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A QUARTIC SURFACE OF INTEGER HEXAHEDRA

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ABSTRACT. We prove that there are infinitely many six-sided polyhedra in $\mathbf{R}^3$, each with four congruent trapezoidal faces and two congruent rectangular faces, so that the faces have integer sides and diagonals, and also the solid has integer length diagonals. The solutions are obtained from the integer points on a particular quartic surface.

A long standing unsolved problem asks whether or not there can be a parallelepiped in $\mathbf{R}^3$ whose sides and diagonals have integer length. If one weakens the requirement and just asks for a six-sided polyhedron with quadrilateral faces, then one can find examples with integer length sides and diagonals. Peterson and Jordan [**1**] described a method for making these 'perfect' hexahedra. We review their method.

Take two congruent rectangles positioned as if they formed the top and bottom parallel faces of a rectangular box. Rotate the top rectangle by 90 degrees around the axis joining the centers of these two rectangles. Now connect the sides of the two rectangles with four congruent trapezoids. (The shape can be viewed as a piecewise linear version of the placement of two cupped hands together, at 90 degrees in clapping position.) If the sides of the rectangle have lengths a, b, then the diagonal has length c, where $a^2 + b^2 = c^2$. The parallel sides of the trapezoids are then also a, b. If the slant side of the trapezoid is, say, e and its diagonal is d, then it follows from Ptolemy's theorem that $d^2 = e^2 + ab$. Consider the trapezoid with base on the top rectangle of side a and other base on the bottom rectangle opposite that edge of side b having the slant sides of length d. Its diagonal is of length f, and also it is the interior diagonal of the hexahedron; thus, $f^2 = d^2 + ab$. We shall refer to such polyhedra, for which these six parameters are integral, as perfect hexahedra.

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Proposition 1. *The simultaneous positive integer solutions to* $a^2 + b^2 = c^2$, $d^2 = e^2 + ab$, $f^2 = d^2 + ab$ *give the edge and diagonal lengths of a perfect hexahedron with two opposite parallel congruent rectangular faces and four congruent trapezoidal faces.*

Peterson and Jordan asked if there are infinitely many such perfect hexahedra. They also gave several examples, including the small example $a = 8$, $b = 15$, $c = 17$, $d = 13$, $e = 7$, $f = 17$ and asked if it is the smallest. We provide a measure of size of solutions and answer both of these questions affirmatively.

We can rewrite this set of equations. Basically, $d^2-e^2 = f^2-d^2 = ab$, so that we have three integer squares in arithmetic progression. This is the same as $e^2 + f^2 = 2d^2$ and $ab = f^2 - d^2$. Now then we have two norm equations $a^2 + b^2 = c^2$, $e^2 + f^2 = 2d^2$ over $Z[i]$.

It is well known that the relatively prime or primitive solutions to the Pythagorean equation, $x^2 + y^2 = z^2$, are given by $x = m^2 - n^2$, $y = 2mn$, $z = m^2 + n^2$ for (relatively prime) integers m and n. All integer solutions are scalar (integer) multiples of the primitive solutions. To obtain solutions to $x^2 + y^2 = 2z^2$, we can use those obtained from the solutions to the Pythagorean equation. Namely, corresponding to a solution we form the complex number, $x + iy$ and observe that $|x+iy|^2 = x^2+y^2$ and $|(x+iy)(1+i)|^2 = 2(x^2+y^2)$; thus, a solution to the Pythagorean equation yields, after multiplication by $1+i$, a solution to the second equation. Conversely, a solution to $x^2 + y^2 = 2z^2$, gives the complex number $x = iy$ so that $|(x+iy)[(1-i)/2]|^2 = ((x^2+y^2)/2)$; thus, the complex number $(x + iy)[(1 - i)/2]$ provides a solution to the Pythagorean equation. This gives a bijection of the sets of solutions of these two equations. (Using this together with sign changes or interchanging of variables accounts for all solutions.)

For the perfect hexahedron then, we parameterize the integer solutions to the equation $a^2 + b^2 = c^2$ as $a = \lambda(r^2 - s^2)$, $b = \lambda(2rs)$. For the solutions to $e^2 + f^2 = 2d^2$, we parameterize with $p, q, e + if = \mu((p^2-q^2)+i2pq)(1+i)$, so that $e = \mu(p^2-q^2-2pq)$, $f = \mu(p^2-q^2+2pq)$. The condition for a perfect hexahedron is that $ab = ((f^2 - e^2)/2)$.

Proposition 2. *Using these parameterizations, a perfect hexahedron is obtained from any integer solution to* $2(p^2 - q^2)pq\mu^2 = (r^2 - s^2)rs\lambda^2$

with $r \neq \pm s$, $p \neq \pm q$, μ, λ, $r, s, p, q \neq 0$ *and conversely.*

Proof. As we have seen, the perfect hexahedron gives rise to the equation $ab = ((f^2 - e^2)/2)$ which, by our parameterization is a multiple of the equation $2(p^2 - q^2)pq\mu^2 = (r^2 - s^2)rs\lambda^2$. Conversely, given any nontrivial integer solution to this equation, we can form $a = \lambda(r^2 - s^2)$, $b = 2rs\lambda$, $c = \lambda(r^2 + s^2)$, $e = \mu(p^2 - q^2 - 2pq)$, $f = \mu(p^2 - q^2 + 2pq)$, $d = \mu(p^2 + q^2)$, which give nonzero integer solutions to the perfect hexahedron.

First we shall determine the smallest solution. We measure the *size* of a solution by the number $(ab/2)$ which is the same as $|(r^2 - s^2)rs\lambda^2|$.

Lemma 1. *The size of any solution of the hexahedra equations is divisible by* 60.

Proof. Consider the equation $a^2 + b^2 = c^2$. Modulo 3 the squares are 0 or 1 so it is impossible that a^2 and b^2 are both 1 mod 3. Thus ab is divisible by 3.

Consider this same equation mod 5. Modulo 5 the squares are 0, 1 or 4. Therefore, the only solutions mod 5 are $0+0 = 0$, $0+1 = 1$, $0+4 = 4$, $1 + 4 = 0$; in either case, abc is divisible by 5. Suppose if possible that ab is not divisible by 5. Using the parameterization described above, then 5 divides $c = \lambda(r^2 + s^2)$, but not any of $\lambda, r, s, r - s, r + s$. Also $(ab/2) = 2pq(p^2 - q^2)\mu^2$ for a nontrivial hexahedron. So therefore 5 does not divide any of $p, q, p + q, p - q, \mu$. However, then the numbers $r, s, r + s, r - s$ are all different mod 5 and nonzero; similarly, for $p, q, p + q, p - q$. Thus, for a solution to the perfect hexahedron, $48\mu^2 = 2(p^2 - q^2)pq\mu^2 = (r^2 - s^2)rs\lambda^2 = 24\lambda^2$ mod 5; this is impossible. Therefore, ab is divisible by 5.

Finally, the size is $|2pq(p^2 - q^2)\mu^2|$ which is certainly even, but also if p, q are both odd, $p^2 - q^2$ is even; so the size is always divisible by 4.

Theorem 1. *There is a unique perfect hexahedron of size* 60; *it has* $a = 8$, $b = 15$, $c = 17$, $d = 13$, $e = 7$, $f = 17$. *The second smallest is a hexahedron of size* 120; *it has* $a = 24$, $b = 10$, $c = 26$, $d = 16$, $e = 7$, $f = 23$.

Proof. Solving the equation $pq(p^2 - q^2)\mu^2 = 30$ in positive integers, we see immediately that $\mu = \pm 1$ and p, q are divisors of 30; say $p > q$ and $p + q > p > q > p - q > 0$ (or possibly $p + q > p > p - q > q > 0$). Hence $p - q = 1$. If $q \geq 3$, then $p \geq 5$ is impossible. It then follows easily that $p = 3$, $q = 2$. Similarly, solving the equation $rs(r^2 - s^2)\lambda^2 = 60$ in positive integers, we see that r, s, λ^2 are divisors of 60. Hence λ^2 is either 1 or 4. When $\lambda = 2$ it is easy to see that there are no solutions in integers to $rs(r^2 - s^2) = 15$. If $\lambda = 1$, suppose r is larger than s. Then either $r + s > r > s > r - s > 0$ or $r + s > r > r - s > s > 0$; if $r \geq 5$, then $r + s \geq 6$, then this is impossible. Thus, $r \leq 4$, and we have the solution $r = 4$, $s = 1$. Up to order and sign, then, these parameters describe the unique smallest positive solution of the theorem.

For the second smallest solution we solve $pq(p^2 - q^2)\mu^2 = 60$ as above to find $p = 4$, $q = 1$, $\mu = 1$. However, to solve $rs(r^2 - s^2)\lambda^2 = 120$, we consider the two possible cases for λ. If $\lambda = 1$ we arrange so that $r + s > r > s > r - s > 0$ or $r + s > r > r - s > s > 0$, and easily find that $r = 5$, $s = 1$ is the only solution. If $\lambda = 2$, then as in the previous case, we find $r = 3$, $s = 2$. These parameters give the solution stated.

Next we describe a method to produce an infinite number of different perfect hexahedra. We consider the 'primitive' equation, where $\lambda = \mu = 1$,

$$2(p^2 - q^2)pq = (r^2 - s^2)rs,$$

and look for a curve lying on the surface, see Figure 1, expressed in terms of the parameters of α, β; for example, in the (p, r) directions this would mean $2(\alpha^2 - q^2)\alpha q = (\beta^2 - s^2)\beta s$.

Here is one way to do that. Suppose that (p_0, q_0, r_0, s_0) is a rational solution; then (q, q_0, s, s_0) for any $(q, s) \in \{(\pm q_0, 0), (0, 0), (0, \pm s_0), (\pm q_0, \pm s_0)\}$ is also a solution. Given any two rational solutions in this set of nine, express the line passing through them as an equation in terms of x, y. The solutions for (x, y) meeting the surface gives new solutions, (x, q_0, y, s_0). Of these nine known points, lines may pass through three of these points and then meet the surface again at its points at infinity. However, several lines meet these nine at only two points; these give rise to new rational solutions. For example, there are lines from the 'first quadrant': the line through $(-q_0, 0), (q_0, s_0)$ or $(0, -s_0)$, (q_0, s_0) or $(0, -s_0)$, $(q_0, 0)$ or $(-q_0, -s_0)$, $(q_0, 0)$. In the first case the line is $y = (s_0/2)[(x/q_0) + 1]$ which meets the surface when $x = (3q_0 s_0^4/(16q_0^4 - s_0^4))$, $y = [s_0(s_0^4 + 8q_0^4)/(16q_0^4 - s_0^4)]$.

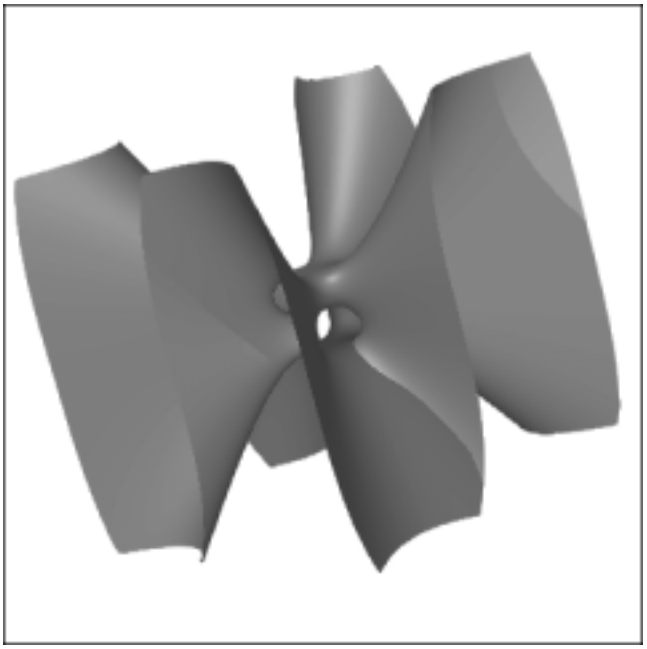

FIGURE 1. Quartic surface.

Now we change q_0, s_0 to the parameters q, s and clear denominators to obtain a parameterized curve on the surface. In a similar way, we obtain the five other parameterized curves.

$$(1) \qquad (3qs^4, q(16q^4 - s^4), s(s^4 + 8q^4), s(16q^4 - s^4))$$
$$(2) \qquad (-3qs^4, q(16q^4 + s^4), -s(-s^4 + 8q^4), s(16q^4 + s^4))$$
$$(3) \qquad (-q(q^4 + 2s^4), q(q^4 - 4s^4), -3qs^4, s(q^4 - 4s^4))$$
$$(4) \qquad (q(q^4 - 2s^4), q(q^4 + 4s^4), -3qs^4, s(q^4 + 4s^4))$$
$$(5) \qquad (-2q(s^4 - q^4), q(2q^4 + s^4), s(-s^4 + 4q^4), s(2q^4 + s^4))$$
$$(6) \qquad (2q(s^4 + q^4), q(2q^4 - s^4), s(s^4 + 4q^4), s(2q^4 - s^4)).$$

These six different rational quintic curves can be slightly modified with sign changes; however, the sign changes yield essentially the same

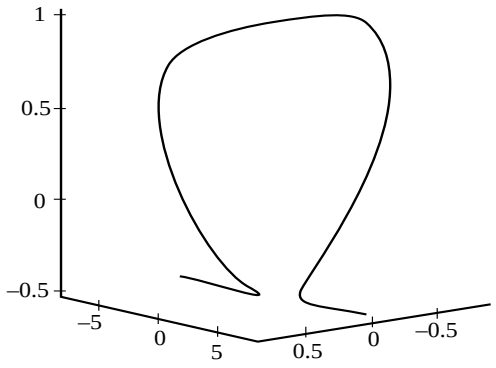

FIGURE 2. Parameterized curve of solutions in 3-space.

solutions for a, b, c, d, e, f. These may be the smallest degree rational curves on the surface which also contain rational points. This could account for the sparsity of solutions as observed in [**1**].

For the second parameterization above, we can view it in 3-space by dehomogenizing with respect to the last variable and letting $t = (q/s)$ give $[(-3t/(16t^4 + 1)), t, ((1 - 8t^4)/(16t^4 + 1))]$, which is unbounded, having asymptote $x = 0$, $z = -(1/2)$, see Figure 2. Moreover, by projection into the plane of the first x and third z variable, we obtain the curve with equation $8x^4+8z^4+4z^3-6z^2-5z-1 = 0$, see Figure 3. The quartic surface, see Figure 1, has an easily discernible hole; the

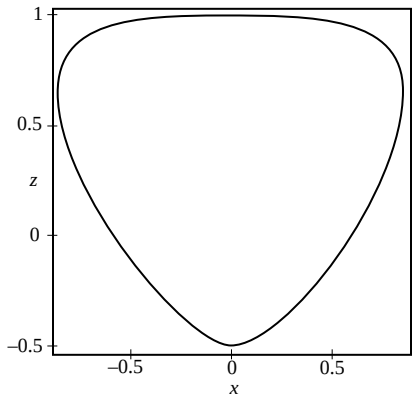

FIGURE 3. Projection of space curve into $x - z$ plane.

three-space curve follows a fold and over the central bridge, then along an opposite fold in the surface.

We obtain integer solutions for a, b, c, d, e, f using the parametrizations discussed above; for an integer $q = n$, $s = 1$ and using this particular curve discussed, we have the positive solutions

$$a = 48n^4(4n^4 + 1), b = 2(-1 + 8n^4)(1 + 16n^4), c = 2 + 16n^4 + 320n^8$$

and

$$\begin{aligned} d &= 2n^2(5 + 16n^4 + 128n^8), \\ e &= 2n^2(-7 - 32n^4 + 128n^8), \\ f &= 2n^2(-1 + 64n^4 + 128n^8) \end{aligned}$$

which give infinitely many distinct integral solutions on the curve since $(a/b), (e/f)$ yield infinitely many distinct rationals.

An easy calculation of the surface area for these examples shows that it grows like a polynomial function in n^{18}. Also we can position the hexahedron in space with the center of both the top and bottom rectangles meeting the z-axis; a corner of the bottom rectangle is placed at $[-(a/2), (b/2), 0]$ and the top rectangle is placed with corner at $[(b/2), (a/2), h]$. We find that the height h satisfies $a^2 + b^2 + 2h^2 = 2d^2 = e^2 + f^2$. From this we see that $h^2 = 2(2n^4 - 1)(-1 + 8n^4)(32n^4 - 1)(64n^8 + 8n^4 + 1)$; thus, the height is nonzero. Summarizing this we have the following.

Theorem 2. *There are infinitely many solutions to $a^2 + b^2 = c^2$, $d^2 = e^2 + ab$, $f^2 = d^2 + ab$. Thus there are infinitely many nontrivial dissimilar perfect hexahedra with two opposite parallel congruent rectangular faces, and four congruent trapezoidal faces.*

REFERENCES

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