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INVERTIBILITY AND POSITIVE INVERTIBILITY OF INTEGRAL OPERATORS IN L^∞

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ABSTRACT. Integral operators on space $L^\infty[0,1]$ are considered. Invertibility conditions, estimates for the norm of the inverse operators and positive invertibility conditions are established. In addition, bounds for the spectral radius are suggested. Applications to nonselfadjoint differential operators and integro-differential ones are discussed.

1. Introduction and statement of the main result. A lot of papers and books are devoted to the spectrum of integral operators. Mainly, the distributions of the eigenvalues are considered, cf. [**6**], [**10**], [**11**] and references therein. However, in many applications, for example, in numerical mathematics and stability analysis, bounds for eigenvalues and invertibility conditions are very important. But the bounds and invertibility conditions are investigated considerably less than the distributions.

In the present paper we consider linear integral operators on space $L^\infty[0,1]$. The following problems are investigated: invertibility conditions, estimates for the norm of the inverse operator, positive invertibility conditions, and estimates for the spectral radius.

A few words about the contents. In the present section we state the main result of the paper, Theorem 1.1 on the invertibility of integral operators. This theorem supplements the well-known results on the invertibility of linear operators, cf. [**5**]. Note that the invertibility conditions of integral operators in space L^2 with the Hilbert-Schmidt and Neumann-Schatten kernels were established in [**4**].

The proof of Theorem 1.1 is divided into a series of lemmas which are presented in Sections 2 and 3. In Section 4, by virtue of the main result,

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we establish new estimates for the spectral radius which supplement the well-known ones, cf. [**8**]. Section 5 deals with positive invertibility of integral operators. Section 6 is devoted to applications of the above-mentioned results to nonselfadjoint differential operators and integro-differential operators. Besides, our results supplement the well-known ones, cf. [**1**], [**2**], [**7**] and references therein.

Recall that $L^\infty \equiv L^\infty[0,1]$ is the space of scalar-valued functions defined on $[0,1]$ and equipped with the norm

$$|h|_{L^\infty} = \text{ess} \sup_{x\in[0,1]} |h(x)|, \quad h \in L^\infty.$$

Everywhere below $\tilde{K}$ is a linear operator in L^∞ defined by

$$(\tilde{K}h)(x) = \int_0^1 K(x,s)h(s)\,ds, \quad h \in L^\infty, \tag{1.1}$$

where $K(x,s)$ is a scalar kernel defined on $[0,1]^2$ and having the property

$$\int_0^1 \text{ess} \sup_{x\in[0,1]} |K(x,s)|\,ds < \infty. \tag{1.2}$$

Define the Volterra operators

$$(V_- h)(x) = \int_0^x K(x,s)h(s)\,ds \tag{1.3}$$

and

$$(V_+ h)(x) = \int_x^1 K(x,s)h(s)\,ds. \tag{1.4}$$

Set

$$w_-(s) \equiv \text{ess} \sup_{0\le s\le x\le 1} |K(x,s)|, \quad w_+(s) \equiv \text{ess} \sup_{0\le x\le s\le 1} |K(x,s)|$$

and

$$M_\infty(V_\pm) \equiv \int_0^1 w_\pm(s)\,ds.$$

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let the conditions* (1.2) *and*

$$e^{M_\infty(V_-)+M_\infty(V_+)} < e^{M_\infty(V_+)} + e^{M_\infty(V_-)} \tag{1.5}$$

hold. Then operator $I-\tilde{K}$ is boundedly invertible in L^∞ and the inverse operator satisfies the inequality

$$|(I-\tilde{K})^{-1}|_{L^\infty} \le \frac{e^{M_\infty(V_-)+M_\infty(V_+)}}{e^{M_\infty(V_+)} + e^{M_\infty(V_-)} - e^{M_\infty(V_-)+M_\infty(V_+)}}. \tag{1.6}$$

Note that condition (1.5) is equivalent to the following one:

$$\theta(K) \equiv (e^{M_\infty(V_+)}-1)(e^{M_\infty(V_-)}-1) < 1. \tag{1.7}$$

Besides (1.6) takes the form

$$|(I-\tilde{K})^{-1}|L^\infty \le \frac{e^{M_\infty(V_-)+M_\infty(V_+)}}{1-\theta(K)}. \tag{1.8}$$

2. Preliminaries. Let X be a Banach space with a norm $\|\cdot\|$. Recall that a linear operator $\tilde{V}$ in X is called a quasinilpotent one if

$$\lim_{n\to\infty} \sqrt[n]{\|\tilde{V}^n\|} = 0.$$

For a quasinilpotent operator $\tilde{V}$ in X, put

$$j(\tilde{V}) \equiv \sum_{k=0}^{\infty} \|\tilde{V}^k\|.$$

Lemma 2.1. *Let A be a bounded linear operator in X of the form*

$$A = I + V + W \tag{2.1}$$

where operators V and W are quasinilpotent. If, in addition, the condition

$$\theta_A \equiv \left\| \sum_{j,k=1}^{\infty} (-1)^{k+j} V^k W^j \right\| < 1 \tag{2.2}$$

is fulfilled, then operator A is boundedly invertible and the inverse operator satisfies the inequality

$$\|A^{-1}\| \le \frac{j(V)j(W)}{1-\theta_A}.$$

Proof. We have

$$A = I + V + W = (I+V)(I+W) - VW. \tag{2.3}$$

Since W and V are quasinilpotent, the operators $I+V$ and $I+W$ are invertible.

$$(I+V)^{-1} = \sum_{k=0}^{\infty} (-1)^k V^k, \qquad (I+W)^{-1} = \sum_{k=0}^{\infty} (-1)^k W^k. \tag{2.4}$$

Thus,

$$\begin{aligned} I + V + W &= (I+V)[I - (I+V)^{-1} VW (I+W)^{-1}](I+W) \\ &= (I+V)(I - B_A)(I+W), \end{aligned} \tag{2.5}$$

where

$$B_A = (I+V)^{-1} VW (I+W)^{-1}. \tag{2.6}$$

But according to (2.4)

$$V(I+V)^{-1} = \sum_{k=1}^{\infty} (-1)^{k-1} V^k, \qquad (I+W)^{-1} = \sum_{k=1}^{\infty} (-1)^{k-1} W^k. \tag{2.7}$$

So

$$B_A = \sum_{j,k=1}^{\infty} (-1)^{k+j} V^k W^j. \tag{2.8}$$

If (2.2) holds, then $\|B_A\| < 1$ and

$$\|(I - B_A)^{-1}\| \le (1 - \theta_A)^{-1}.$$

So by (2.5) $I + V + W$ is invertible. Moreover, due to (2.3)

$$A^{-1} = (I + W)^{-1}(I - B_A)^{-1}(I + V)^{-1}. \tag{2.9}$$

But (2.4) implies

$$\|(I + W)^{-1}\| \le j(W), \qquad \|(I + V)^{-1}\| \le j(V).$$

Now the required inequality for A^{-1} follows from (2.9). □

Furthermore, take into account that by (2.7)

$$\|V(I + V)^{-1}\| \le \sum_{k=1}^{\infty} \|V^k\| \le j(V) - 1. \tag{2.10}$$

Similarly,

$$\|W(I + W)^{-1}\| \le j(W) - 1. \tag{2.11}$$

Thus,

$$\theta_A \le (j(W) - 1)(j(V) - 1).$$

So condition (2.2) is provided by the inequality

$$(j(W) - 1)(j(V) - 1) < 1.$$

The latter inequality is equivalent to the following one:

$$j(W)j(V) < j(W) + j(V). \tag{2.12}$$

Lemma 2.1 yields

Corollary 2.2. *Let V, W be quasinilpotent and condition* (2.12) *be fulfilled. Then operator A defined by* (2.1) *is boundedly invertible and the inverse operator satisfies the inequality*

$$\|A^{-1}\| \le \frac{j(V)j(W)}{j(W) + j(V) - j(W)j(V)}.$$

Let us turn now to integral operator $\tilde{K}$. Under condition (1.2), operators $V_\pm$ are quasinilpotent due to the well-known theorem V.6.2 [**12**, p. 153]. Now Corollary 2.2 yields

Corollary 2.3. *With the notation*

$$j(V_\pm) \equiv \sum_{k=0}^{\infty} |V_\pm^k|_{L^\infty},$$

let the conditions (1.2) *and*

$$j(V_+)j(V_-) < j(V_+) + j(V_-)$$

be fulfilled. Then $I - \tilde{K}$ is boundedly invertible in L^∞ and the inverse operator satisfies the inequality

$$|(I - \tilde{K})^{-1}|_{L^\infty} \le \frac{j(V_-)j(V_+)}{j(V_-) + j(V_+) - j(V_-)j(V_+)}.$$

3. Powers of Volterra operators.

Lemma 3.1. *Under condition* (1.2)*, operator V_- defined by* (1.3) *satisfies the inequality*

$$|V_-^k|_{L^\infty} \le \frac{M_\infty^k(V_-)}{k!}, \quad k = 1, 2, \dots . \tag{3.1}$$

Proof. We have

$$|V_- h|_{L^\infty} = \text{ess} \sup_{x \in [0,1]} \left| \int_0^x K(x,s)h(s)\, ds \right| \le \int_0^1 w_-(s)|h(s)|\, ds.$$

Repeating these arguments, we arrive at the relation

$$|V_-^k h|_{L^\infty} \le \int_0^1 w_-(s_1) \int_0^{s_1} w_-(s_2) \dots \int_0^{s_k} |h(s_k)|\, ds_k \dots ds_2 ds_1.$$

Taking $|h|_{L^\infty} = 1$, we get

$$|V_-^k|_{L^\infty} \le \int_0^1 w_-(s_1) \int_0^{s_1} w_-(s_2) \dots \int_0^{s_{k-1}} ds_k \dots ds_2 ds_1. \tag{3.2}$$

It is simple to see that

$$\int_0^1 w_-(s_1) \dots \int_0^{s_{k-1}} w_0(s_k)\, ds_k \dots ds_1$$
$$= \int_0^{\tilde{\mu}} \int_0^{z_1} \dots \int_0^{z_{k-1}} dz_k dz_{k-1} \dots dz_1 = \frac{\tilde{\mu}^k}{k!},$$

where

$$z_j = z_k(s_j) \equiv \int_0^{s_j} w_-(s)\, ds, \quad j = 1, \dots, k$$

and

$$\tilde{\mu} = \int_0^1 w_-(s)\, ds.$$

Thus (3.2) gives

$$|V_-^k|_{L^\infty} \le \frac{(\int_0^1 w_-(s) ds)^k}{k!} = \frac{M_\infty^k(V_-)}{k!}$$

as claimed. □

Similarly, the inequality

$$|V_+^k|_{L^\infty} \le \frac{M_\infty^k(V_+)}{k!}, \quad k = 1, 2, \dots , \tag{3.3}$$

can be proved. Note that in the case L^2 estimates of powers of general Volterra operators are considered in [**3**, Section 17.2].

The assertion of Theorem 1.1 follows from Corollary 2.3 and relations (3.1), (3.3).

4. The spectral radius. Clearly,

$$\lambda I - \tilde{K} = \lambda(I - \lambda^{-1}\tilde{K}), \quad \lambda \ne 0.$$

Consequently, if

$$e^{(M_\infty(V_-)+M_\infty(V_+))|\lambda|^{-1}} \;<\; e^{|\lambda|^{-1}M_\infty(V_+)} + e^{|\lambda|^{-1}M_\infty(V_-)},$$

then due to Theorem 1.1, $\lambda I - \tilde{K}$ is boundedly invertible. Thus, we get

Lemma 4.1. *Under condition* (1.2), *any point* $\lambda \neq 0$ *of the spectrum* $\sigma(\tilde{K})$ *of operator* $\tilde{K}$ *satisfies the inequality*

$$e^{(M_\infty(V_-)+M_\infty(V_+))|\lambda|^{-1}} \;\geq\; e^{|\lambda|^{-1}M_\infty(V_+)} + e^{|\lambda|^{-1}M_\infty(V_-)}. \tag{4.1}$$

Let $r_s(\tilde{K}) = \sup|\sigma(\tilde{K})|$ *be the spectral radius of* $\tilde{K}$. *Then* (4.1) *yields*

$$e^{r_s^{-1}(\tilde{K})(M_\infty(V_-)+M_\infty(V_+))} \;\geq\; e^{r_s^{-1}(\tilde{K})M_\infty(V_+)} + e^{r_s^{-1}}(\tilde{K})M_\infty(V_-). \tag{4.2}$$

Clearly, if $V_+ = 0$ *or* (*and*) $V_- = 0$, *then* $r_s(\tilde{K}) = 0$.

Theorem 4.2. *Under condition* (1.2), *let* $V_+ \neq 0$, $V_- \neq 0$. *Then the equation*

$$e^{(M_\infty(V_-)+M_\infty(V_+))z} \;=\; e^{zM_\infty(V_+) + e^{zM_\infty}(V_-)}, \quad z \geq 0 \tag{4.3}$$

has a unique positive zero $z(K)$. *Moreover, the inequality* $r_s(\tilde{K}) \leq z^{-1}(K)$ *is valid.*

Proof. Equation (4.3) is equivalent to the following one:

$$(e^{M_\infty(V_+)z} - 1)(e^{zM_\infty(V_-)} - 1) \;=\; 1. \tag{4.4}$$

In addition, (4.2) is equivalent to the relation

$$(e^{r_s^{-1}(\tilde{K})M_\infty(V_+)} - 1)(e^{r_s^{-1}(\tilde{K})M_\infty(V_-)} - 1) \;\geq\; 1.$$

Hence, the result follows since the left part of equation (4.4) monotonically increases. $\square$

To estimate $z(K)$, let us consider the equation

$$\sum_{k=1}^{\infty} a_k z^k \;=\; 1 \tag{4.5}$$

where the coefficients a_k are nonnegative and have the property

$$\theta_0 \equiv 2 \max_k \sqrt[k]{a_k} < \infty.$$

Lemma 4.3. *The unique nonnegative root* z_0 *of equation* (4.5) *satisfies the estimate* $z_0 \geq 1/\theta_0$.

Proof. Set in (4.5) $z = x\theta_0^{-1}$. We have

$$1 = \sum_{k=1}^{\infty} a_k \theta_0^{-k} x^k. \tag{4.6}$$

But

$$\sum_{k=1}^{\infty} a_k \theta_0^{-k} \leq \sum_{k=1}^{\infty} 2^{-k} = 1,$$

and therefore the unique positive root x_0 of (4.6) satisfies the inequality $x_0 \geq 1$. Hence, $z_0 = \theta_0^{-1} x_0 \geq \theta_0^{-1}$, as claimed. $\square$

Note that the latter lemma generalizes the well-known result for algebraic equations, cf. [**9**, p. 277].

Rewrite (4.4) as

$$\sum_{k=1}^{\infty} \frac{z^k M_\infty^k(V_-)}{k!} \sum_{j=1}^{\infty} \frac{z^j M_\infty^j(V_+)}{j!} = 1.$$

Or

$$\sum_{k=1}^{\infty} C_k z^k = 1$$

with

$$C_k = \sum_{j=1}^{k-1} \frac{M_\infty^{k-j}(V_-) M_\infty^j(V_+)}{j!(k-j)!}.$$

Due to the previous lemma, with the notation

$$\delta(K) = 2 \max_{j=1,2,\dots} \sqrt[j]{C_j}$$

we get $z(K) \geq \delta^{-1}(K)$. Now Theorem 4.2 yields

Corollary 4.4. *Under condition* (1.2), *the inequality* $r_s(\tilde{K}) \leq \delta(K)$ *is true.*

Corollary 4.4 improves the well-known estimate

$$r_s(\tilde{K}) \leq \tilde{\delta}_0(K) \equiv \sup_x \int_0^1 |K(x,s)|\, ds \tag{4.7}$$

([**8**, Theorem 16.2]) if $\tilde{\delta}_0(K) > \delta(K)$. That is, Corollary 4.4 improves estimate (4.7) for operators which are "close" to Volterra ones.

5. Nonnegative invertibility. We will say that $h \in L^\infty$ is nonnegative if $h(t)$ is nonnegative for almost all $t \in [0,1]$; a linear operator A in L^∞ is nonnegative if Ah is nonnegative for each nonnegative $h \in L^\infty$. Recall that I is the identity operator.

Theorem 5.1. *Let the conditions* (1.2), (1.5) *and*

$$K(t,s) \geq 0, \quad 0 \leq t,\ s \leq 1 \tag{5.1}$$

hold. Then operator $I - \tilde{K}$ *is boundedly invertible and the inverse operator is nonnegative. Moreover,*

$$(I - \tilde{K})^{-1} \geq I. \tag{5.2}$$

Proof. Relation (2.9) with $A = I - \tilde{K}$, $W = V_-$ and $V = V_+$ implies

$$(I - \tilde{K})^{-1} = (I - V_+)^{-1}(I - B_K)^{-1}(I - V_-)^{-1} \tag{5.3}$$

where

$$B_K = (I - V_+)^{-1} V_+ V_- (I - V_-)^{-1}.$$

Moreover, by (5.1) we have $V_\pm \geq 0$. So due to (2.4), $(I - V_\pm)^{-1} \geq 0$ and $B_K \geq 0$. Relations (2.7) and (2.8) according to (2.10) and (2.11) imply

$$|B_K|_{L^\infty} \leq (e^{M_\infty(V_+)} - 1)(e^{M_\infty(V_-)} - 1).$$

But (1.5) is equivalent to (1.7). We thus get $|B_K|_{L^\infty} < 1$. Consequently,

$$(I - B_K)^{-1} = \sum_{k=0}^{\infty} B_K^k \geq 0.$$

Now (5.3) implies the inequality $(I - \tilde{K})^{-1} \geq 0$. In addition, since $I - \tilde{K} \leq I$ and $(I - \tilde{K})^{-1} \geq 0$, we have inequality (5.2). □

6. Applications.

6.1. A nonselfadjoint differential operator. Consider a differential operator A defined by

$$(Ah)(x) = -\frac{d^2h(x)}{dx^2} + g(x)\frac{dh(x)}{dx} + m(x)h(x), \quad 0 < x < 1,\ h \in D(A) \tag{6.1}$$

on the domain

$$\begin{aligned} D(A) = \{h \in L^\infty,\ h'' \in L^\infty,\ a_0h(0) + b_0h'(0) = 0, \\ a_1h(1) + b_1h'(1) = 0\} \\ a_jb_j \equiv \text{const},\ a_j^2 + b_j^2 > 0, \quad j = 0, 1. \end{aligned} \tag{6.2}$$

In addition,

(6.3) the coefficients $g, w \in L^\infty$ and are complex, in general.

Let an operator S be defined on $D(A)$ by

$$(Sh)(x) = -h''(x), \quad h \in D(A).$$

Assume S has the Green function $G(t, s)$, so that

$$(S^{-1}h)(x) \equiv \int_0^1 G(x, s)h(s)\,ds \in D(A)$$

for any $h \in L^\infty$. Besides,

(6.4)

$$\beta_\infty(S) \equiv \int_0^1 \sup_x |G(x, s)|\,ds < \infty \quad \text{and} \quad \int_0^1 \sup_x |G_x(x, s)|\,ds < \infty.$$

Thus, $A=(I-\tilde{K})S$ where

$$(\tilde{K}h)(x) \;=\; -\Big(g(x)\frac{d}{dx}+m(x)\Big)\int_0^1 G(x,s)h(s)\,ds \;=\; \int_0^1 K(x,s)h(s)\,ds,$$

with

$$K(x,s) \;=\; -\,g(x)G_x(x,s)-m(x)G(x,s). \tag{6.5}$$

According to (6.3) and (6.4), condition (1.2) holds. Take into account that

$$|S^{-1}h|_{L^\infty} \;\le\; \beta_\infty(S)|h|_{L^\infty}.$$

Since

$$A^{-1} \;=\; S^{-1}(I-\tilde{K})^{-1},$$

Theorem 1.1 immediately implies the following result.

Proposition 6.1. *Under* (6.3)–(6.5), *let condition* (1.5) *hold. Then operator A defined by* (6.1) *and* (6.2) *is boundedly invertible in L^∞. In addition,*

$$|A^{-1}|_{L^\infty} \;\le\; \frac{\beta_\infty(S)e^{M_\infty(V_-)+M_\infty(V_+)}}{e^{M_\infty(V_+)}+e^{M_\infty(V_-)}-e^{M_\infty(V_-)+M_\infty(V_+)}}.$$

6.2 An integro-differential operator. On domain (6.2), let us consider the operator

(6.6)
$$(Eu)(x) \;=\; -\frac{d^2u(x)}{dx^2}+\int_0^1 K_0(x,s)u(s)\,ds, \quad u\in D(A),\ 0<x<1,$$

where K_0 is a kernel with the property

$$\operatorname*{ess\,sup}_x \int_0^1 |K_0(x,s)|\,ds \;<\; \infty. \tag{6.7}$$

Let S and G be the same as in the previous subsection. Then we can write $E=(I-\tilde{K})S$ where $\tilde{K}$ is defined by (1.1) with

$$K(x,s) \;=\; -\int_0^1 K_0(x,x_1)G(x_1,s)\,dx_1. \tag{6.8}$$

So if $I - \tilde{K}$ is invertible, then E is invertible as well. Clearly, under (6.4) and (6.7), condition (1.2) holds. Since

$$E^{-1} = S^{-1}(I - \tilde{K})^{-1},$$

Theorems 1.1 and 5.1 yield

Proposition 6.2. *Under* (6.4), (6.7) *and* (6.8), *let condition* (1.5) *hold. Then operator* E *defined by* (6.6) *and* (6.2) *is boundedly invertible in* L^∞ *and*

$$|E^{-1}|_{L^\infty} \le \frac{\beta_\infty(S) e^{M_\infty(V_-)+M_\infty(V_+)}}{e^{M_\infty(V_+)} + e^{M_\infty(V_-)} - e^{M_\infty(V_-)+M_\infty(V_+)}}.$$

If, in addition, $G \ge 0$ *and* $K_0 \le 0$, *then* E^{-1} *is positive. Moreover,*

$$(E^{-1}h)(x) \ge (S^{-1}h)(x) = \int_0^1 G(x,s)h(s)\,ds$$

for any nonnegative $h \in L^\infty$.

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REFERENCES

1. D.E. Edmunds and W.D. Evans, *Spectral theory and differential operators*, Clarendon Press, Oxford, 1990.

2. Y. Egorov and V. Kondratiev, *Spectral theory of elliptic operators*, Birkhäuser Verlag, Basel, 1996.

3. M.I. Gil', *Stability of finite and infinite dimensional systems*, Kluwer Academic Publishers, Boston-Dordrecht-London, 1998.

4. ———, *Invertibility conditions and bounds for spectra of matrix integral operators*, Monat. Math. **129** (2000), 15–24.

5. R. Harte, *Invertibility and singularity for bounded linear operators*, Marcel Dekker, Inc., New York, 1988.

6. H. König, *Eigenvalue distribution of compact operators*, Birkhäuser Verlag, Basel-Boston-Stuttgart, 1986.

7. J. Locker, *Spectral theory of non-self-adjoint two point differential operators*, Math. Surveys Monographs **73**, Amer. Math. Soc., Providence, RI, 1999.

8. M.A. Krasnosel'skii, J. Lifshits and A. Sobolev, *Positive linear systems. The method of positive operators*, Heldermann Verlag, Berlin, 1989.

9. A.M. Ostrowski, *Solution of equations in Euclidean and Banach spaces*, Academic Press, New York, 1973.

10. A. Pietsch, *Eigenvalues and s-numbers*, Cambridge University Press, Cambridge, 1987.

11. S. Prössdorf, *Linear integral equations*, Itogi Nauki i Tekhniki, Fundamental'nye napravleniya **27** (1988), VINITI, Moscow (in Russian).

12. P.P. Zabreiko, A.I. Koshelev, M.A. Krasnosel'skii, S.G. Mikhlin, L.S. Rakovshik, B.Ya. Stetzenko, *Integral equations*, Nauka, Moscow, 1968 (in Russian).

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