} \in D
\end{aligned} \tag{1.7}
$$

where

$$
\begin{aligned}
T_i &= (\max\{x_i - a_i, b_i - x_i\})^{r_i}, \\
S_i &= \frac{(x_i - a_i)^{1+r_i} + (b_i - x_i)^{1+r_i}}{(1 + r_i)(b_i - a_i)} \quad i = 1, \dots, m.
\end{aligned} \tag{1.8}
$$

This result is a modification of the result of Pečarić and Savić previously proved in [**13**, p. 196]. Namely, instead of the condition (1.4), the following assumption can be used:

$$
0 < c_i \le G_i(y_i) \le \lambda_i c_i \quad \text{for all } y_i \in [a_i, b_i], \quad i = 1, \dots, m,
$$

where

$$
G_i(y_i) := \int_{D_i} g(\mathbf{y})\, d\mathbf{y}_i, \qquad D_i := \prod_{\substack{j=1 \\ j \ne i}}^{m} [a_j, b_j],
$$

$$
d\mathbf{y}_i := \prod_{\substack{j=1 \\ j \ne i}}^{m} dy_j, \quad i = \mathbf{1}, \dots, m.
$$

In that case, instead of (1.7) we get

$$
\begin{aligned}
\left| f(\mathbf{x}) - \frac{\int_D f(\mathbf{y}) g(\mathbf{y})\, d\mathbf{y}}{\int_D g(\mathbf{y})\, d\mathbf{y}} \right| &\le \sum_{i=1}^{m} N_i \frac{\int_D |y_i - x_i|^{r_i} g(\mathbf{y})\, d\mathbf{y}}{\int_D g(\mathbf{y})\, d\mathbf{y}} \\
&\le \sum_{i=1}^{m} N_i \frac{\lambda_i T_i S_i}{T_i + (\lambda_i - 1) S_i}, \quad \mathbf{x} \in D
\end{aligned} \tag{1.9}
$$

where T_i and S_i are given by (1.8). Further, if f satisfies the condition (1.2) with $r_1 = \cdots = r_m = 1$, that is, if

$$|f(\mathbf{x}) - f(\mathbf{y})| \le \sum_{i=1}^{m} N_i\,|x_i - y_i| \quad \text{for all } \mathbf{x}, \mathbf{y} \in Q, \tag{1.10}$$

then (1.9) can be rewritten as

$$\begin{aligned} \left| f(\mathbf{x}) - \frac{\int_D f(\mathbf{y}) g(\mathbf{y})\, d\mathbf{y}}{\int_D g(\mathbf{y})\, d\mathbf{y}} \right| &\le \sum_{i=1}^{m} N_i\, \frac{\int_D |y_i - x_i| g(\mathbf{y})\, d\mathbf{y}}{\int_D g(\mathbf{y})\, d\mathbf{y}} \\ &\le \sum_{i=1}^{m} N_i\, \frac{\lambda_i t_i s_i}{t_i + (\lambda_i - 1) s_i}, \quad \mathbf{x} \in D \end{aligned} \tag{1.11}$$

where

$$t_i = \max\{x_i - a_i, b_i - x_i\},$$

$$s_i = \left[\frac{1}{4} + \frac{\big(x_i - ((a_i + b_i)/2)\big)^2}{(b_i - a_i)^2}\right](b_i - a_i), \quad i = 1, \dots, m.$$

When $g(\mathbf{y}) = 1$ for all $\mathbf{y} \in D$, then (1.11) reduces to

$$\left| f(\mathbf{x}) - \frac{\int_D f(\mathbf{y})\, d\mathbf{y}}{\prod_{i=1}^{m} (b_i - a_i)} \right| \le \sum_{i=1}^{m} \left[\frac{1}{4} + \frac{\big(x_i - ((a_i + b_i)/2)\big)^2}{(b_i - a_i)^2}\right](b_i - a_i) N_i$$

which is for $m = 1$ just the Ostrowski's inequality (1.1). Milovanović [**9**, p. 27] proved (1.11), but under more restrictive assumptions on f than (1.10), that is, under the assumption that f is differentiable on D and

$$\left| \frac{\partial f(\mathbf{x})}{\partial x_i} \right| \le N_i \quad \text{for all } \mathbf{x} \in D, \quad i = 1, \dots, m.$$

Also, it should be noted that the inequality (1.6), along with the special case when $g(\mathbf{y}) = \mathbf{1}$ for all $\mathbf{y} \in D$, was rediscovered in the recent paper [**3**].

The proof of the second part of Theorem B is based on the following result obtained by Pečarić and Savić [**14**, p. 247].

Theorem C. *Assume that a space X and a linear functional $A : X \to \mathbf{R}$ are as in Theorem* B. *Let $f, g \in X$ be such that $fg \in X$. Suppose*

$$m_1 \le f(\mathbf{x}) \le M_1, \quad m_1 \ne M_1$$

and

$$0 < m_2 \le g(\mathbf{x}) \le M_2 \quad \text{for all } \mathbf{x} \in Q,$$

where m_1, M_1, m_2 and M_2 are constants. If

$$D(f) := M_1 - A(f) \quad \text{and} \quad d(f) := A(f) - m_1,$$

then

$$\frac{m_1 M_2 D(f) + M_1 m_2 d(f)}{M_2 D(f) + m_2 d(f)} \le \frac{A(fg)}{A(g)} \le \frac{M_1 M_2 d(f) + m_1 m_2 D(f)}{M_2 d(f) + m_2 D(f)}.$$

Theorem C is a generalization of one result of Lupaş [**6**, Theorem 1]. We actually need the following corollary.

Corollary D. *Let Q be a convex and compact subset of $\mathbf{R}^m$, $m \in \mathbf{N}$ such that $\mathrm{Vol}\,(Q) = \int_Q d\mathbf{y} > 0$. Let $g, w : Q \to \mathbf{R}$ be integrable on Q. Suppose*

$$k \le g(\mathbf{y}) \le K, \quad k \ne K$$

and

$$0 < c \le w(\mathbf{y}) \le \lambda c, \quad \mathbf{y} \in Q,$$

for some constants k, K, c and λ. If $G := (1/\mathrm{Vol}\,(Q)) \int_Q g(\mathbf{y})\, d\mathbf{y}$, then

$$\frac{\lambda k(K-G) + K(G-k)}{\lambda(K-G) + (G-k)} \le \frac{\int_Q g(\mathbf{y}) w(\mathbf{y})\, d\mathbf{y}}{\int_Q w(\mathbf{y})\, d\mathbf{y}} \le \frac{k(K-G) + \lambda K(G-k)}{(K-G) + \lambda(G-k)}.$$

The result stated in the above corollary is a generalization of an analogous result proved by Karamata [**5**] for integrable real-valued functions defined on $[0, \mathbf{1}]$. It is easily proved; we simply take X to be a space of all integrable functions defined on Q and define $A : X \to \mathbf{R}$ as

$$A(f) := \frac{1}{\mathrm{Vol}\,(Q)} \int_Q f(\mathbf{y})\,d\mathbf{y}, \quad f \in X,$$

and then apply Theorem C.

Another possibility to generalize the inequality (1.1) is to use the higher order derivatives. This idea was used by Anastassiou [**1**], [**2**]. The main result from [**2**] concerning functions of several variables is:

Theorem E. *Let Q be a compact and convex subset of $\mathbf{R}^m$, $m \geq 1$. Let $f \in C^{n+1}(Q)$, $n \in N$ and $\mathbf{x} \in Q$ be fixed such that all partial derivatives $f_{\boldsymbol{\alpha}} := (\partial^{\alpha} f/\partial z^{\alpha})$, where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$, $\alpha_i \in \mathbf{N} \cup \{0\}$, $i = \mathbf{1}, \dots, m$, $|\boldsymbol{\alpha}| = \sum_{i=1}^m \alpha_i = j$, $j = \mathbf{1}, \dots, n$ fulfill $f_{\boldsymbol{\alpha}}(\mathbf{x}) = 0$. Then*

$$\begin{aligned}
&\left|\frac{1}{\mathrm{Vol}\,(Q)} \int_Q f(\mathbf{y})\,d\mathbf{y} - f(\mathbf{x})\right| \\
(1.12)\qquad &\leq \frac{1}{(n+\mathbf{1})!\mathrm{Vol}\,(Q)} \int_Q \left(\sum_{i=1}^m |y_i - x_i| \left\|\frac{\partial}{\partial z_i}\right\|_\infty\right)^{n+1} f(\mathbf{y})\,d\mathbf{y} \\
&\leq \frac{D_{n+1}(f)}{(n+1)!\mathrm{Vol}\,(Q)} \int_Q (\|\mathbf{y} - \mathbf{x}\|_1)^{n+1}\,d\mathbf{y},
\end{aligned}$$

where

$$D_{n+1}(f) := \max_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|=n+1} \|f_{\boldsymbol{\alpha}}\|_\infty$$

and

$$\|\mathbf{y} - \mathbf{x}\|_1 := \sum_{i=1}^m |y_i - x_i|.$$

In this paper we extend the above results of Pečarić and Savić as well as the results of Anastassiou. Similar results for the functions of one variable can be found in the recent paper [**7**].

2. Two integral identities. We consider an open interval $I \subset \mathbf{R}$ and fixed $a, b \in I$, $a < b$. Suppose that a function $g : I \to \mathbf{R}$ is given.

If, for some $n \in \mathbf{N}$, $g^{(n)}(t)$ exists for all $t \in [a,b]$, then we define

$$R_n(x,y;g) := g(y) - g(x) - \sum_{j=1}^{n} \frac{g^{(j)}(x)}{j!}(y-x)^j, \quad x,y \in [a,b].$$

Also we set

$$R_0(x,y;g) := g(y) - g(x), \quad x,y \in [a,b].$$

Lemma 1. *Let $g : I \to \mathbf{R}$ be a function defined on an open interval $I \subset \mathbf{R}$, and let $a,b \in I$, $a < b$.*

(i) *If (for some $n \in \mathbf{N}$) $g^{(n)}(t)$ exists for all $t \in [a,b]$ and $g^{(n)}(\cdot)$ is integrable on $[a,b]$, then for all $x,y \in [a,b]$ we have*

$$R_n(x,y;g) = \frac{1}{(n-1)!}\int_x^y [g^{(n)}(t) - g^{(n)}(x)](y-t)^{n-1}\,dt. \tag{2.1}$$

(ii) *If (for some $n \in \mathbf{N} \cup \{0\}$) $g^{(n+1)}(t)$ exists for all $t \in [a,b]$ and $g^{(n+1)}(\cdot)$ is integrable on $[a,b]$, then for all $x,y \in [a,b]$ we have*

$$R_n(x,y;g) = \frac{1}{n!}\int_x^y g^{(n+1)}(t)(y-t)^n\,dt. \tag{2.2}$$

Proof. For $x,y \in [a,b]$ and for $j \in \mathbf{N}$, denote

$$\Delta_j(x,y;g) := \frac{1}{(j-1)!}\int_x^y [g^{(j)}(t) - g^{(j)}(x)](y-t)^{j-1}\,dt.$$

We have

$$\begin{aligned}\Delta_1(x,y;g) &= \int_x^y [g'(t) - g'(x)]\,dt \\ &= g(y) - g(x) - g'(x)(y-x) \\ &= R_1(x,y;g)\end{aligned}$$

which shows that (2.1) is true for $n = 1$. Further, by partial integration we get for any $j \in \mathbf{N}$

$$\begin{aligned}\Delta_j(x,y;g) &= -\left[g^{(j)}(t) - g^{(j)}(x)\right] \frac{(y-t)^j}{j!}\bigg|_x^y \\ &\quad + \frac{1}{j!}\int_x^y g^{(j+1)}(t)(y-t)^j\,dt \\ &= \frac{1}{j!}\int_x^y g^{(j+1)}(t)(y-t)^j\,dt. \end{aligned} \tag{2.3}$$

Using this we also have

$$\begin{aligned}\Delta_j(x,y;g) &= \frac{1}{j!}\int_x^y [g^{(j+1)}(t) - g^{(j+1)}(x)](y-t)^j\,dt \\ &\quad - g^{(j+1)}(x)\,\frac{(y-t)^{j+1}}{(j+1)!}\bigg|_x^y \\ &= \Delta_{j+1}(x,y;g) + \frac{g^{(j+1)}(x)}{(j+1)!}\,(y-x)^{j+1}\end{aligned}$$

or

$$\Delta_{j+1}(x,y;g) = \Delta_j(x,y;g) - \frac{g^{(j+1)}(x)}{(j+1)!}\,(y-x)^{j+1}. \tag{2.4}$$

Suppose (2.1) is valid for some $n \in \mathbf{N}$ and $g^{(n+1)}(\cdot)$ is integrable on $[a,b]$. Then by (2.4)

$$\begin{aligned}R_{n+1}(x,y;g) &= R_n(x,y;g) - \frac{g^{(n+1)}(x)}{(n+1)!}\,(y-x)^{n+1} \\ &= \Delta_n(x,y;g) - \frac{g^{(n+1)}(x)}{(n+1)!}\,(y-x)^{n+1} \\ &= \Delta_{n+1}(x,y;g)\end{aligned}$$

and we conclude by induction that (2.1) is true for any $n \in \mathbf{N}$. Now combining (2.1) and (2.3) we get, for $n \in \mathbf{N}$,

$$R_n(x,y;g) = \Delta_n(x,y;g) = \frac{1}{n!}\int_x^y g^{(n+1)}(t)(y-t)^n\,dt.$$

So (2.2) is valid for any $n \in \mathbf{N}$. It is easy to check that (2.2) holds for $n = 0$ too. $\square$

Let Q be any compact and convex subset of $\mathbf{R}^m$, $m \in \mathbf{N}$. A weight function on Q is any function $w : Q \to [0, \infty)$ which is integrable on Q and

$$\int_Q w(\mathbf{y})\, d\mathbf{y} > 0.$$

For given m-tuple $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$, $\alpha_i \in \mathbf{N} \cup \{0\}$, $i = 1, \dots, m$, we use the notation

$$|\boldsymbol{\alpha}| := \sum_{i=1}^m \alpha_i \quad \text{and} \quad \alpha! := \prod_{i=1}^m \alpha_i! = \alpha_1! \cdots \alpha_m!.$$

Also, for any $\mathbf{z} = (z_1, \dots, z_m) \in \mathbf{R}^m$, we set

$$\mathbf{z}^{\boldsymbol{\alpha}} := \prod_{i=1}^m z_i^{\alpha_i} = z_1^{\alpha_1} \cdots z_m^{\alpha_m}.$$

Here we assume the convention $0^0 = 1$. With such notation the following multinomial formula is valid:

$$\left(\sum_{i=1}^m z_i \right)^n = \sum_{|\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!}\, \mathbf{z}^{\boldsymbol{\alpha}}, \quad n \in \mathbf{N}.$$

Also, for given m-tuple $\mathbf{r} = (r_1, \dots, r_m)$, $r_i \in [0, \infty)$, $i = 1, \dots, m$, we set

$$|\mathbf{z}| := (|z_1|, \dots, |z_m|) \quad \text{and} \quad |\mathbf{z}|^{\mathbf{r}} := \prod_{i=1}^m |z_i|^{r_i} = |z_1|^{r_1} \cdots |z_m|^{r_m},$$

again with convention $0^0 = 1$.

If a weight function $w : Q \to [0, \infty)$ is given, then we define the moment $m_{\boldsymbol{\alpha}}(Q; w)$ of order $\boldsymbol{\alpha}$, of the set Q with respect to w as

$$m_{\boldsymbol{\alpha}}(Q; w) := \int_Q \mathbf{y}^{\boldsymbol{\alpha}} w(\mathbf{y})\, d\mathbf{y}.$$

For any fixed $\mathbf{x} \in Q$ we define the $\mathbf{x}$-centered moment $E(\mathbf{x}, Q; w)$ of order α, of the set Q with respect to w as

$$E_{\boldsymbol{\alpha}}(\mathbf{x}, Q; w) : \int_Q (\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} w(\mathbf{y})\, d\mathbf{y}.$$

Also, for any fixed $\mathbf{x} \in Q$ and for any m-tuple $\mathbf{r} \in [0,\infty)^m$ we define the $\mathbf{x}$-centered absolute moment $M_{\mathbf{r}}(\mathbf{x}, Q; w)$ of order $\mathbf{r}$, of the set Q with respect to w as

$$M_{\mathbf{r}}(\mathbf{x}, Q; w) := \int_Q |\mathbf{y}-\mathbf{x}|^{\mathbf{r}} w(\mathbf{y})\, d\mathbf{y}.$$

Note that

$$m_{\mathbf{0}}(Q; w) = E_0(\mathbf{x}, Q; w) = M_{\mathbf{0}}(\mathbf{x}, Q; w) = \int_Q w(\mathbf{y})\, d\mathbf{y},$$

where $\mathbf{0} = (0, \dots, 0)$. In the special case when $w(\mathbf{y}) = \mathbf{1}$ for all $\mathbf{y} \in Q = [\mathbf{a}, \mathbf{b}]$, where $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, b_m)$ are such that $a_i < b_i$, $i = \mathbf{1}, \dots, m$ and

$$[\mathbf{a}, \mathbf{b}] := \prod_{i=1}^{m} [a_i, b_i] = \{(x_1, \dots, x_m) : a_i \le x_i \le b_i,\ i = \mathbf{1}, \dots, m\},$$

we shall use the notations

$$m_{\boldsymbol{\alpha}} := m_{\boldsymbol{\alpha}}([\mathbf{a}, \mathbf{b}]; \mathbf{1}), \qquad E_{\boldsymbol{\alpha}}(\mathbf{x}) := E_{\boldsymbol{\alpha}}(\mathbf{x}, [\mathbf{a}, \mathbf{b}]; \mathbf{1})$$

and

$$M_{\mathbf{r}}(\mathbf{x}) := M_{\mathbf{r}}(\mathbf{x}, [\mathbf{a}, \mathbf{b}]; \mathbf{1}).$$

An easy calculation gives

$$m_{\boldsymbol{\alpha}} = \prod_{i=1}^{m} \int_{a_i}^{b_i} y_i^{\alpha_i}\, dy_i = \prod_{i=1}^{m} \frac{b_i^{\alpha_i+1} - a_i^{\alpha_i+1}}{\alpha_i + 1}$$

and

$$\begin{aligned} E_{\boldsymbol{\alpha}}(\mathbf{x}) &= \prod_{i=1}^{m} \int_{a_i}^{b_i} (y_i - x_i)^{\alpha_i}\, dy_i \\ &= \prod_{i=1}^{m} \frac{(b_i - x_i)^{\alpha_i+1} + (-1)^{\alpha_i}(x_i - a_i)^{\alpha_i+1}}{\alpha_i + 1}, \end{aligned}$$

while for $\mathbf{r} \in [0,\infty)^m$ we have

$$M_{\mathbf{r}}(\mathbf{x}) = \prod_{i=1}^{m} \int_{a_i}^{b_i} |y_i - x_i|^{r_i}\, dy_i = \prod_{i=1}^{m} \frac{(x_i - a_i)^{r_i+1} + (b_i - x_i)^{r_i+1}}{r_i + 1}.$$

Next suppose that $f : V \to \mathbf{R}$ is any function defined on an open set $V \subset \mathbf{R}^m$ which contains Q as a subset. If for some $k \in \mathbf{N}$ partial derivatives $f_{\boldsymbol{\alpha}}(\mathbf{y})$ exist for all $\mathbf{y} \in Q$ and for all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \le k$, then we can define

$$\mathcal{R}_k(\mathbf{x}, f; w) := \int_Q f(\mathbf{y})w(\mathbf{y})\, d\mathbf{y} - f(\mathbf{x}) \int_Q w(\mathbf{y})\, d\mathbf{y} - \sum_{j=1}^{k} \sum_{|\boldsymbol{\alpha}|=j} \frac{f_{\boldsymbol{\alpha}}(\mathbf{x})}{\boldsymbol{\alpha}!} E_{\boldsymbol{\alpha}}(\mathbf{x}, Q; w),$$

where $\mathbf{x} \in Q$ is any fixed element. Also, we set

$$\mathcal{R}_0(\mathbf{x}, f; w) := \int_Q f(\mathbf{y})w(\mathbf{y})\, d\mathbf{y} - f(\mathbf{x}) \int_Q w(\mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in Q.$$

Theorem 1. *Let $f : V \to \mathbf{R}$ be a function defined on an open subset V of $\mathbf{R}^m$, $m \in \mathbf{N}$. Let Q be any compact and convex subset of V, and let $w : Q \to [0,\infty)$ be a weight function on Q.*

(i) *If $f \in C^n(Q)$ for some $n \in \mathbf{N}$, then for any $\mathbf{x} \in Q$ we have*

$$\mathcal{R}_n(\mathbf{x}, f; w) = \sum_{|\boldsymbol{\alpha}|=n} \frac{n}{\boldsymbol{\alpha}!} \int_Q (\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \cdot \left\{ \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x} + t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^{n-1}\, dt \right\} w(\mathbf{y})\, d\mathbf{y}. \tag{2.5}$$

(ii) *If $f \in C^{n+1}(Q)$ for some $n \in \mathbf{N} \cup \{0\}$, then for any $\mathbf{x} \in Q$ we have*

$$\mathcal{R}_n(\mathbf{x}, f; w) = \sum_{|\boldsymbol{\alpha}|=n+1} \frac{n+1}{\boldsymbol{\alpha}!} \int_Q (\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \cdot \left\{ \int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x} + t(\mathbf{y}-x))(1-t)^{n}\, dt \right\} w(\mathbf{y})\, d\mathbf{y}. \tag{2.6}$$

Proof. For fixed $\mathbf{x}, \mathbf{y} \in Q$, $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$, we can define $g_{\mathbf{x},\mathbf{y}} : [0,1] \to \mathbf{R}$ as

$$g_{\mathbf{x},\mathbf{y}}(t) := f(\mathbf{x} + t(\mathbf{y} - \mathbf{x})), \quad t \in [0, \mathbf{1}].$$

We have $g_{\mathbf{x},\mathbf{y}}(0) = f(\mathbf{x})$ and $g_{\mathbf{x},\mathbf{y}}(\mathbf{1}) = f(\mathbf{y})$. Also, if $f \in C^k(Q)$ for some $k \in \mathbf{N}$, then $g_{\mathbf{x},\mathbf{y}} \in C^k([0,1])$ and for $j = \mathbf{1}, \ldots, k$

$$\begin{aligned} g^{(j)}_{\mathbf{x},\mathbf{y}}(t) &= \left(\sum_{i=1}^{m} (y_i - x_i) \frac{\partial}{\partial z_i} \right)^j f(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) \\ &= \sum_{|\boldsymbol{\alpha}|=j} \frac{j!}{\boldsymbol{\alpha}!} (\mathbf{y} - \mathbf{x})^{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) \end{aligned} \tag{2.7}$$

and

$$g^{(j)}_{\mathbf{x},y}(0) = \sum_{|\boldsymbol{\alpha}|=j} \frac{j!}{\boldsymbol{\alpha}!} (\mathbf{y} - \mathbf{x})^{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}}(\mathbf{x}). \tag{2.8}$$

Further,

$$\begin{aligned} R_k(0, \mathbf{1}; g_{\mathbf{x},\mathbf{y}}) &= g_{\mathbf{x},\mathbf{y}}(\mathbf{1}) - g_{\mathbf{x},\mathbf{y}}(0) - \sum_{j=1}^{k} \frac{g^{(j)}_{\mathbf{x},\mathbf{y}}(0)}{j!} \\ &= f(\mathbf{y}) - f(\mathbf{x}) - \sum_{j=1}^{k} \sum_{|\alpha|=j} \frac{f_{\boldsymbol{\alpha}}(\mathbf{x})}{\boldsymbol{\alpha}!} (\mathbf{y} - \mathbf{x})^{\boldsymbol{\alpha}}. \end{aligned}$$

Multiplying this by $w(\mathbf{y})$ and integrating over $\mathbf{y} \in Q$, we get

$$\begin{aligned} &\int_Q R_k(0, \mathbf{1}; g_{\mathbf{x},\mathbf{y}}) w(\mathbf{y})\, d\mathbf{y} \\ &\quad = \int_Q f(\mathbf{y}) w(\mathbf{y})\, d\mathbf{y} - f(\mathbf{x}) \int_Q w(\mathbf{y})\, d\mathbf{y} \\ &\qquad - \sum_{j=1}^{k} \sum_{|\boldsymbol{\alpha}|=j} \frac{f_{\boldsymbol{\alpha}}(\mathbf{x})}{\boldsymbol{\alpha}!} \int_Q (\mathbf{y} - \mathbf{x})^{\boldsymbol{\alpha}} w(\mathbf{y})\, d\mathbf{y} \\ &\quad = \int_Q f(\mathbf{y}) w(\mathbf{y})\, d\mathbf{y} - f(\mathbf{x}) \int_Q w(\mathbf{y})\, d\mathbf{y} \\ &\qquad - \sum_{j=1}^{k} \sum_{|\boldsymbol{\alpha}|=j} \frac{f_{\boldsymbol{\alpha}}(\mathbf{x})}{\boldsymbol{\alpha}!} E_{\boldsymbol{\alpha}}(\mathbf{x}, Q; w) \\ &\quad = \mathcal{R}_k(\mathbf{x}, f; w). \end{aligned} \tag{2.9}$$

Now if $f \in C^n(Q)$ for some $n \in \mathbf{N}$, then by (2.7) and (2.8) we have

$$g_{\mathbf{x},\mathbf{y}}^{(n)}(t) - g_{\mathbf{x},\mathbf{y}}^{(n)}(0) = \sum_{|\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} (\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} [f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})].$$

Using this and (2.1) we have

$$\begin{aligned} R_n(0,\mathbf{1}; g_{\mathbf{x},\mathbf{y}}) &= \frac{1}{(n-1)!} \int_0^1 [g_{\mathbf{x},\mathbf{y}}^{(n)}(t) - g_{\mathbf{x},\mathbf{y}}^{(n)}(0)](1-t)^{n-1}\, dt \\ &= \sum_{|\boldsymbol{\alpha}|=n} \frac{n}{\boldsymbol{\alpha}!} (\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \\ &\quad \cdot \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^n\, dt. \end{aligned}$$

Multiplying this by $w(\mathbf{y})$ and integrating over $\mathbf{y} \in Q$ and then using (2.9), we get (2.5). If $f \in C^{n+1}(Q)$ for some $n \in \mathbf{N} \cup \{0\}$, then by (2.2) and (2.7) we have

$$\begin{aligned} R_n(0,\mathbf{1}; g_{\mathbf{x},\mathbf{y}}) &= \frac{1}{n!} \int_0^1 g_{\mathbf{x},\mathbf{y}}^{(n+1)}(t)(\mathbf{1}-t)^n\, dt \\ &= \sum_{|\boldsymbol{\alpha}|=n+1} \frac{n+\mathbf{1}}{\boldsymbol{\alpha}!} (\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}} \\ &\quad \cdot \int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(\mathbf{1}-t)^n\, dt. \end{aligned}$$

Multiplying this by $w(\mathbf{y})$ and integrating over $\mathbf{y} \in Q$, and then using (2.9), we get (2.6). □

3. Some Ostrowski-type inequalities. In this section we use integral identities (2.5) and (2.6) to deduce some Ostrowski-type inequalities. As in the preceding section we consider a function $f : V \to \mathbf{R}$ defined on an open set $V \subset \mathbf{R}^m$, $m \in \mathbf{N}$, and which is integrable on a convex and compact subset Q of V. We also consider a weight function $w : Q \to [0, \infty)$ defined on $Q \subset V$. We can define

$$\mathcal{O}_0(x, f; w) := \frac{\int_Q f(\mathbf{y}) w(\mathbf{y})\, d\mathbf{y}}{m_{\mathbf{0}}(Q; w)} - f(\mathbf{x}), \quad \mathbf{x} \in Q.$$

Further, under the assumption that for some $k \in \mathbf{N}$ partial derivatives $f_{\boldsymbol{\alpha}}(\mathbf{y})$ exist for all $\mathbf{y} \in Q$ and for all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \le k$, we can define

$$\begin{aligned}\mathcal{O}_k(\mathbf{x}, f; w) &:= \frac{\mathcal{R}_k(\mathbf{x}, f; w)}{m_{\mathbf{0}}(Q; w)} \\ &= \mathcal{O}_0(\mathbf{x}, f; w) - \frac{1}{m_{\mathbf{0}}(Q; w)} \sum_{j=1}^{k} \sum_{|\boldsymbol{\alpha}|=j} \frac{f_{\boldsymbol{\alpha}}(\mathbf{x})}{\boldsymbol{\alpha}!} E_{\boldsymbol{\alpha}}(\mathbf{x}, Q; w), \\ &\qquad \mathbf{x} \in Q.\end{aligned}$$

3.1 Inequalities involving functions of class $C^n(Q)$. We use the integral identity (2.5) to obtain some upper bounds on the quantity $|\mathcal{O}_n(\mathbf{x}, f; w)|$. The basic estimations are given in the following theorem.

Theorem 2. *Let $f : V \to \mathbf{R}$ be a function defined on an open subset V of $\mathbf{R}^m$, $m \in \mathbf{N}$. Let Q be any compact and convex subset of V such that* $\operatorname{Vol}(Q) = \int_Q d\mathbf{y} > 0$, *and let $w : Q \to [0, \infty)$ be a weight function on Q. Suppose $f \in C^n(Q)$ for some $n \in \mathbf{N}$.*

(i) *For any $\mathbf{x} \in Q$ we have*

$$\begin{aligned}|\mathcal{O}_n(\mathbf{x}, f; w)| &\le \frac{1}{m_{\mathbf{0}}(Q; w)} \sum_{|\boldsymbol{\alpha}|=n} \frac{\|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty}{\boldsymbol{\alpha}!} M_{\boldsymbol{\alpha}}(\mathbf{x}; Q; w) \\ &\le \frac{D_n(\mathbf{x}, f)}{n! m_0(Q; w)} \int_Q \|\mathbf{y} - \mathbf{x}\|_1^n \, w(\mathbf{y}) \, d\mathbf{y} \qquad (3.1) \\ &\le \frac{D_n(\mathbf{x}, f)}{n!} \mu_n(\mathbf{x}, Q),\end{aligned}$$

where

$$\mu_s(\mathbf{x}, Q) := \max_{\mathbf{y} \in Q} \|\mathbf{y} - \mathbf{x}\|_1^s = \Big(\max_{y \in Q} \|\mathbf{y} - \mathbf{x}\|_1 \Big)^s, \quad s > 0 \qquad (3.2)$$

and

$$D_n(\mathbf{x}, f) := \max_{\boldsymbol{\alpha} : |\boldsymbol{\alpha}|=n} \|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty.$$

(ii) *If for some constants c and λ*

$$0 < c \le w(\mathbf{y}) \le \lambda c \quad \textit{for all } \mathbf{y} \in Q, \qquad (3.3)$$

then for any $\mathbf{x} \in Q$

$$|\mathcal{O}_n(\mathbf{x}, f; w)| \le \frac{D_n(\mathbf{x}, f)}{n!} \cdot \frac{\lambda \mu_n(\mathbf{x}, Q) S_n(\mathbf{x}, Q)}{\mu_n(\mathbf{x}, Q) + (\lambda - 1) S_n(\mathbf{x}, Q)}, \tag{3.4}$$

where

$$S_s(\mathbf{x}, Q) := \frac{1}{\mathrm{Vol}\,(Q)} \int_Q \|\mathbf{y} - \mathbf{x}\|_1^s \, d\mathbf{y}, \quad s > 0. \tag{3.5}$$

Proof. For fixed $\mathbf{x} \in Q$ and for any $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$, $\alpha_i \in \mathbf{N} \cup \{0\}$, $i = 1, \dots, m$, such that $|\boldsymbol{\alpha}| = n$, we have

$$\begin{aligned} \left| \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^{n-1} \, dt \right| & \\ \le \|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty \int_0^1 (1-t)^{n-1} \, dt & \\ = \frac{1}{n} \|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty. & \end{aligned}$$

So from (2.5) we get the estimation

$$\begin{aligned} |\mathcal{R}_n(\mathbf{x}, f; w)| &\le \sum_{|\boldsymbol{\alpha}|=n} \frac{\|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty}{\boldsymbol{\alpha}!} \int_Q |\mathbf{y} - \mathbf{x}|^{\boldsymbol{\alpha}} w(\mathbf{y}) \, d\mathbf{y} \\ &= \sum_{|\boldsymbol{\alpha}|=n} \frac{\|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty}{\boldsymbol{\alpha}!} M_{\boldsymbol{\alpha}}(\mathbf{x}, Q; w) \end{aligned}$$

and dividing this by $m_{\mathbf{0}}(Q; w) = \int_Q w(\mathbf{y}) \, d\mathbf{y} > 0$ we get the first inequality in (3.1). Further,

$$\begin{aligned} \sum_{|\boldsymbol{\alpha}|=n} \frac{\|f_{\boldsymbol{\alpha}}(\cdot) - f_{\boldsymbol{\alpha}}(\mathbf{x})\|_\infty}{\boldsymbol{\alpha}!} M_{\boldsymbol{\alpha}}(\mathbf{x}, Q; w) & \\ \le D_n(\mathbf{x}, f) \sum_{|\boldsymbol{\alpha}|=n} \frac{1}{\boldsymbol{\alpha}!} \int_Q |\mathbf{y} - \mathbf{x}|^{\boldsymbol{\alpha}} w(\mathbf{y}) \, d\mathbf{y} & \\ = \frac{D_n(\mathbf{x}, f)}{n!} \int_Q \left(\sum_{|\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} |\mathbf{y} - \mathbf{x}|^{\boldsymbol{\alpha}} \right) w(\mathbf{y}) \, d\mathbf{y} & \\ = \frac{D_n(\mathbf{x}, f)}{n!} \int_Q \|\mathbf{y} - \mathbf{x}\|_1^n \, w(\mathbf{y}) \, d\mathbf{y} & \end{aligned}$$

and this implies the second inequality in (3.1). The third inequality in (3.1) is a consequence of the following simple estimation

$$\begin{aligned}\int_Q \|\mathbf{y}-\mathbf{x}\|_1^n\, w(\mathbf{y})\, d\mathbf{y} &\le \max_{\mathbf{y}\in Q} \|\mathbf{y}-\mathbf{x}\|_1^n \int_Q w(\mathbf{y})\, d\mathbf{y} \\ &= \mu_n(\mathbf{x},Q) m_{\mathbf{0}}(Q;w).\end{aligned}$$

To obtain (3.4) set $g(\mathbf{y}) = \|\mathbf{y}-\mathbf{x}\|_1^n$, $y \in Q$, and note that

$$0 \le g(\mathbf{y}) \le \max_{\mathbf{y}\in Q} \|\mathbf{y}-\mathbf{x}\|_1^n = \mu_n(\mathbf{x},Q)$$

and

$$\frac{1}{\mathrm{Vol}\,(Q)} \int_Q g(\mathbf{y})\, d\mathbf{y} = S_n(\mathbf{x},Q).$$

Now apply Corollary D with $k=0$, $K=\mu_n(\mathbf{x},Q)$ and $G=S_n(\mathbf{x},Q)$ to get

$$\begin{aligned}\frac{1}{m_0(Q;w)} \int_Q \|\mathbf{y}-\mathbf{x}\|_1^n\, w(\mathbf{y})\, d\mathbf{y} &= \frac{\int_Q \|\mathbf{y}-\mathbf{x}\|_1^n\, w(\mathbf{y})\, d\mathbf{y}}{\int_Q w(\mathbf{y})\, d\mathbf{y}} \\ &\le \frac{\lambda\mu_n(\mathbf{x},Q)S_n(\mathbf{x},Q)}{\mu_n(\mathbf{x},Q)-S_n(\mathbf{x},Q)+\lambda S_n(\mathbf{x},Q)}\end{aligned}$$

so that (3.4) follows from the second inequality in (3.1). $\square$

Remark 1. Upper bound on $|\mathcal{O}_n(\mathbf{x},f;w)|$ given by the inequality (3.4) is better than one given by the third inequality in (3.1). Namely, it is obvious that

$$S_n(\mathbf{x},Q) := \frac{1}{\mathrm{Vol}\,(Q)} \int_Q \|\mathbf{y}-\mathbf{x}\|_1^n\, d\mathbf{y} \le \max_{\mathbf{y}\in Q} \|\mathbf{y}-\mathbf{x}\|_1^n = \mu_n(\mathbf{x},Q)$$

and

$$\begin{aligned}&\frac{\lambda\mu_n(\mathbf{x},Q)S_n(\mathbf{x},Q)}{\mu_n(\mathbf{x},Q)+(\lambda-1)S_n(\mathbf{x},Q)} \\ &\qquad = \frac{\lambda S_n(\mathbf{x},Q)}{\mu_n(\mathbf{x},Q)-S_n(\mathbf{x},Q)+\lambda S_n(\mathbf{x},Q)}\,\mu_n(\mathbf{x},Q) \\ &\qquad \le \mu_n(\mathbf{x},Q).\end{aligned}$$

It is easy to see that this last inequality can be strict.

Remark 2. In the case when $Q = [\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{m} [a_i, b_i]$, where $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, b_m)$ are such that $a_i < b_i$, $i = 1, \dots, m$, the quantity $\mu_n(\mathbf{x}, Q)$ can be calculated. If $\mathbf{x} \in Q$ is fixed, then for any $\mathbf{y} \in Q$

$$\begin{aligned}\|\mathbf{y} - \mathbf{x}\|_1 &= \sum_{i=1}^{m} |y_i - x_i| \\ &\le \sum_{i=1}^{m} \max\{x_i - a_i, b_i - x_i\} \\ &= \sum_{i=1}^{m} \left(\frac{b_i - a_i}{2} + \left| x_i - \frac{a_i + b_i}{2} \right| \right) \\ &= \left\| \frac{\mathbf{b} - \mathbf{a}}{2} + \left| \mathbf{x} - \frac{\mathbf{a} + \mathbf{b}}{2} \right| \right\|_1 .\end{aligned}$$

If we define $\mathbf{y}_0 = (y_{01}, \dots, y_{0m}) \in \mathbf{R}^m$ by

$$y_{0i} := \begin{cases} a_i & \text{if } x_i - a_i \ge b_i - xi \\ b_i & \text{if } x_i - a_i < b_i - x_i \end{cases}, \quad i = 1, \dots, m$$

then $\mathbf{y}_0 \in [\mathbf{a}, \mathbf{b}]$ and it is easy to check that $\|\mathbf{y}_0 - \mathbf{x}\|_1 = \|((\mathbf{b} - \mathbf{a})/2) + |\mathbf{x} - ((\mathbf{a} + \mathbf{b})/2)|\|_1$. We conclude that

$$\max_{\mathbf{y} \in [\mathbf{a}, \mathbf{b}]} \|\mathbf{y} - \mathbf{x}\|_1 = \left\| \frac{\mathbf{b} - \mathbf{a}}{2} + \left| \mathbf{x} - \frac{\mathbf{a} + \mathbf{b}}{2} \right| \right\|_1$$

and

$$\mu_n(\mathbf{x}, [\mathbf{a}, \mathbf{b}]) = \left\| \frac{\mathbf{b} - \mathbf{a}}{2} + \left| \mathbf{x} - \frac{\mathbf{a} + \mathbf{b}}{2} \right| \right\|_1^n .$$

We proceed with some estimations which can be obtained when additional assumptions are made on f.

Definition 1. Consider a function $g : Q \to \mathbf{R}$ defined on a subset Q of $\mathbf{R}^m$, $m \in \mathbf{N}$. Let $\mathbf{L} = (L_1, \dots, L_m)$ and $\mathbf{r} = (r_1, \dots, r_m)$ where $L_i \ge 0$, $r_i > 0$, $i = \mathbf{1}, \dots, m$. We say that g is of class $C_{\mathbf{r}, \mathbf{L}}(Q)$ if

$$|g(\mathbf{u}) - g(\mathbf{v})| \le \sum_{i=1}^{m} L_i \, |u_i - v_i|^{r_i} \quad \text{for all } \mathbf{u}, \mathbf{v} \in Q.$$

If g is of class $C_{(1,\dots,1),\mathbf{L}}(Q)$, that is, if

$$|g(\mathbf{u}) - g(\mathbf{v})| \le \sum_{i=1}^{m} L_i \, |u_i - v_i| \quad \text{for all } \mathbf{u}, \mathbf{v} \in Q,$$

then we say that g is $\mathbf{L}$-Lipschitzian. If g is $(L, \dots, L)$-Lipschitzian for some $L > 0$, that is, if

$$|g(\mathbf{u}) - g(\mathbf{v})| \le L \sum_{i=1}^{m} |u_i - v_i| = L\|\mathbf{u} - \mathbf{v}\|_1 \quad \text{for all } \mathbf{u}, \mathbf{v} \in Q,$$

then we say that g is L-Lipschitzian. Finally we say that g is of class $C_{r,L}(Q)$ for some $r > 0$ and $L > 0$ if

$$|g(\mathbf{u}) - g(\mathbf{v})| \le L \, \|\mathbf{u} - \mathbf{v}\|_1^r \quad \text{for all } \mathbf{u}, \mathbf{v} \in Q.$$

Theorem 3. *Let $f : V \to \mathbf{R}$ be a function defined on an open subset V of $\mathbf{R}^m$, $m \in \mathbf{N}$. Let Q be any compact and convex subset of V such that $\operatorname{Vol}(Q) = \int_Q d\mathbf{y} > 0$, and let $w : Q \to [0, \infty)$ be a weight function on Q. Suppose $f \in C^n(Q)$ for some $n \in \mathbf{N}$.*

(i) *Suppose $f_{\boldsymbol{\alpha}}$ is of class $C_{\mathbf{r},\mathbf{L}}(Q)$ for all $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$, $\alpha_i \in \mathbf{N} \cup \{0\}$, $i = \mathbf{1}, \dots, m$, such that $|\boldsymbol{\alpha}| = n$. Then for any $\mathbf{x} \in Q$ we have*

$$\begin{aligned} &|\mathcal{O}_n(\mathbf{x}, f; w)| \\ &\le \frac{1}{m_0(Q; w)} \sum_{i=1}^{m} \frac{L_i \Gamma(r_i + \mathbf{1})}{\Gamma(r_i + \mathbf{1} + n)} \int_Q \|\mathbf{y} - \mathbf{x}\|_1^n \, |y_i - x_i|^{r_i} \, w(\mathbf{y}) \, d\mathbf{y} \qquad (3.6) \\ &\le \sum_{i=1}^{m} \frac{L_i \Gamma(r_i + 1)}{\Gamma(r_i + 1 + n)} \, \mu_{n,i}(\mathbf{x}, Q), \end{aligned}$$

where

$$\mu_{n,i}(\mathbf{x}, Q) = \max_{\mathbf{y} \in Q} \|\mathbf{y} - \mathbf{x}\|_1^n \, |y_i - x_i|^{r_i}, \quad i = \mathbf{1}, \dots, m.$$

If, additionally, w satisfies (3.3)*, then*

$$|\mathcal{O}_n(\mathbf{x},f;w)| \leq \sum_{i=1}^{m} \frac{L_i\Gamma(r_i+1)}{\Gamma(r_i+1+n)} \cdot \frac{\lambda\mu_{n,i}(\mathbf{x},Q)S_{n,i}(\mathbf{x},Q)}{\mu_{n,i}(\mathbf{x},Q)+(\lambda-1)S_{n,i}(\mathbf{x},Q)}, \tag{3.7}$$

where

$$S_{n,i}(\mathbf{x},Q) = \frac{1}{\mathrm{Vol}\,(Q)} \int_Q \|\mathbf{y}-\mathbf{x}\|_1^n \, |y_i - x_i|^{r_i}\, d\mathbf{y}, \quad i = \mathbf{1}, \dots, m.$$

(ii) *Suppose $f_{\boldsymbol{\alpha}}$ is of class $C_{r,L}(Q)$ for all $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$, $\alpha_i \in \mathbf{N} \cup \{0\}$, $i = 1, \dots, m$, such that $|\boldsymbol{\alpha}| = n$. Then, for any $\mathbf{x} \in Q$, we have*

$$\begin{aligned} |\mathcal{O}_n(\mathbf{x},f;w)| &\leq \frac{L\Gamma(r+1)}{\Gamma(r+1+n)m_{\mathbf{0}}(Q;w)} \int_Q \|\mathbf{y}-\mathbf{x}\|_1^{n+r}\, w(\mathbf{y})\, d\mathbf{y} \\ &\leq \frac{L\Gamma(r+1)}{\Gamma(r+1+n)}\, \mu_{n+r}(\mathbf{x},Q), \end{aligned} \tag{3.8}$$

where $\mu_{n+r}(\mathbf{x},Q)$ is defined by (3.2)*. If additionally w satisfies* (3.3)*, then*

$$|\mathcal{O}_n(\mathbf{x},f;w)| \leq \frac{L\Gamma(r+1)}{\Gamma(r+1+n)} \cdot \frac{\lambda\mu_{n+r}(\mathbf{x},Q)S_{n+r}(\mathbf{x},Q)}{\mu_{n+r}(\mathbf{x},Q)+(\lambda-1)S_{n+r}(\mathbf{x},Q)}, \tag{3.9}$$

where $S_{n+r}(\mathbf{x},Q)$ is defined by (3.5)*.*

Proof. If $f_{\boldsymbol{\alpha}}$ is of class $C_{\mathbf{r},\mathbf{L}}(Q)$, then

$$\begin{aligned} |f_{\boldsymbol{\alpha}}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})| &\leq \sum_{i=1}^{m} L_i\, |t(y_i - x_i)|^{r_i} \\ &= \sum_{i=1}^{m} L_i\, t^{r_i} |y_i - x_i|^{r_i}, \end{aligned}$$

for any two $\mathbf{x}, \mathbf{y} \in Q$ and for any $t \in [0,1]$. This implies

$$\left| \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](\mathbf{1} - t)^{n-1}\, dt \right| \le \sum_{i=1}^m L_i \, |y_i - x_i|^{r_i} \int_0^1 t^{r_i} (1-t)^{n-1}\, dt = \sum_{i=1}^m L_i \, |y_i - x_i|^{r_i} \, B(r_i + 1, n),$$

where $B(u,v) := \int_0^1 t^{u-1}(1-t)^{v-1}\, dt$, $u > 0$, $v > 0$ is a beta function. We know that $B(u,v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$, where Γ is a gamma function. Also $\Gamma(n) = (n-1)!$, $n \in \mathbf{N}$ so that

$$\left| \int_0^1 [f_{\boldsymbol{\alpha}}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1 - t)^{n-1}\, dt \right| \le (n-1)! \sum_{i=1}^m \frac{L_i \Gamma(r_i + \mathbf{1})}{\Gamma(r_i + \mathbf{1} + n)} \, |y_i - x_i|^{r_i}.$$

Using this estimation we get from (2.5)

$$\begin{aligned} &|\mathcal{R}_n(\mathbf{x}, f; w)| \\ &\le \sum_{|\boldsymbol{\alpha}|=n} \frac{n}{\boldsymbol{\alpha}!} \int_Q |\mathbf{y} - \mathbf{x}|^{\boldsymbol{\alpha}} \left((n-1)! \sum_{i=1}^m \frac{L_i \Gamma(r_i+1)}{\Gamma(r_i + 1 + n)} \, |y_i - x_i|^{r_i} \right) w(\mathbf{y})\, d\mathbf{y} \\ &= \sum_{i=1}^m \frac{L_i \Gamma(r_i+1)}{\Gamma(r_i + 1 + n)} \int_Q \left(\sum_{|\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} \, |\mathbf{y} - \mathbf{x}|^{\boldsymbol{\alpha}} \right) |y_i - x_i|^{r_i} \, w(\mathbf{y})\, d\mathbf{y} \\ &= \sum_{i=1}^m \frac{L_i \Gamma(r_i+1)}{\Gamma(r_i + 1 + n)} \int_Q \|\mathbf{y} - \mathbf{x}\|_1^n \, |y_i - x_i|^{r_i} \, w(\mathbf{y})\, d\mathbf{y} \end{aligned}$$

and dividing this by $m_{\mathbf{0}}(Q; w) = \int_Q w(\mathbf{y})\, d\mathbf{y} > 0$ we get the first inequality in (3.6). To obtain the second inequality in (3.6) we only have to note that

$$\int_Q \|\mathbf{y} - \mathbf{x}\|_1^n \, |y_i - x_i|^{r_i} \, w(\mathbf{y})\, d\mathbf{y} \le \max_{\mathbf{y} \in Q} \|\mathbf{y} - \mathbf{x}\|_1^n \, |y_i - x_i|^{r_i} \int_Q w(\mathbf{y})\, d\mathbf{y} = \mu_{n,i}(\mathbf{x}, Q) m_{\mathbf{0}}(Q; w), \quad i = \mathbf{1}, \dots, m.$$

Further, for fixed $i \in \{\mathbf{1}, \dots, m\}$ we can set $g(\mathbf{y}) = \|\mathbf{y}-\mathbf{x}\|_1^n \, |y_i - x_i|^{r_i}$, $\mathbf{y} \in Q$ and note that

$$0 \le g(\mathbf{y}) \le \max_{\mathbf{y}\in Q} \|\mathbf{y}-\mathbf{x}\|_1^n \, |y_i - x_i|^{r_i} = \mu_{n,i}(\mathbf{x}, Q)$$

and

$$\begin{aligned} \frac{1}{\operatorname{Vol}(Q)} \int_Q g(\mathbf{y})\, dy &= \frac{1}{\operatorname{Vol}(Q)} \int_Q \|\mathbf{y}-\mathbf{x}\|_1^n \, |y_i - x_i|^{r_i}\, dy \\ &= S_{n,i}(\mathbf{x}, Q). \end{aligned}$$

If w satisfies (3.3), then we can apply Corollary D to obtain

$$\begin{aligned} &\frac{1}{m_{\mathbf{0}}(Q;w)} \sum_{i=1}^m \frac{L_i \Gamma(r_i+1)}{\Gamma(r_i+1+n)} \int_Q \|\mathbf{y}-\mathbf{x}\|_1^n \, |y_i - x_i|^{r_i}\, w(\mathbf{y})\, d\mathbf{y} \\ &\quad \le \sum_{i=1}^m \frac{L_i \Gamma(r_i+1)}{\Gamma(r_i+1+n)} \cdot \frac{\lambda \mu_{n,i}(\mathbf{x}, Q) S_{n,i}(\mathbf{x}, Q)}{\mu_{n,i}(\mathbf{x}, Q) - S_{n,i}(\mathbf{x}, Q) + \lambda S_{n,i}(\mathbf{x}, Q)} \end{aligned}$$

so that (3.7) follows from the first inequality in (3.6). Further, suppose $f_{\boldsymbol{\alpha}}$ is one of class $C_{r,L}(Q)$. Then

$$\left| f_{\boldsymbol{\alpha}}\big(\mathbf{x} + t(\mathbf{y}-\mathbf{x})\big) - f_{\boldsymbol{\alpha}}(\mathbf{x}) \right| \le L \, \|t(\mathbf{y}-\mathbf{x})\|_1^r = L t^r \, \|\mathbf{y}-\mathbf{x}\|_1^r$$

for any two $\mathbf{x}, \mathbf{y} \in Q$ and for any $t \in [0,1]$. This implies

$$\begin{aligned} &\left| \int_0^1 [f_{\boldsymbol{\alpha}}\big(\mathbf{x} + t(\mathbf{y}-\mathbf{x})\big) - f_{\boldsymbol{\alpha}}(\mathbf{x})](1-t)^{n-1}\, dt \right| \\ &\qquad \le L \, \|\mathbf{y}-\mathbf{x}\|_1^r \int_0^1 t^r (1-t)^{n-1}\, dt \\ &\qquad = L \, \|\mathbf{y}-\mathbf{x}\|_1^r \, B(r+1, n) \\ &\qquad = \frac{L\Gamma(r+1)(n-1)!}{\Gamma(r+1+n)} \, \|\mathbf{y}-\mathbf{x}\|_1^r. \end{aligned}$$

Using this estimation and (2.5), we have

$$\begin{aligned} |\mathcal{R}_n(\mathbf{x}, f; w)| &\le \sum_{|\boldsymbol{\alpha}|=n} \frac{n}{\boldsymbol{\alpha}!} \int_Q |\mathbf{y}-\mathbf{x}|^{\boldsymbol{\alpha}} \, \frac{L\Gamma(r+1)(n-1)!}{\Gamma(r+1+n)} \, \|\mathbf{y}-\mathbf{x}\|_1^r \, w(\mathbf{y})\, d\mathbf{y} \\ &= \frac{L\Gamma(r+1)}{\Gamma(r+1+n)} \int_Q \Bigg(\sum_{|\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} \, |\mathbf{y}-\mathbf{x}|^{\boldsymbol{\alpha}} \Bigg) \|\mathbf{y}-\mathbf{x}\|_1^r \, w(\mathbf{y})\, d\mathbf{y} \\ &= \frac{L\Gamma(r+1)}{\Gamma(r+1+n)} \int_Q \|\mathbf{y}-\mathbf{x}\|_1^{n+r} \, w(\mathbf{y})\, d\mathbf{y} \end{aligned}$$

and dividing this by $m_{\mathbf{0}}(Q;w) = \int_Q w(\mathbf{y})\,d\mathbf{y} > 0$ we get the first inequality in (3.8). The second inequality in (3.8) is a consequence of the estimation

$$\begin{aligned}\int_Q \|\mathbf{y}-\mathbf{x}\|_1^{n+r}\,w(\mathbf{y})\,d\mathbf{y} &\leq \max_{\mathbf{y}\in Q}\|\mathbf{y}-\mathbf{x}\|_1^{n+r}\int_Q w(\mathbf{y})\,d\mathbf{y}\\ &= \mu_{n+r}(\mathbf{x},Q)m_{\mathbf{0}}(Q;w).\end{aligned}$$

Finally, if $g(\mathbf{y}) = \|\mathbf{y}-\mathbf{x}\|_1^{n+r}$, $\mathbf{y}\in Q$, then

$$0 \leq g(\mathbf{y}) \leq \max_{\mathbf{y}\in Q}\|\mathbf{y}-\mathbf{x}\|_1^{n+r} = \mu_{n+r}(\mathbf{x},Q)$$

and

$$\frac{1}{\mathrm{Vol}\,(Q)}\int_Q g(\mathbf{y})\,d\mathbf{y} = \frac{1}{\mathrm{Vol}\,(Q)}\int_Q \|\mathbf{y}-\mathbf{x}\|_1^{n+r}\,d\mathbf{y} = S_{n+r}(\mathbf{x},Q),$$

so that when w satisfies (3.3) we can apply Corollary D to get

$$\begin{aligned}&\frac{1}{m_{\mathbf{0}}(Q;w)}\int_Q \|\mathbf{y}-\mathbf{x}\|_1^{n+r}\,w(\mathbf{y})\,d\mathbf{y}\\ &\qquad\leq \frac{\lambda\mu_{n+r}(\mathbf{x},Q)S_{n+r}(\mathbf{x},Q)}{\mu_{n+r}(\mathbf{x},Q)-S_{n+r}(\mathbf{x},Q)+\lambda S_{n+r}(\mathbf{x},Q)}.\end{aligned}$$

Now (3.9) follows from the first inequality in (3.8). □

It should be noted that, under the assumption that f is of class $C_{\mathbf{r},\mathbf{L}}(Q)$, inequalities (3.6) and (3.7) are valid for $n=0$ too. When $n=0$, (3.6) becomes

$$\begin{aligned}|\mathcal{O}_0(\mathbf{x},f;w)| &\leq \frac{1}{m_{\mathbf{0}}(Q;w)}\sum_{i=1}^m L_i\int_Q |y_i-x_i|^{r_i}\,w(\mathbf{y})\,d\mathbf{y}\\ &\leq \sum_{i=1}^m L_i\,\mu_{0,i}(\mathbf{x},Q),\end{aligned}$$

where

$$\mu_{0,i}(\mathbf{x},Q) = \max_{\mathbf{y}\in Q}|y_i-x_i|^{r_i},\quad i=1,\ldots,m;$$

while (3.7) reduces to

$$|\mathcal{O}_0(\mathbf{x}, f; w)| \le \sum_{i=1}^{m} L_i \cdot \frac{\lambda \mu_{0,i}(\mathbf{x}, Q) S_{0,i}(\mathbf{x}, Q)}{\mu_{0,i}(\mathbf{x}, Q) + (\lambda - 1) S_{0,i}(\mathbf{x}, Q)},$$

where

$$S_{0,i}(\mathbf{x}, Q) = \frac{1}{\mathrm{Vol}\,(Q)} \int_Q |y_i - x_i|^{r_i}\, d\mathbf{y}, \quad i = \mathbf{1}, \dots, m.$$

These inequalities can be obtained by applying Theorem B. It is enough to set X to be a linear space of all functions $f : Q \to \mathbf{R}$ integrable on Q and define $A : X \to \mathbf{R}$ by

$$A(f) := \frac{1}{\mathrm{Vol}\,(Q)} \int_Q f(\mathbf{y})\, d\mathbf{y}, \quad f \in X.$$

Moreover, under the assumption that f is of class $C_{r,L}(Q)$, the second part of Theorem 3 is valid for $n = 0$ too. When $n = 0$, (3.8) and (3.9) reduce to

$$\begin{aligned} |\mathcal{O}_0(\mathbf{x}, f; w)| &\le \frac{L}{m_{\mathbf{0}}(Q; w)} \int_Q \|\mathbf{y} - \mathbf{x}\|_1^r\, w(\mathbf{y})\, d\mathbf{y} \\ &\le L\, \mu_r(\mathbf{x}, Q), \end{aligned} \tag{3.10}$$

and

$$|\mathcal{O}_0(\mathbf{x}, f; w)| \le L \cdot \frac{\lambda \mu_r(\mathbf{x}, Q) S_r(\mathbf{x}, Q)}{\mu_r(\mathbf{x}, Q) + (\lambda - 1) S_r(\mathbf{x}, Q)}, \tag{3.11}$$

respectively. It is not hard to prove (3.10) and (3.11) directly. Namely, we have

$$|f(\mathbf{y}) - f(\mathbf{x})| \le L\, \|\mathbf{y} - \mathbf{x}\|_1^r, \quad \mathbf{x}, \mathbf{y} \in Q.$$

Multiplying by $w(\mathbf{y}) \ge 0$ and integrating over $\mathbf{y} \in Q$, we get

$$\begin{aligned} \left| \int_Q f(\mathbf{y}) w(\mathbf{y})\, d\mathbf{y} - f(\mathbf{x}) \int_Q w(\mathbf{y})\, d\mathbf{y} \right| &\le \int_Q |f(\mathbf{y}) - f(\mathbf{x})|\, w(\mathbf{y})\, d\mathbf{y} \\ &\le L \int_Q \|\mathbf{y} - \mathbf{x}\|_1^r\, w(\mathbf{y})\, d\mathbf{y}. \end{aligned}$$

Dividing this by $m_{\mathbf{0}}(Q;w) = \int_Q w(\mathbf{y})\,d\mathbf{y} > 0$, we get the first inequality in (3.10). The rest of the proof is the same as for (3.8) and (3.9).

In connection with Theorem 3, observations similar to those given in Remarks 1 and 2 can be made.

3.2 Inequalities involving functions of class $C^{n+1}(Q)$. In this section we use the integral identity (2.6) to obtain some further bounds on the quantity $|\mathcal{O}_n(\mathbf{x}, f; w)|$.

Theorem 4. *Let $f : V \to \mathbf{R}$ be a function defined on an open subset V of $\mathbf{R}^m$, $m \in \mathbf{N}$. Let Q be any compact and convex subset of V such that $\mathrm{Vol}\,(Q) = \int_Q d\mathbf{y} > 0$, and let $w : Q \to [0, \infty)$ be a weight function on Q. Suppose $f \in C^{n+1}(Q)$ for some $n \in \mathbf{N} \cup \{0\}$.*

(i) *For any $\mathbf{x} \in Q$ we have*

$$
\begin{aligned}
|\mathcal{O}_n(\mathbf{x}, f; w)| &\le \frac{1}{m_{\mathbf{0}}(Q;w)} \sum_{|\boldsymbol{\alpha}|=n+1} \frac{\|f_{\boldsymbol{\alpha}}\|_\infty}{\boldsymbol{\alpha}!} M_{\boldsymbol{\alpha}}(\mathbf{x}, Q; w) \\
&\le \frac{D_{n+1}(f)}{(n+1)!\,m_{\mathbf{0}}(Q;w)} \int_Q \|\mathbf{y} - \mathbf{x}\|_1^{n+1} w(\mathbf{y})\,d\mathbf{y} \qquad (3.12) \\
&\le \frac{D_{n+1}(f)}{(n+1)!} \mu_{n+1}(\mathbf{x}, Q),
\end{aligned}
$$

where $\mu_{n+1}(\mathbf{x}, Q)$ is defined by (3.2) *and*

$$
D_{n+1}(f) := \max_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|=n+1} \|f_{\boldsymbol{\alpha}}\|_\infty .
$$

(ii) *If, additionally, w satisfies* (3.3), *then for any $\mathbf{x} \in Q$,*

$$
|\mathcal{O}_n(\mathbf{x}, f; w)| \le \frac{D_{n+1}(f)}{(n+\mathbf{1})!} \cdot \frac{\lambda \mu_{n+1}(\mathbf{x}, Q) S_{n+1}(\mathbf{x}, Q)}{\mu_{n+1}(\mathbf{x}, Q) + (\lambda - 1) S_{n+1}(\mathbf{x}, Q)}, \qquad (3.13)
$$

where $\mu_{n+1}(\mathbf{x}, Q)$ is defined by (3.2) *and $S_{n+1}(\mathbf{x}, Q)$ is defined by* (3.5).

Proof. If $f \in C^{n+1}(Q)$ for some $n \in \mathbf{N} \cup \{0\}$, then for any partial derivative $f_{\boldsymbol{\alpha}}$ with $|\boldsymbol{\alpha}| = n+1$ and for any $\mathbf{x} \in Q$

$$\left| \int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x} + t(\mathbf{y}-\mathbf{x}))(\mathbf{1}-t)^n \, dt \right| \leq \|f_{\boldsymbol{\alpha}}\|_\infty \int_0^1 (\mathbf{1}-t)^n \, dt = \frac{\|f_{\boldsymbol{\alpha}}\|_\infty}{n+\mathbf{1}}.$$

Using this estimation we get, from (2.6),

$$\begin{aligned} |\mathcal{R}_n(\mathbf{x}, f; w)| &\leq \sum_{|\boldsymbol{\alpha}|=n+1} \frac{\|f_{\boldsymbol{\alpha}}\|_\infty}{\boldsymbol{\alpha}!} \int_Q |\mathbf{y}-\mathbf{x}|^{\boldsymbol{\alpha}} w(\mathbf{y}) \, d\mathbf{y} \\ &= \sum_{|\boldsymbol{\alpha}|=n+1} \frac{\|f_{\boldsymbol{\alpha}}\|_\infty}{\boldsymbol{\alpha}!} M_{\boldsymbol{\alpha}}(\mathbf{x}, Q; w) \\ &\leq D_{n+1}(f) \sum_{|\boldsymbol{\alpha}|=n+1} \frac{M_{\boldsymbol{\alpha}}(\mathbf{x}, Q; w)}{\boldsymbol{\alpha}!} \\ &= \frac{D_{n+1}(f)}{(n+1)!} \int_Q \Bigg(\sum_{|\boldsymbol{\alpha}|=n+1} \frac{(n+1)!}{\boldsymbol{\alpha}!} |\mathbf{y}-\mathbf{x}|^{\boldsymbol{\alpha}} \Bigg) w(\mathbf{y}) \, d\mathbf{y} \\ &= \frac{D_{n+1}(f)}{(n+1)!} \int_Q \|\mathbf{y}-\mathbf{x}\|_1^{n+1} w(\mathbf{y}) \, d\mathbf{y} \\ &\leq \frac{D_{n+1}(f)}{(n+1)!} \max_{\mathbf{y} \in Q} \|\mathbf{y}-\mathbf{x}\|_1^{n+1} \int_Q w(\mathbf{y}) \, d\mathbf{y} \\ &= \frac{D_{n+1}(f)}{(n+1)!} \mu_{n+1}(\mathbf{x}, Q) m_0(Q; w) \end{aligned}$$

and dividing this by $m_{\mathbf{0}}(Q; w) = \int_Q w(\mathbf{y}) \, d\mathbf{y} > 0$ we get (3.12). To prove (3.13), we use the argument similar to that used for (3.4). ◻

Corollary 1. *Let the assumptions of Theorem* 4 *be satisfied. Additionally, suppose that, for some* $\mathbf{x} \in Q$ *all partial derivatives* $f_{\boldsymbol{\alpha}}$*,*

$\mathbf{1} \leq |\boldsymbol{\alpha}| \leq n$ *fulfill* $f_{\boldsymbol{\alpha}}(\mathbf{x}) = 0$. *Then we have*

$$\begin{aligned}\left|\frac{\int_Q f(\mathbf{y})w(\mathbf{y})\,d\mathbf{y}}{m_{\mathbf{0}}(Q;w)} - f(\mathbf{x})\right| \\ &\leq \frac{1}{m_{\mathbf{0}}(Q;w)} \sum_{|\boldsymbol{\alpha}|=n+1} \frac{\|f_{\boldsymbol{\alpha}}\|_\infty}{\boldsymbol{\alpha}!} M_{\boldsymbol{\alpha}}(\mathbf{x},Q;w) \\ &\leq \frac{D_{n+1}(f)}{(n+1)!m_{\mathbf{0}}(Q;w)} \int_Q \|\mathbf{y}-\mathbf{x}\|_1^{n+1} w(\mathbf{y})\,d\mathbf{y} \\ &\leq \frac{D_{n+1}(f)}{(n+1)!}\,\mu_{n+1}(\mathbf{x},Q).\end{aligned} \tag{3.14}$$

If additionally w *satisfies* (3.3), *then*

$$\begin{aligned}\left|\frac{\int_Q f(\mathbf{y})w(\mathbf{y})\,d\mathbf{y}}{m_{\mathbf{0}}(Q;w)} - f(\mathbf{x})\right| \\ &\leq \frac{D_{n+1}(f)}{(n+\mathbf{1})!} \cdot \frac{\lambda\mu_{n+1}(\mathbf{x},Q)S_{n+1}(\mathbf{x},Q)}{\mu_{n+1}(\mathbf{x},Q) + (\lambda-1)S_{n+1}(\mathbf{x},Q)}.\end{aligned}$$

Proof. Since $f_{\boldsymbol{\alpha}}(\mathbf{x}) = 0$ for $1 \leq |\boldsymbol{\alpha}| \leq n$, we have

$$\mathcal{O}_n(\mathbf{x},f;w) = \frac{\int_Q f(\mathbf{y})w(\mathbf{y})\,dy}{m_{\mathbf{0}}(Q;w)} - f(\mathbf{x})$$

and the desired result follows by Theorem 4. □

Remark 3. When $w(\mathbf{y}) = \mathbf{1}$ for all $\mathbf{y} \in Q$, the first part of the above corollary coincides with Theorem E. Namely, in that case we have $m_{\mathbf{0}}(Q;\mathbf{1}) = \int_Q d\mathbf{y} = \mathrm{Vol}\,(Q)$. Also

$$\begin{aligned}\sum_{|\boldsymbol{\alpha}|=n+1} \frac{\|f_{\boldsymbol{\alpha}}\|_\infty}{\boldsymbol{\alpha}!} M_{\boldsymbol{\alpha}}(\mathbf{x},Q;1) \\ &= \frac{1}{(n+1)!} \int_Q \left(\sum_{|\boldsymbol{\alpha}|=n+1} \frac{(n+1)!}{\boldsymbol{\alpha}!} |\mathbf{y}-\mathbf{x}|^{\boldsymbol{\alpha}} \|f_{\boldsymbol{\alpha}}\|_\infty \right) d\mathbf{y} \\ &= \frac{1}{(n+1)!} \int_Q \left(\sum_{i=1}^m |y_i - x_i| \left\| \frac{\partial}{\partial z_i} \right\|_\infty \right)^{n+1} f(\mathbf{y})\,d\mathbf{y},\end{aligned}$$

so that the first and the second inequality in (3.14) coincide with (1.12).

Remark 4. In connection with Theorem 4 and with Corollary 1, observations similar to those given in Remarks 1 and 2 can be made.

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