JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 4, Number 2, Spring 1992

POSITIVE SOLUTIONS OF A CLASS OF NONLINEAR INTEGRAL EQUATIONS AND APPLICATIONS

LYNN ERBE, DAJUN GUO AND XINZHI LIU

1. Introduction. It is known that many real world problems such as chemical reactors, neutron transport, infectious diseases and etc. can be modeled by nonlinear integral equations [**2,4,5**]. In the study of such problems, we are often interested only in finding positive solutions of a nonlinear integral equation due to the practical meaning of the physical model concerned. In this note we study a class of nonlinear integral equations given by

$$u(x) = \lambda \int_\Omega K(x,y) f(y, u(y))\, dy \tag{1.1}$$

and its nonlinear perturbation

$$u(x) = \lambda \int_\Omega K(x,y) f(y, u(y))\, dy + G(u(x)), \tag{1.2}$$

where $f(x,u)$ is a reciprocal of a polynomial. A prototype of (1.1) is the following integral equation

$$\varphi(x) = \int_0^1 \frac{R(x,y)}{x^2 - y^2} \frac{1}{1+\varphi(y)}\, dy, \tag{1.3}$$

which comes from the integral equation

$$1 = \psi(x) + \psi(x) \int_0^1 \frac{R(x,y)}{x^2-y^2} \psi(y)\, dy \tag{1.4}$$

by a change of variable $\varphi(x) = (1/\psi(x)) - 1$. Equation (1.4) is of interest in nuclear physics [**5**]. This paper is organized as follows. In Section 2 we consider equation (1.1) and prove that for any positive number λ, equation (1.1) has exactly one positive solution which can

Received by the editors on June 21, 1991.
Research partially supported by NSERC–Canada.

be obtained by iteration, that is, every positive number is an eigenvalue, and corresponding to every eigenvalue there is a unique eigenfunction which can be obtained by iteration. Furthermore, the solution of (1.1) is continuous and strictly increasing in λ and its norm tends to zero (infinity) as λ becomes small (large). Then we apply the result to equation (1.4) and a second order boundary value problem for an ordinary differential equation. In Section 3 we investigate equation (1.2). Due to the nonlinear perturbation, many of the nice properties that equation (1.1) has are lost. Even the basic existence is, in general, very difficult to obtain. However, under certain assumptions, we can construct a strict set contraction operator and establish the existence and uniqueness of positive solutions of (1.2) by using the Darbo fixed point theorem. Finally, when the uniqueness is not required, we employ the topological degree theory and cone compression and cone expansion technique to prove the existence of positive solutions of (1.2) under relaxed assumptions imposed on the perturbation term.

2. Positive eigenvectors of nonlinear integral equations. In this section we consider the following nonlinear integral equation

$$u(x) = \lambda \int_\Omega K(x,y) f(y, u(y))\, dy, \tag{2.1}$$

where $\lambda > 0$ is a parameter, Ω is a bounded closed domain in $\mathbf{R}^N$, K is a nonnegative function, and

$$f(x,u) = \left[a_0(x) + \sum_{i=1}^m a_i(x) u^{\alpha_i} \right]^{-1}. \tag{2.2}$$

If we let

$$Au(x) = \int_\Omega K(x,y) f(y, u(y))\, dy, \tag{2.3}$$

then (2.1) is equivalent to the following eigenvalue problem

$$\tilde{\lambda} u = Au, \qquad \tilde{\lambda} = 1/\lambda. \tag{2.4}$$

The following theorem shows that any positive real number is an eigenvalue of (2.4) corresponding to a unique positive eigenvector which depends continuously on λ and can be obtained from an iteration procedure.

Theorem 2.1. *Assume that*

(A1) $0 \not\equiv \int_\Omega K(x,y)\,dy \geq 0$ *on* Ω *and for any* $x, x_0 \in \Omega$

$$\lim_{x\to x_0} \int_\Omega |K(x,y) - K(x_0,y)|\,dy = 0 \quad \text{uniformly in } x_0;$$

(A2) $a_i \in C[\Omega, \mathbf{R}_+]$ *and* $a_0(x) > 0$ *on* Ω, $i = 0, 1, \ldots, m$;

(A3) $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m \leq 1$ *and* $m \geq 1$.

Then

(i) *for any* $\lambda > 0$, *equation* (2.1) *has exactly one positive solution* u_λ *on* Ω. *Moreover,*

$$\lim_{n\to\infty} ||u_n - u_\lambda|| = 0, \tag{2.5}$$

where $u_n(x) = \lambda \int_\Omega K(x,y) f(y, u_{n-1}(y))\,dy$, $n = 1, 2, \ldots$ *with* $u_0(x)$ *being an arbitrary nonnegative function in* $C(\Omega)$;

(ii) u_λ *is strictly increasing in* λ, *i.e.,* $0 < \lambda_1 < \lambda_2$ *implies* $u_{\lambda_1} < u_{\lambda_2}$, *where* $u_{\lambda_1}(x), u_{\lambda_2}(x)$ *are the unique positive solutions of (2.1) with* $\lambda = \lambda_1$ *and* $\lambda = \lambda_2$, *respectively*;

(iii) u_λ *is continuous in* λ, *i.e., for any* $\lambda_0 > 0$,

$$\lim_{\lambda\to\lambda_0} ||u_\lambda - u_{\lambda_0}|| = 0; \tag{2.6}$$

(iv) $\lim_{\lambda\to+0} ||u_\lambda|| = 0$ *and* $\lim_{\lambda\to+\infty} ||u_\lambda|| = +\infty$.

Proof. (i) Let $E = C(\Omega)$, where $C(\Omega)$ denotes the Banach space of all continuous functions $u(x)$ on Ω with norm $||u|| = \max_{x\in\Omega} |u(x)|$, $|\cdot|$ being any convenient norm in $\mathbf{R}^N$. Set $P = \{u \in C(\Omega); u(x) \geq 0, x \in \Omega\}$. Then P is a normal cone of E. It is easy to see that the operator defined by (2.3) maps P into P and is decreasing and completely continuous. Let $v_0(x) \equiv 0$, i.e., $v_0 = \theta$ and $v_1 = \lambda A v_0$. Then it follows from (A1) and (A2) that

$$v_1(x) = \lambda \int_\Omega K(x,y)[a_0(y)]^{-1}\,dy \geq 0 \quad \text{and} \quad v_1(x) \not\equiv 0, \quad x \in \Omega,$$

i.e., $v_1 > v_0$. Define $v_n(x) = \lambda A v_{n-1}$ and suppose that

$$v_0 \leq v_2 \leq \cdots \leq v_{2l} \leq v_{2l+1} \leq \cdots \leq v_1. \tag{2.7}$$

Since A is decreasing, we see from (2.7) that

$$v_{2l} \leq \lambda A v_{2l-1} \leq v_{2l+2} \leq \lambda A v_{2l} = v_{2l+1}$$

which in turn implies

$$v_{2l+2} = \lambda A v_{2l+1} \leq \lambda A v_{2l+2} \leq \lambda A v_{2l} = v_{2l+1}.$$

Thus,

$$v_{2l} \leq v_{2l+2} \leq v_{2l+3} \leq v_{2l+1}. \tag{2.8}$$

Hence, it follows by induction that

$$v_0 \leq v_2 \leq \cdots \leq v_{2n} \leq v_{2n+1} \leq \cdots \leq v_1, \quad n = 0, 1, 2, \dots . \tag{2.9}$$

Since A is completely continuous, the set $\{v_{2n}\}_{n=0}^{\infty}$ is relatively compact, which implies that there exists a subsequence $\{v_{2n_l}\} \subset \{v_{2n}\}$ such that $v_{2n_l} \to v_*$ as $l \to \infty$. Since $\{v_{2n}\}$ is a monotone sequence, it follows that $v_{2n} \to v_*$ as $n \to \infty$. Similarly, we can prove $v_{2n+1} \to v^*$ as $n \to \infty$. It can be seen easily from (2.9) that

$$v_0 \leq v_2 \leq \cdots \leq v_{2n} \leq v_* \leq v^* \leq v_{2n+1} \leq \cdots \leq v_1. \tag{2.10}$$

Since $v_{2n+1} = \lambda A v_{2n}$, $v_{2n} = \lambda A v_{2n-1}$ and A is continuous, we see that

$$v^* = \lambda A v_* \quad \text{and} \quad v_* = \lambda A v^*. \tag{2.11}$$

We next show that $v_* = v^*$. From (2.10), we obtain

$$\begin{aligned} v_* \geq v_2 &= \lambda \int_\Omega K(x,y) f(y, v_1(y))\, dy \\ &= \lambda \int_\Omega K(x,y) \bigg[a_0(y) + \sum_{i=1}^m a_i(y) v_1(y)^{\alpha_i} \bigg]^{-1} dy. \end{aligned}$$

Let $M = ||v_1|| = \max_{x\in\Omega} \lambda \int_\Omega K(x,y)[a_0(y)]^{-1}\,dy$ and

$$\varepsilon_0 = \min_{x\in\Omega} a_0 \bigg/ \bigg(a_0(x) + \sum_{i=1}^m a_i(x) M^{\alpha_i} \bigg).$$

Then $0 < \varepsilon_0 \leq 1$ and

$$v_* \geq v_2 \geq \varepsilon_0 \lambda \int_\Omega K(x,y)[a_0(y)]^{-1}\,dy = \varepsilon_0 v_1 \geq \varepsilon_0 v^*. \tag{2.12}$$

Set $T = \{t > 0 | v_* \geq t v^*\}$. T is nonempty by (2.12). Let $t_0 = \sup T$. We claim $t_0 \geq 1$. If, otherwise, $0 < t_0 < 1$, then by (2.11)

$$\begin{aligned} v^* &= \lambda A v_* \leq \lambda A(t_0 v^*) \\ &= \lambda \int_\Omega K(x,y) \bigg[a_0(y) + \sum_{i=1}^m a_i(y) t_0^{\alpha_i} v^*(y)^{\alpha_i} \bigg]^{-1} dy \\ &\leq t_0^{-1}(1+\eta)^{-1} v_*, \end{aligned} \tag{2.13}$$

where

$$\eta = \min_{y\in\Omega} \frac{a_0(y)(t_0^{-1} - 1) + \sum_{i=1}^m a_i(y)(t_0^{\alpha_i - 1} - 1) v^*(y)^{\alpha_1}}{a_0(y) + \sum_{i=1}^m a_i(y) v^*(y)^{\alpha_i}} > 0, \tag{2.14}$$

which implies that $v_* \geq t_0(1+\eta)v^*$. Since $t_0(1+\eta) > t_0$, this contradicts the choice of t_0. Thus, we have $v_* \geq v^*$. This, together with (2.10), implies $v_* = v^*$. Moreover, $v_* \geq v_2 \geq \varepsilon_0 v_1 > v_0 = \theta$. Thus, we see from (2.11) that $v_* = v^* = u$ is a positive solution of equation (2.1). Finally, for any $u_0 \in P$, we define

$$u_n = \lambda A u_{n-1}, \quad n = 1, 2, \dots . \tag{2.15}$$

Since $u_0 \geq \theta$, we have $\theta \leq \lambda A u_0 \leq \lambda A v_0$, i.e., $v_0 \leq u_1 \leq v_1$.

Applying the operator λA, we derive $v_2 \leq u_2 \leq v_1$. Continuing this process, we obtain

$$v_{2n} \leq u_{2n} \leq v_{2n-1}, \qquad v_{2n} \leq u_{2n+1} \leq v_{2n+1}, \quad n = 1, 2, \dots . \tag{2.16}$$

Since $v_{2n} \to u$, $v_{2n-1} \to u$ and $v_{2n+1} \to u$, it follows that $u_{2n} \to u$ and $u_{2n+1} \to u$, i.e., $||u_n - u|| \to 0$ as $n \to \infty$. If $\bar{u}$ is another solution of (2.1), one can easily conclude that $v_{2n} \le \bar{u} \le v_{2n+1}$, which implies $\bar{u} = u$. Thus, we have proved that for any $\lambda > 0$, (2.1) admits a unique positive solution u_λ on Ω and $\lim_{n\to\infty} ||u_n - u_\lambda|| = 0$, where $u_n = \lambda A u_{n-1}$, $n = 1, 2, \dots$, and $u_0 \in P$ is arbitrary.

(ii) Let $0 < \lambda_1 < \lambda_2$, u_{λ_1} and u_{λ_2} be the unique positive solutions of (2.1) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. Define

$$u_n = \lambda_1 A u_{n-1}, \qquad \bar{u}_n = \lambda_2 A \bar{u}_{n-1}, \quad n = 1, 2, \dots \tag{2.17}$$

with $u_0 = \bar{u}_0 = \theta$. Then $\lim_{n\to\infty} u_n(x) = u_{\lambda_1}(x)$ and $\lim_{n\to\infty} \bar{u}_n(x) = u_{\lambda_2}(x)$ uniformly on Ω. We claim that

$$u_{2n} \le \bar{u}_{2n}, \quad n = 0, 1, 2, \dots . \tag{2.18}$$

By definition $u_0 = \bar{u}_0$. We assume that $u_{2k} \le \bar{u}_{2k}$, $k \ge 0$. Then

$$u_{2k+1} = \lambda_1 A u_{2k} \ge \lambda_1 A \bar{u}_{2k} = (\lambda_1/\lambda_2)\bar{u}_{2k+1}. \tag{2.19}$$

Since $0 < \lambda_1/\lambda_2 < 1$, it follows from (2.13) and (2.19) that

$$u_{2k+2} = \lambda_1 A u_{2k+1} \le \lambda_1 A((\lambda_1/\lambda_2)\bar{u}_{2k+1}) \le \lambda_2 (1+\eta)^{-1} A \bar{u}_{2k+1} < \bar{u}_{2k+2}.$$

Thus, our claim is verified by induction. Letting $n \to \infty$ in (2.18), we get

$$u_{\lambda_1} \le u_{\lambda_2}. \tag{2.20}$$

Applying A to (2.20), we derive

$$\frac{1}{\lambda_1} u_{\lambda_1} \ge \frac{1}{\lambda_2} u_{\lambda_2}, \quad \text{i.e.} \quad u_{\lambda_1} \ge \frac{\lambda_1}{\lambda_2} u_{\lambda_2}.$$

Thus

$$\frac{1}{\lambda_1} u_{\lambda_1} = A u_{\lambda_1} \le A((\lambda_1/\lambda_2) u_{\lambda_2}) \le \frac{\lambda_2}{\lambda_1}(1+\eta)^{-1} A u_{\lambda_2} = \frac{1}{\lambda_1}(1+\eta)^{-1} u_{\lambda_2}$$

which implies $u_{\lambda_1} < u_{\lambda_2}$. Hence, u_λ is strictly increasing in λ.

(iii) Let $\lambda_0 > 0$ and $\lambda > \lambda_0$. It is easy to see from (2.19) that $u_\lambda \leq (\lambda/\lambda_0)u_{\lambda_0}$. Thus, we have

$$||u_\lambda - u_{\lambda_0}|| \leq (\lambda/\lambda_0 - 1)||u_{\lambda_0}|| \to 0 \quad \text{as } \lambda \to \lambda_0^+.$$

Similarly, we can show

$$||u_\lambda - u_{\lambda_0}|| \leq (\lambda_0/\lambda - 1)||u_{\lambda_0}|| \to 0 \quad \text{as } \lambda \to \lambda_0^-.$$

Thus, $\lim_{\lambda\to\lambda_0} ||u_\lambda - u_{\lambda_0}|| = 0$, i.e., u_λ is continuous in λ.

(iv) It is easy to see from (2.10) that, for any $\lambda > 0$, $\theta < u_\lambda \leq v_1 = \lambda A v_0$. Thus $||u_\lambda|| \leq \lambda ||Av_0|| \to 0$ as $\lambda \to 0^+$, i.e., $\lim_{\lambda\to 0^+} ||u_\lambda|| = 0$. On the other hand, if $\lambda > 1$, $(1/\lambda)u_\lambda \leq \bar{u}_1$, where $\bar{u}_1$ is the unique solution of (2.1) with $\lambda = 1$. Thus

$$\bar{u}_1 = A\bar{u}_1 \leq A((1/\lambda)u_\lambda) \leq \lambda(1+\eta)^{-1}Au_\lambda = (1+\eta)^{-1}u_\lambda,$$

which implies

$$u_\lambda \geq (1+\eta)\bar{u}_1, \tag{2.21}$$

where

$$\eta = \eta(\lambda, \bar{u}_1) = \min_{y\in\Omega} \frac{a_0(y)(\lambda - 1) + \sum_{i=1}^m a_i(y)(\lambda^{1-\alpha_i} - 1)\bar{u}_1(y)^{\alpha_i}}{a_0(y) + \sum_{i=1}^m a_i(y)\bar{u}_1(y)^{\alpha_i}}.$$

Since $a_0(y) > 0$, $a_i(y) \geq 0$, $i = 1, 2, \dots, m$, and $\bar{u}_1(y) \geq 0$ on Ω, it follows that $\eta \to +\infty$ as $\lambda \to +\infty$. Thus, we obtain from (2.21) that

$$\lim_{\lambda\to+\infty} ||u_\lambda|| \geq \lim_{\lambda\to+\infty} (1+\eta)||\bar{u}_1|| = +\infty,$$

which yields $\lim_{\lambda\to+\infty} ||u_\lambda|| = +\infty$. The proof of the theorem is complete. $\square$

As an application of Theorem 2.1, we consider the following nonlinear integral equation

$$1 = \psi(x) + \lambda\psi(x) \int_0^1 \frac{R(x,y)}{x^2 - y^2}\psi(y)\, dy, \quad 0 \leq x \leq 1, \tag{2.22}$$

which arises in nuclear physics. The solution $\psi(x)$ represents a certain probability distribution and so $0 < \psi(x) \le 1$.

The following result is an easy consequence of Theorem 2.1.

Theorem 2.2. *Assume that*

(B1) $R \in C([0,1] \times [0,1])$, *$R(x,y) \ge 0$ for $x > y$ and $R(x,y) \le 0$ for $x < y$ and $R(x,y) \not\equiv 0$;*

(B2) *there exists an $r > 0$ such that*

$$|R(x,y)| \le C|x-y|^r S(x,y), \quad 0 \le x,\ y \le 1,\ x \ne y,$$

where C is a constant, $S(x,y)$ is a bounded and nonnegative function such that

$$\limsup_{x,y\to 0} S(x,y)/(x+y) < +\infty.$$

Then

(i) *for any $\lambda > 0$, equation* (2.22) *has exactly one positive solution $\psi_\lambda(x)$. Moreover, $0 < \psi_\lambda(x) \le 1$ and, constructing successively the sequence of function*

$$\psi_n(x) = \left[1 + \lambda \int_0^1 \frac{R(x,y)}{x^2-y^2} \psi_{n-1}(y)\,dy\right]^{-1}, \quad n = 1,2,\dots$$

for any initial function $\psi_0 \in C([0,1])$, $0 < \psi_0(x) \le 1$, we have

$$||\psi_n - \psi_\lambda|| = \max_{0\le x\le 1} |\psi_n(x) - \psi_\lambda(x)| \to 0 \quad \text{as } n \to \infty;$$

(ii) *ψ_λ is strictly decreasing in λ, i.e., $0 < \lambda_1 < \lambda_2$ implies $\psi_{\lambda_1} > \psi_{\lambda_2}$;*

(iii) *ψ_λ is continuous in λ;*

(iv) $\lim_{\lambda\to 0^+} ||\psi_\lambda|| = 1$ *and* $\lim_{\lambda\to+\infty} ||\psi_\lambda|| = 0$.

Proof. Letting $\varphi(t) = 1/\psi(x) - 1$, equation (2.22) then reduces to

$$\varphi(x) = \lambda \int_0^1 \frac{R(x,y)}{x^2-y^2} \frac{1}{1+\varphi(y)}\,dy. \tag{2.23}$$

It is clear that $0 < \psi(x) \le 1$ is equivalent to $\varphi(x) \ge 0$. By (B2), we see

$$\left|\frac{R(x,y)}{x^2-y^2}\right| \le \frac{C_1}{|x-y|^{1-r}}, \quad 0 \le x,\ y \le 1,\ x \ne y,$$

and

$$\frac{R(x,y)}{x^2-y^2} \ge 0, \quad 0 \le x,\ y \le 1,\ x \ne y,$$

which implies

$$0 \not\equiv \int_0^1 \frac{R(x,y)}{x^2-y^2}\,dy < +\infty, \quad t \in [0,1].$$

It is not difficult to check that equation (2.22) satisfies all assumptions of Theorem 2.1. Then the conclusion of Theorem 2.2 follows easily from Theorem 2.1 and the definition of $\varphi(x)$. □

Remark. It is easy to give some elementary functions $R(x,y)$ which satisfy (B1) and (B2). For example, $R(x,y) = C(x-y)^{1/3}(x+3y-\sin x + 2x^2y)$, $C > 0$ is a constant, $R(x,y) = C(x-y)^{1/5}\ln(1+x+y)$, etc.

We next consider a two-point boundary value problem for an ordinary differential equation

(2.24)

$$\begin{cases} -d^2u/dx^2 = \lambda[a_0(x) + \sum_{i=1}^m a_i(x)u(x)^{\alpha_i}]^{-1}, & 0 \le x \le 1, \\ u(0) = u(1) = 0, & \lambda > 0 \text{ is a parameter.} \end{cases}$$

Theorem 2.3. *Assume that*

(C1) $a_i \in C[[0,1], R_+]$, $i = 0,1,2,\dots,m$, *and* $a_0(x) > 0$ *on* $[0,1]$;

(C2) $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m \le 1$ *and* $m \ge 1$.

Then

(i) *for any* $\lambda > 0$, *the boundary value problem* (2.24) *has exactly one positive solution* $u_\lambda \in C^2(0,1) \cap C^1[0,1]$, *and* $u_\lambda(x) = \lim_{n\to\infty} u_n(x)$ *uniformly on* $[0,1]$, *where* $u_n(x)$ *is the solution of the boundary value problem*

$$\begin{cases} -d^2u/dx^2 = \lambda[a_0(x) + \sum_{i=1}^m a_i(x)u_{n-1}^{\alpha_i}(x)]^{-1}, \\ u(0) = u(1) = 0, \end{cases} \tag{2.25}$$

and $u_0(x) \in C^2(0,1) \cap C^1[0,1]$ *is an arbitrary nonnegative function,*

(ii) $0 < \lambda_1 < \lambda_2$ *implies* $u_{\lambda_1} < u_{\lambda_2}$, *where* u_{λ_1} *and* u_{λ_2} *are the unique solutions of* (2.24) *with* $\lambda = \lambda_1$ *and* $\lambda = \lambda_2$, *respectively*;

(iii) $\lim_{\lambda \to \lambda_0} ||u_\lambda - u_{\lambda_0}|| = 0$ *for any* $\lambda_0 > 0$;

(iv) $\lim_{\lambda \to 0^+} ||u_\lambda|| = 0$ *and* $\lim_{\lambda \to +\infty} ||u_\lambda|| = +\infty$.

Proof. It is well known that (2.24) is equivalent to the following integral equation

$$u(x) = \lambda \int_0^1 K(x,y) f(y, u(y))\, dy, \tag{2.26}$$

where $K(x,y)$ denotes the Green's function of (2.24) and is given by

$$K(x,y) = \begin{cases} x(1-y), & x \le y; \\ y(1-x), & x > y, \end{cases}$$

and

$$f(x,u) = \left[a_0(x) + \sum_{i=1}^m a_i(x) u^{\alpha_i} \right]^{-1}.$$

It is easy to see that equation (2.26) satisfies all the assumptions of Theorem 2.1 and therefore the conclusions of the theorem follow easily. □

Remark. Let $v(x) \equiv 0$ and $w(x) = u_1(x)$, where $u_1(x)$ is the unique solution of the boundary value problem (2.25) with $n = 1$. Then it is easy to see that $v(x)$ and $w(x)$ are lower and upper solutions of the boundary value problem (2.24) and $v(x) \le w(x)$ on $[0,1]$. With the help of lower and upper solutions and modifying the boundary value problem (2.24) suitably, see [**1**], one can prove the existence of a positive solution to the boundary value problem (2.24) using Schauder's fixed point theorem. However, this method gives existence only and does not provide any concrete procedure for the computation of the solution and the dependence properties of the solution on the parameter λ in contrast to our present approach.

3. Nonlinear perturbations. We investigate, in this section, the perturbed integral equation

$$u(x) = \lambda \int_\Omega K(x,y) f(y, u(y))\, dy + G(u(x)), \tag{3.1}$$

where $\lambda > 0$, $f(x,u) = [a_0(x) + \sum_{i=1}^m a_i(x) u^{\alpha_i}]^{-1}$ and $G : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous.

Unlike equation (2.1), the perturbed equation (3.1) is, in general, very difficult to deal with. Even the existence is hard to obtain without severe restrictions on G.

Even if one can, under certain assumptions on G, construct sequences like (2.9), nothing can be obtained except pointwise convergence since the sequence does not possess, in general, any compactness properties. Thus, the argument used in Section 2 is no longer applicable to equation (3.1). However, other methods like the Darbo fixed point theorem for strict set contraction operators and cone compression and expansion techniques can be applied if we impose suitable assumptions on G. Our first result gives a set of sufficient conditions which guarantee the existence and uniqueness of a positive solution of equation (3.1).

Theorem 3.1. *Assume that*

(D1) (A1)–(A3) *hold with* $\int_\Omega K(x,y)\, dy > 0$ *for all* $x \in \Omega$;

(D2) $G(u)$ *is nonincreasing and there exists* $0 < L < 1$ *such that*

$$|G(u) - G(v)| \le L|u - v|, \quad u, v \in \mathbf{R}_+; \tag{3.2}$$

(D3) *for any* $u > 0$ *and* $0 < t < 1$

$$G(tu) < t^{-1} G(u). \tag{3.3}$$

Then equation (3.1) has exactly one positive solution on Ω.

Proof. Equation (3.1) can be written in the form

$$u = Bu \tag{3.4}$$

where $B = \lambda A + G$, and

$$Au(x) = \int_\Omega k(x,y) f(y, u(y))\, dy \tag{3.5}$$

$$Gu(x) = G(u(x)). \tag{3.6}$$

Let $P = \{u \in C(\Omega) | u(x) \geq 0 \text{ for } x \in \Omega\}$, then P is a normal cone in the Banach space $C(\Omega)$, and the operators $A : P \to P$ and $G : P \to P$ are decreasing, and so $B : P \to P$ is also decreasing. From the proof of Theorem 2.1, we know that A is completely continuous and for any $u > \theta$, $0 < t < 1$, there exists $\eta = \eta(u,t) > 0$ such that

$$A(tu) \leq [t(1+\eta)]^{-1} Au. \tag{3.7}$$

Now, for any $u > \theta$ and $0 < t < 1$, it follows from (3.3) that

$$G(tu(x)) < t^{-1} G(u(x)) \quad \text{for all } x \in \Omega,$$

so, note that G is continuous,

$$\max_{x \in \Omega} \frac{G(tu(x))}{t^{-1} G(u(x))} = \frac{1}{1+\eta_1}, \quad \eta_1 > 0.$$

Hence,

$$G(tu(x)) \leq [t(1+\eta_1)]^{-1} G(u(x)), \quad \text{for all } x \in \Omega. \tag{3.8}$$

It then follows from (3.7) and (3.8) that

$$B(tu) \leq [t(1+\eta_0)]^{-1} Bu, \quad \text{for } u > \theta,\ 0 < t < 1, \tag{3.9}$$

where $\eta_0 = \eta_0(u,t) = \min\{\eta, \eta_1\} > 0$. Let $u_1 = B\theta$. Since B is decreasing, $\theta \leq u \leq u_1$ implies $\theta \leq Bu \leq B\theta = u_1$. Hence B maps $[\theta, u_1]$ into itself. Clearly, $[\theta, u_1]$ is a bounded closed set. From (3.2), we see

$$||Gu - Gv|| \leq L||u - v||, \quad \text{for any} \quad u, v \in P, \tag{3.10}$$

which implies that G is a strict set contraction, and so B is also a strict set contraction. Hence, by the Darbo fixed point theorem, see Theorem

5.2.9 in [**3**], B has a fixed point $\bar{u}$ in $[\theta, u_1]$. Next, we prove that B has exactly one fixed point in P. In fact, if $u^* \in P$ is also a fixed point of B, $u^* \geq \theta$ implies $\theta \leq u^* = Bu^* \leq B\theta = u_1$. Observing

$$B\bar{u}(x) = \lambda \int_\Omega K(x,y)\left[a_0(y) + \sum_{i=1}^m a_i(y)\bar{u}(y)^{\alpha_i}\right]^{-1} dy + G(\bar{u}(x)),$$

and

$$a_0(x) > 0, \qquad \int_\Omega K(x,y)\,dy > 0 \quad \text{for } x \in \Omega,$$

we see that there exist $0 < \tau < 1$, $0 < \sigma < 1$ such that $B\bar{u} \geq \tau A\theta$, $A\theta \geq \sigma G\theta$, and so

$$B\bar{u} \geq \rho u_1, \quad \text{where } \rho = (1/2)\tau\sigma > 0. \tag{3.11}$$

Hence, $\bar{u} = B\bar{u} \geq \rho u_1 \geq \rho u^*$. Let $t_0 = \sup\{t > 0 | \bar{u} \geq tu^*\}$, then $0 < \rho \leq t_0 < +\infty$. We claim that $t_0 \geq 1$. If otherwise, $0 < t_0 < 1$, then $\bar{u} \geq t_0 u^*$ and by (3.9)

$$\bar{u} = B\bar{u} \leq B(t_0 u^*) \leq [t_0(1+\eta_0)]^{-1} Bu^* = [t_0(1+\eta_0)]^{-1} u^*$$

for some $\eta_0 > 0$, so $u^* \geq t_0(1+\eta_0)\bar{u} \geq t_0\bar{u}$. Hence,

$$u^* = Bu^* \leq B(t_0\bar{u}) \leq [t_0(1+\bar{\eta}_0)]^{-1} B\bar{u} = [t_0(1+\bar{\eta}_0)]^{-1}\bar{u}$$

for some $\bar{\eta}_0 > 0$, i.e., $\bar{u} \geq t_0(1+\bar{\eta}_0)u^*$, which contradicts the definition of t_0. Thus, $t_0 \geq 1$ and $\bar{u} \geq u^*$. In the same way, we can show $u^* \geq \bar{u}$. Hence, $\bar{u} = u^*$ and the theorem is proved. $\square$

Remark. It is not difficult to give some elementary nonincreasing functions $G(u)$ which satisfy conditions (3.2) and (3.3). For example,

$$G(u) = \frac{1}{2+u}, \quad \text{with } L = \frac{1}{4},$$
$$G(u) = \frac{1}{\sqrt{1+u}}, \quad \text{with } L = \frac{1}{2}.$$

We note that the question of continuous dependence on the parameter λ for (3.1) remains. Furthermore, it would be of interest to investigate smooth dependence (on λ) for (1.1).

If the uniqueness is not required, then we can relax the conditions imposed on G. Of course, one should not expect that the previous arguments apply to this situation. In the following, we employ the cone expansion and compression technique and prove the existence of a positive solution of equation (3.1) under much weaker conditions on G. We need the following lemma; for a proof, see [**6**].

Lemma 3.1. *Let E be an ordered Banach space with respect to a cone P, and let Ω_1, Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. For $i = 1, 2$, define*

$$r_i = \inf\{||u|| : u \in P \cap \partial\Omega_i\}, \qquad R_i = \sup\{||u|| : u \in P| \cap \partial\Omega_i\},$$
$$C_i = S(P \cap \partial\Omega_i),$$

where $S(P \cap \partial\Omega_i)$ denotes the Kuratowski measure of noncompactness $a = S(P_1)$, $P_1 = \{u \in P : ||u|| = 1\}$. Let $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ be an L-strict set contraction. Then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$ if one of the following conditions is satisfied:

$$\text{(i)} \qquad \begin{cases} ||Au|| \geq ||u||, & \text{for all } u \in P \cap \partial\Omega_1, \\ ||Au|| \leq ||u||, & \text{for all } u \in P \cap \partial\Omega_2, \\ r_1 > L(R_1 + 2C_1/a); \end{cases}$$

$$\text{(ii)} \qquad \begin{cases} ||Au|| \leq ||u||, & \text{for all } u \in P \cap \partial\Omega_1, \\ ||Au|| \geq ||u||, & \text{for all } u \in P \cap \partial\Omega_2, \\ r_2 > L(R_2 + 2C_2/a); \end{cases}$$

L is a constant such that

$$S(AQ) \leq LS(Q)$$

for any bounded set Q in P.

Theorem 3.2. *Assume that*

(E1) (A1)–(A3) *hold with $\int_\Omega K(x, y)\, dy > 0$ for all $x \in \Omega$;*

(E2) *there exists $0 < L < 1/3$ such that*

$$|G(u) - G(v)| \leq L|u - v|, \quad u, v \in \mathbf{R}_+;$$

(E3) $\lim_{u\to\infty} G(u)/u = \alpha < 1$.

Then equation (3.1) *has at least one positive solution on* Ω.

Proof. We again write equation (3.1) in the form $u = Bu$ where $B = \lambda A + G$, $\lambda > 0$ and

$$\begin{aligned} Au(x) &= \int_\Omega K(x,y) f(y, u(y))\, dy, \\ Gu(x) &= G(u(x)). \end{aligned}$$

Let $P = \{u \in C(\Omega) | u(x) \geq 0 \text{ for } x \in \Omega\}$, then P is a normal cone in the Banach space $C(\Omega)$, $A : P \to P$ is completely continuous and $G : P \to P$ is an L-strict set contraction as proved in the previous theorem. Consequently, $B : P \to P$ is an L-strict set contraction. Let

$$M_i = \max_{x\in\Omega} a_i(x), \quad i = 0, 1, 2, \dots, m, \quad \text{and} \quad m_0 = \min_{x\in\Omega} a_0(x).$$

Since $a_0 \in C(\Omega)$ and $a_0(x) > 0$ on Ω, we have $m_0 > 0$.

Let

$$K = \max_{x\in\Omega} \int_\Omega K(x,y)\, dy \quad \text{and} \quad k = \min_{x\in\Omega} \int_\Omega K(x,y)\, dy.$$

It follows from (A1) that $\int_\Omega K(x,y)\, dy$ is continuous on Ω. Since $\int_\Omega K(x,y)\, dy > 0$ on Ω, we have $k > 0$. By (E3), there exists $N > 0$ such that

$$\frac{G(u)}{u} \leq \frac{1+\alpha}{2} < 1, \quad \text{whenever} \quad u \geq N. \tag{3.12}$$

Define $\Omega_1 = \{u \in C(\Omega), ||u|| < r\}$ and $\Omega_2 = \{u \in C(\Omega), ||u|| < R\}$, where

$$r = \min\left\{1, \frac{\lambda k}{\sum_{i=1}^m Mi}\right\} \quad \text{and} \quad R = \max\left\{\frac{\lambda K}{m_0} + \frac{1+\alpha}{2}, 1 + \frac{1}{N}, N\right\}. \tag{3.13}$$

Clearly, Ω_1, Ω_2 are two bounded open sets in $C(\Omega)$, $\theta \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Moreover, $B : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ is an L-strict set contraction. Let $u \in P \cap \partial\Omega_1$. It follows from (3.13) that

$$||\lambda Au|| \geq \frac{\lambda \int_\Omega K(x,y)\, dy}{M_0 + \sum_{i=1}^m M_i ||u||^{\alpha_i}} \geq \frac{\lambda k}{\sum_{i=0}^m M_i} \geq ||u||.$$

Thus,

$$||Bu|| \geq ||\lambda Au|| \geq ||u||, \quad \text{for all} \quad u \in P \cap \partial\Omega_1. \tag{3.14}$$

If $u \in P \cap \partial\Omega_2$, then by (3.12), we have

$$||\lambda Au|| \leq \lambda K/m_0 \quad \text{and} \quad ||Gu|| \leq (1+\alpha)/2.$$

This, together with (3.13), implies

$$||Bu|| \leq \frac{\lambda K}{m_0} + \frac{1+\alpha}{2} \leq ||u||, \quad \text{for all} \quad u \in P \cap \partial\Omega_2. \tag{3.15}$$

It is easy to see

$$r_1 = R_1 = r, \quad C_1 = 2r \quad \text{and} \quad a = 2.$$

Thus

$$L(R_1 + 2C_1/a) = L(r + 4r/2) < r = r_1. \tag{3.16}$$

Hence, the conclusion of the theorem follows from (3.14)–(3.16) and Lemma 3.1. The proof is therefore complete. □

Remark. It is easy to see that functions such as $G(u) = \ln(5+u) + 1/(3+u)$, $G(u) = (1/6)u + 1/\sqrt{3+u}$, etc., satisfy conditions (E2) and (E3).

REFERENCES

1. S.R. Bernfeld and V. Lakshmikantham, *An introduction to nonlinear boundary value problems*, Academic Press, Inc., New York and London, 1974.

2. Dajun Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones*, Academic Press, Boston, New York, 1988.

3. V.I. Istrąțescu, *Fixed point theory*, D. Reidel Publishing Co., Boston, 1981.

4. A. Pazy and P.H. Rabinowitz, *A nonlinear integral equation with applications to neutron transport theory*, Arch. Rational Mech. Anal. **32** (1969), 226–246.

5. C.A. Stuart, *Positive solutions of a nonlinear integral equation*, Math. Ann. **192** (1971), 119–124.

6. Yong Sun, *Extension of Guo's theorem on domain compression and expansion*, Numer. Funct. Anal. Optim. **10** (1989), 607–617.

7. L.R. Williams and R.W. Leggett, *Nonzero solutions of nonlinear integral equations modeling infectious disease*, SIAM J. Math. Anal. **13** (1982), 112–121.

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Shandong University, P.R. China and Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1