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EXISTENCE AND UNIQUENESS THEOREMS FOR INTEGRO-DIFFERENTIAL EQUATIONS ON THE HALF-AXIS WITH NON-DIFFERENCE KERNEL OF A CERTAIN TYPE. UPPER AND LOWER BOUNDS FOR SOLUTIONS.

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ABSTRACT. The integro-differential equation

$$-\frac{d^2y}{dx^2}+y=\int_0^\infty R(x-t)y(t)\,dt + \int_0^\infty R_1(x+t)y(t)\,dt, \qquad x>0$$

is considered under the following hypotheses: (i) $R(x)$ is an even function with positive range which decreases everywhere on the positive semi-axis; (ii) there exist constants $s^*>1$ and $A>0$ such that $R(x)\le Ae^{-s^*|x|}$, $-\infty<x<\infty$; (iii) the function $R_1(x)$ satisfies the inequality $|R_1(x)|\le R(x)$, $-\infty<x<\infty$; (iv) $\int_{-\infty}^\infty R(x)\,dx<1$. The general solution is found in a class of twice differentiable functions $y(x)$ satisfying conditions of the type: $|y(x)|\le \text{const}\cdot e^{\lambda x}$, $x>0$, where λ is a real number bounded from above by a number σ^* determined in a rather complicated way by the function $R(x)$. The initial value and boundary value problems are formulated and the existence and uniqueness theorems for these problems are proved. For two particular solutions of the equation, upper and lower bounds are not found.

1. Introduction. Formulation of the problem. There are many applied problems which lead to equations of the form:

$$-\frac{d^2y}{dx^2}+y=\int_0^\infty R(x-t)y(t)\,dt + \int_0^\infty R_1(+t)y(t)\,dt, \qquad x>0. \tag{1}$$

Equations of this kind arise in various fields of physics. As such, we may mention radiative equilibrium of stars [**3**], anomalous skin-effect

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in metals [**9, 4, 2**] stationary neutron density in multiplying media [**1, 11, 12**], wave propagation in acoustic and electrodynamic waveguides [**12, 8, 10**]. In all these fields of research, there are many particular problems which lead to the equation (1) with $R_1 \equiv 0$. These cases have been exhaustively treated with the standard Wiener-Hopf technique. However, there are many problems which cannot be simplified in this way. That is why the equation (1) in its general form deserves an independent investigation.

Asymptotic behavior of solutions of equation (1) was investigated in the author's previous paper [**6**] under the assumption that such solutions exist. In the present paper the existence theorem is proved and initial value and boundary value problems are considered.

We restrict the class of considered equations by the following four conditions: (i) $R(x)$ is an even function with positive range which decreases everywhere on the positive semi-axis; (ii) there exist constants $s^* > 1$ and $A > 0$ such that

$$R(x) \leq Ae^{-s^*|x|}, \qquad -\infty < x < \infty; \tag{2}$$

(iii) the function $R_1(x)$ is real-valued and satisfies the inequality

$$|R_1(x)| \leq R(x), \qquad -\infty < x < \infty; \tag{3}$$

(iv) $\int_{-\infty}^{\infty} R(x)\,dx < 1$.

In many applied problems there is a natural restriction on desired solution, i.e.,

$$|y(x)| \leq \text{const} \cdot e^{\lambda x}, \qquad x > 0$$

where λ is an a priori determined positive or negative number.

Let us consider the class Q_λ of all twice differentiable functions satisfying the last condition with the same λ but not necessarily the same constant. In this paper we are concerned with solutions of the equation (1) only in the class Q_λ provided $\lambda < 1$ and satisfies a condition which we describe in the following way. Let $\phi(\alpha)$ be a Fourier transform of the kernel $R(x)$

$$\phi(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} R(x)\,dx, \qquad -s^* < \alpha < s^*$$

and

$$G(p) = 1 - p^2 - \phi(ip). \tag{4}$$

In the previous author's paper [**6**] it was shown that under condition (iv) the function $G(p)$ has exactly two zeros $p = \pm\sigma^*$ in the strip $-\sigma^* \le \mathrm{Re}\,p \le \sigma^*$. The above mentioned condition on λ can now be formulated as: if $\lambda > \sigma^*$, then the strip $\sigma^* < \mathrm{Re}\,p \le \lambda$ is free from zeros of the function $G(p)$. Therefore, the strip $-\sigma^* \le \mathrm{Re}\,p \le \lambda$ contains only two zeros $p = \pm\sigma^*$ of function $G(p)$.

2. Main result. The following statements may be considered as the main result of the paper.

The form of the correct initial value problem for equation (1) depends on the value of λ. If $\lambda \ge \sigma^*$, then initial value problem

$$y(0) = y_0, \qquad y'(0) = y_0' \tag{5}$$

for equation (1) is well posed, i.e., has one and only one solution in class Q_λ. If $-\sigma^* \le \lambda < \sigma^*$ we must replace the pair of initial conditions (5) with one initial condition of the type:

$$\alpha y(0) + \beta y'(0) = \gamma, \qquad \alpha, \beta \in \Re. \tag{6}$$

This problem has one and only one solution in Q_λ except for the case $\alpha/\beta = \mu$, where μ is some negative number which is evaluated later. In the last case the initial value problem has a solution only if $\gamma = 0$, and this solution is the trivial one: $y(x) \equiv 0$. The solution of the initial value problem in case $-\sigma^* \le \lambda < \sigma^*$ is at the same time the unique solution of the boundary value problem

$$\alpha y(0) + \beta y'(0) = \gamma, \qquad y(\infty) = 0.$$

Equation (1) has only trivial solution in Q_λ if $\lambda < -\sigma^*$.

If $\lambda \ge \sigma^*$, class Q_λ contains two linearly independent solutions $w_0(x)$ and $w_1(x)$ of the equation (1) which satisfy the following inequalities:

$$e^{\sigma^* x} - e^{-\sigma^* x} \le w_0(x), w_1(x) \le e^{\sigma^* x} + e^{-\sigma^* x} \tag{7}$$

$$0 \le w_1(x) - w_0(x) \le 2e^{-\sigma^* x}. \tag{8}$$

There is a constant $c > 0$ such that

$$w_1(x) - w_0(x) > ce^{-\sigma^* x}. \tag{9}$$

For all above mentioned solutions, there were obtained formulae for sequences of lower and upper estimates which approximate a solution with an arbitrary accuracy. These formulae except (7) and (8) cannot be used immediately (if ever) for getting a numerical solution.

3. Preliminary remarks. As the first step we replace equation (1) with the equivalent system of two equations:

$$z(x) = \int_0^\infty R(x-t)y(t)\,dt + \int_0^\infty R_1(x+t)y(t)\,dt, \qquad x > 0 \tag{10}$$

$$-\frac{d^2y}{dx^2} + y(x) = z(x), \qquad x > 0. \tag{11}$$

The following statements are true:

i) If $y(x)$ is any function in Q_λ but not necessarily a solution of (1), then the function $z(x)$ defined by (10) is also in Q_λ.

ii) If $y(x) \geq 0$ for $x > 0$, then $z(x) \geq 0$ for $x > 0$.

iii) If $z(x)$ is any function in Q_λ, then the general solution $y(x) \in Q_\lambda$ of equation (11) has the form

$$y(x) = \frac{1}{2}\int_0^\infty z(s)e^{-|s-x|}\,ds + Be^{-x} \tag{12}$$

where B is an arbitrary constant, and vice versa, each function given by (12) is a solution of (11) in Q_λ. Therefore, equation (1) is equivalent in Q_λ to the family of inhomogeneous integral equations:

$$\begin{aligned} y(x) = \frac{1}{2}\int_0^\infty e^{-|s-x|}\,dx\bigg(\int_0^\infty R(s-t)y(t)\,dt \\ + \int_0^\infty R_1(s+t)y(t)\,dt\bigg) + Be^{-x}. \end{aligned} \tag{13}$$

There is a simple relation between B and initial values $y(0)$ and $y'(0)$:

$$B = \frac{1}{2}(y(0) - y'(0)). \tag{14}$$

Due to the linearity of equation (1) it is sufficient to consider only $B=0$ and $B=1$.

Equation (13) determines the integral operator P defined on Q_λ for any $\lambda<1$:

$$P[y(t)] = \frac{1}{2}\int_0^\infty e^{-|s-x|}\,ds\left(\int_0^\infty R(s-t)y(t)\,dt + \int_0^\infty R_1(s+t)y(t)\,dt\right) + Be^{-x}. \tag{15}$$

We will need the following evident property of the operator P:

$$P[y_2(t)] \geq P[y_1(t)] \quad \text{if } y_2(t)\geq y_1(t) \tag{16}$$

and the following equality:

$$\begin{aligned} P[e^{\sigma t}] = e^{\sigma x} &- \frac{G(\sigma)}{1-\sigma^2}e^{\sigma x} - \frac{e^{-x}}{2(1+\sigma)}\phi(i\sigma) \\ &- \frac{1}{2}\int_0^\infty e^{-|s-x|}\,ds\int_0^\infty (R(s+t)e^{-\sigma t} \\ &- R_1(s+t)e^{\sigma t})\,dt + Be^{-x}. \end{aligned} \tag{17}$$

4. Uniqueness lemma.

Lemma 1 (Uniqueness lemma). *If $y(x)$ is a solution of integral equation* (13) *for $B=0$ and*

$$\lim_{x\to\infty} y(x) = 0,$$

then $y(x)\equiv 0$.

Proof. Let $y(x)$ be a nontrivial solution of (13) for $B=0$ such that $\lim_{x\to\infty} y(x)=0$. We may assume that for some values of x function $y(x)$ is positive and so the upper bound y^* of $y(x)$ is positive. There exists a point x^* such that $y(x^*)=y^*$. Let us estimate y^*. Since

$$y(x^*) = \frac{1}{2}\int_0^\infty e^{-|s-x^*|}\,ds\int_0^\infty (R(s-t)+R_1(s+t))y(t)\,dt$$

Then

$$y^* \le y^* \cdot \frac{1}{2}\int_0^\infty e^{-|s-x^*|}\, ds \int_0^\infty (R(s-t)+R_1(s+t))\, dt. \tag{18}$$

The integral on the righthand side of this inequality coincides with $P[e^{\sigma t}]$ when $\sigma = 0$ and $B = 0$. Denoting this integral by $P_0[1]$, we have

$$y^* \le y^* \cdot P_0[1]. \tag{19}$$

According to (17), we have

$$P_0[1] = \phi(0)\left(1 - \frac{e^{-x}}{2}\right) - \frac{1}{2}\int_0^\infty e^{-|s-x|}\, ds \\ \int_0^\infty (R(s+t)-R_1(s+t))\, dt.$$

Since $\phi(0) < 1$, we get

$$P_0[1] < 1,$$

which contradicts (19). The theorem is proved. $\square$

5. Existence of solutions. The proof of the existence theorem is based on the following lemma.

Lemma 2. *Let Ω be a set of all continuous functions $\omega(x)$ such that*

$$e^{\sigma^* x} - e^{-\sigma^* x} \le \omega(x) \le e^{\sigma^* x} + e^{-\sigma^* x}.$$

Operator P maps Ω into itself.

Proof. To prove Lemma 2, it is sufficient to establish two inequalities:

$$P[\omega(t)] \le e^{\sigma^* x} + e^{-\sigma^* x}, \qquad P[\omega(t)] \ge e^{\sigma^* x} - e^{-\sigma^* x}$$

for any function $\omega(x) \in \Omega$.

Taking into account (16), we may write:

$$P[\omega(t)] \le P[e^{\sigma^* t} + e^{-\sigma^* t}], \qquad P[\omega(t)] \ge P[e^{\sigma^* t} - e^{\sigma^* t}]. \tag{20}$$

According to identity (17) and the definition of σ^*, we have:

$$
\begin{aligned}
P[e^{\sigma^* t} + e^{-\sigma^* t}] &= e^{\sigma^* x} + e^{-\sigma^* x} - (1-B)e^{-x} \\
&\quad - \frac{1}{2}\int_0^\infty e^{-|s-x|}\, ds \\
&\quad \int_0^\infty (e^{\sigma^* t} + e^{-\sigma^* t})(R(s+t) - R_1(s+t))\, dt \\
P[e^{\sigma^* t} - e^{-\sigma^* t}] &= e^{\sigma^* x} - e^{-\sigma^* x} + (B+\sigma^*)e^{-x} \\
&\quad + \frac{1}{2}\int_0^\infty e^{-|s-x|}\, ds \\
&\quad \int_0^\infty (e^{\sigma^* t} - e^{-\sigma^* t})(R(s+t) + R_1(s+t))\, dt.
\end{aligned} \tag{21}
$$

The last two relations make obvious the two following inequalities:

$$
\begin{aligned}
P(e^{\sigma^* t} + e^{-\sigma^* t}) &\le e^{\sigma^* x} + e^{-\sigma^* x} \\
P(e^{\sigma^* t} - e^{-\sigma^* t}) &\ge e^{\sigma^* x} - e^{-\sigma^* x},
\end{aligned}
$$

which together with inequality (20) prove Lemma 2. Later we will need a stronger version of the last inequality, namely,

$$
P[e^{\sigma^* t} - e^{-\sigma^* t}] \ge e^{\sigma^* x} - e^{-\sigma^* x} + (B+\sigma^*)e^{-x}, \tag{22}
$$

which also follows immediately from (21). □

Theorem 1 (Existence Theorem). *Under the hypotheses of Section* 1 *the integro-differential equation* (1) *has at least two linearly independent solutions* $w_0(x)$ *and* $w_1(x)$ *in the set* Ω.

Proof. Let us define two sequences of functions:

$$
u_n(x) = P[u_{n-1}(t)], \quad n = 1, 2, \dots, \qquad u_0(x) = e^{\sigma^* x} - e^{-\sigma^* x} \tag{23}
$$

$$
v_n(x) = P[v_{n-1}(t)], \quad n = 1, 2, \dots, \qquad v_0(x) = e^{\sigma^* x} + e^{-\sigma^* x}. \tag{24}
$$

Due to Lemma 2, all these functions are in Ω. Due to (16) we may write

$$\begin{gathered}
u_0(x) \leq u_1(x) \leq \cdots \leq u_n(x) \leq \cdots \\
v_0(x) \geq v_1(x) \geq \cdots \geq v_n(x) \geq \cdots \\
u_n(x) \leq v_n(x), \qquad n = 0, 1, 2, \dots ,
\end{gathered}$$

so that for any value of x there exist two limits:

$$\lim_{n\to\infty} u_n(x) = u(x), \qquad \lim_{n\to\infty} v_n(x) = v(x). \tag{25}$$

Let us consider the sequence of derivatives $\{u_n'(x)\}_{n=1}^{\infty}$. According to (15) we have:

$$\begin{aligned}
\frac{d}{dx}\{P[u_n(t)]\} = -\frac{1}{2}\int_0^x e^{-|s-x|}\, ds \\
\int_0^\infty u_n(t)(R(s-t) + R_1(s+t))\, dt \\
+ \frac{1}{2}\int_x^\infty e^{-|sx|}\, ds \\
\int_0^\infty u_n(t)(R(s+t) - R_1(s+t))\, dt - Be^{-x}.
\end{aligned}$$

Taking into account that $u_n(t)$ and $(R(s-t) + R_1(s+t))$ are positive functions, we get:

$$\left|\frac{d}{dx}\{P[u_n(t)]\}\right| \leq P[u_n(t)]$$

or

$$\left|\frac{du_{n+1}(x)}{dx}\right| \leq u_{n+1}(x) \leq e^{\sigma^* x} + e^{-\sigma^* x}.$$

Thus, all functions $u_n(x)$ have derivatives bounded by the same number $e^{\sigma^* N} + e^{-\sigma^* N}$ on any finite interval $[0, N]$ and, therefore, converge uniformly on this interval. Since the difference of any two functions of the sequence $\{u_n(x)\}$ is not greater than $2e^{-\sigma^* x}$, we get that this difference is less than $2e^{-\sigma^* N}$ on the interval (N, ∞). These two properties show that the sequence of functions $\{P[u_n(t)]\}$ converges uniformly to $P[u(t)]$ and therefore

$$P[u(t)] = P[\lim_{n\to\infty} u_n(t)] = \lim_{n\to\infty} P[u_n(t)] = \lim_{n\to\infty} u_{n+1}(x) = u(x).$$

Thus, $u(x)$ is a solution of (13). Similarly, function $v(x)$ defined by (24) is also a solution of (13). According to the uniqueness lemma, it is the same solution. This is true for any value of B. Setting $B = 0$ and $B = 1$ we obtain two linearly independent functions w_0 and w_1 each of which is a solution of (1). The theorem is proved. $\square$

6. Properties of the solutions $w_0(x)$ and $w_1(x)$. According to the proof of the existence theorem, the solutions $w_0(x)$ and $w_1(x)$ may be estimated from below and above with the aid of sequences $\{u_{0n}(x)\}, \{v_{0n}(x)\}, \{u_{1n}(x)\}, \{v_{1n}(x)\}$:

(26)

$$u_{0n}(x) \le w_0(x) \le v_{0n}(x), \qquad u_{1n}(x) \le w_1(x) \le v_{1n}(x), \quad n = 1, 2, \dots .$$

These estimates may be made as accurate as one wishes by choosing a sufficiently large n.

It is worthwhile to point out a couple of inequalities which connect these two solutions:

$$w_1(x) - w_0(x) \le 2e^{-\sigma^* x} \tag{27}$$

which is common for any pair of functions in Ω, and

$$w_1(x) - w_0(x) \ge e^{-x}. \tag{28}$$

To prove the last inequality it is helpful to write down the definitions of the sequences $\{u_{0,n}(x)\}$ and $\{u_{1,n}(x)\}$

$$u_{0,n+1}(x) = P_0[u_{0,n}(t)], \qquad u_{1,n+1}(x) = P_1[u_{1,n}(t)],$$
$$u_{0,0}(x) = u_{1,0}(x) = e^{\sigma^* x} - e^{-\sigma^* x}$$

where operator P_0 is operator P with $B = 0$ and operator P_1 is operator P with $B = 1$.

If the inequality

$$u_{0,n}(x) \le u_{1,n}(x)$$

holds (it obviously holds for $n = 0$), then the similar inequality holds for $u_{0,n+1}(x)$ and $u_{1,n+1}(x)$. Thus, taking the limit we get

$$w_0(x) \le w_1(x).$$

Therefore,

$$w_0(x) = P_0[w_0(t)] \le P_0[w_1(t)] = P_1[w_1(t)] - e^{-x} = w_1(x) - e^{-x}.$$

Notice an obvious consequence of the inequality (28):

$$\begin{gathered} e^{\sigma^* x} - e^{-\sigma^* x} \le w_0(x) \le e^{\sigma^* x} + e^{-\sigma^* x} - e^{-x} \\ e^{\sigma^* x} + e^{-\sigma^* x} \ge w_1(x) \ge e^{\sigma^* x} - e^{-\sigma^* x} + e^{-x}. \end{gathered}$$

In particular, at $x = 0$, we have

$$0 < w_0(0) \le 1 \le w_1(0) \le 2. \tag{29}$$

Moreover,

$$w_0(0) > \sigma^*. \tag{30}$$

Indeed, by the definition (23)

$$w_0(x) \ge u_{0,1}(x) = P[e^{\sigma^* t} - e^{-\sigma^* t}]$$

and, due to (22),

$$w_0(x) \ge e^{\sigma^* x} - e^{-\sigma^* x} + (B + \sigma^*) e^{-x}.$$

Taking $x = 0$ we obtain (30). Also, due to (14),

$$w_0(0) = w_0'(0), \qquad w_1(0) - w_1'(0) = 2.$$

7. Uniqueness theorem for (1). To establish the uniqueness theorem in Q_λ we need the following theorem from [**6**] which may be formulated in the form:

Theorem 2. *If λ is any number from the interval $(-s^*, s^*)$, then any solution of* (1) *in Q_λ has asymptotic behavior of the form*:

$$y(x) = \sum_k P_k(x) e^{p_k x} + O(e^{\nu x})$$

where summation is over all zeros p_k *of the function* $G(p)$ (4) *lying in the open strip* $\nu < \mathrm{Re}\, p < \lambda$, $\nu > -s^*$.

We will use this theorem for some ν such that $-s^* < \nu < 0$. Taking into consideration that the strip $-\sigma^* \le \mathrm{Re}\, p \le \lambda$ contains only two zeros $p = \pm\sigma^*$ and that they are simple zeros of function $G(p)$, we get:

$$y(x) = P_y e^{\sigma^* x} + O(e^{\nu x}), \qquad \nu < 0, \tag{31}$$

where P_y is a constant depending on a solution. Now we will use the last relation to prove the following theorem.

Theorem 3. *If* $\lambda \ge \sigma^*$ *the general solution of equation* (1) *in* Q_λ *is a linear combination of the solutions* $w_0(x)$ *and* $w_1(x)$:

$$y(x) = c_0 w_0(x) + c_1 w_1(x), \tag{32}$$

where c_0 *and* c_1 *are arbitrary constants.*

Proof. The solution $y(x)$ has the form (31) and satisfies the integral equation (13) for some value of $B = B_y$. The solutions $w_0(x)$ and $w_1(x)$ as functions in Ω may be represented by the formulae:

$$w_0(x) = e^{\sigma^* x} + O(e^{\nu x})$$
$$w_1(x) = e^{\sigma^* x} + O(e^{\nu x}).$$

Let us consider the function

$$Y(x) = y(x) - B_y w_1(x) - w_0(x)(P_y - B_y).$$

The function $Y(x)$ is a solution of (13) with $B = 0$. Its asymptotic behavior is of the form

$$Y(x) = O(e^{\nu x}), \qquad \nu < 0.$$

According to the uniqueness lemma, $Y(x) \equiv 0$. The theorem is proved. □

Theorem 4. *If $-\sigma^* \le \lambda < \sigma^*$ the general solution of equation* (1) *in Q_λ has the form*

$$y(x) = c(w_1(x) - w_0(x))$$

where c is an arbitrary constant.

Proof. If $\lambda < \sigma^*$ the solutions $w_0(x)$ and $w_1(x)$ do not belong to Q_λ and so we are left with the only one (up to a constant factor) solution $w_1(x) - w_0(x)$. It belongs to Q_λ because $\lambda > -\sigma^*$. $\square$

Theorem 5. *If $\lambda < -\sigma^*$ the class Q_λ contains only a trivial solution of equation* (1).

The proof of this theorem requires a statement which could be easily obtained in [**6**] but was not. To get this statement out of the frame of [**6**] seems to be too cumbersome. Therefore, we omit the proof.

8. Initial value problem.

Theorem 6. *Initial value problem for equation* (1) *with initial conditions $y(0) = y_0$, $y'(0) = y_0'$ has one and only one solution in Q_λ if $\lambda \ge \sigma^*$. Namely,*

$$y(x) = c_0 w_0(x) + c_1 w_1(x)$$

where

$$c_0 = \frac{2y_0 - w_1(0)(y_0 - y_0')}{w_0(0)} \tag{33}$$

$$c_1 = \frac{1}{2}(y_0 - y_0'). \tag{34}$$

Proof. Let us assume the existence of a solution $y(x)$ of the initial value problem. According to Theorem 3 it has the form (32). Coefficients c_0 and c_1 are uniquely determined by the initial conditions since the denominator $w_0(0)$ is greater than σ^* (see (30)). Thus we get expressions (33) and (34). The uniqueness part of the theorem is proved. The function $y(x)$ defined by (32), (33) and (34) is obviously a solution of the initial value problem. The theorem is proved. □

Theorem 7. *Initial value problem for equation* (1) *and initial conditions* (6) *has a unique solution* $y(x)$ *in* Q_λ *if* $-\sigma^* \geq \lambda < \sigma^*$ *and*

$$y(x) = c(w_1(x) - w_0(x)) \tag{35}$$

where

$$c = \frac{\gamma}{\alpha(w_1(0) - w_0(0)) + \beta(w_1(0) - w_0(0) - 2)} \tag{36}$$

provided

$$\alpha(w_1(0) - w_0(0)) + \beta(w_1(0) - w_0(0) - 2) \neq 0. \tag{37}$$

If this condition does not hold, the initial value problem has no solutions unless $\gamma = 0$. *In this last case it has only trivial solution* $y(x) \equiv 0$.

Proof. According to Theorem 4, the general solution of equation (1) has the form:

$$y(x) = c(w_1(x) - w_0(x)).$$

Initial condition (6) can be satisfied if inequality (37) holds. In this case the coefficient c is uniquely determined by (36). Thus the defined function $y(x)$ is obviously the solution of the initial value problem. The theorem is proved. Condition (37) may be rewritten in the form:

$$\alpha/\beta \neq \mu$$

where

$$\mu = -\frac{w_1(0) - w_0(0) - 2}{w_1(0) - w_0(0)}.$$

Inequalities (29) show that μ is negative.

Solution (35) is at the same time the unique solution of the boundary value problem:

$$\alpha y(0) + \beta y'(0) = \gamma, \qquad y(\infty) = 0. \tag{38}$$

Theorem 8. *Boundary value problem* (1)–(38) *is equivalent to the initial value problem* (1)–(6) *in the class* Q_λ *if* $-\sigma^* \leq \lambda < \sigma^*$.

This theorem is valid because any solution $y(x)$ of equation (1) such that $\lim_{x\to\infty} y(x) = 0$ is proportional to the difference $w_1(x) - w_0(x)$.

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