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ABOUT THE EQUATION $k * u^2 = u$

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ABSTRACT. The aim of this note is to solve the general equation $k * u^2 = u$ for k subject to the conditions $k(-x) = k(x)$, $k \geq 0$, $k \in L^{3/2}(\mathbf{R}) \cap L^3(\mathbf{R})$ and under the constraints $u \geq 0$, $u(\pm\infty) = 0$.

1. Statement of the problem. We consider the following convolution problem in $L^3(\mathbf{R})$:

Find $u \in L^3(\mathbf{R})$, $u \geq 0$, such that $e^{-|x|} * u^2 = u$ and satisfies the initial conditions $u(\pm\infty) = 0$.

Throughout this note, $f * g$ stands for the standard convolution product,

$$(f * g)(s) = \int_{\mathbf{R}} f(t)g(s-t)\,dt.$$

If we take the derivative of the equation $e^{-|x|} * u^2 = u$ in the general sense, using the fact that $(e^{-|x|})'' = e^{-|x|} - 2\delta$, where δ is the Dirac measure, we obtain

$$u'' = (e^{-|x|} - 2\delta) * u^2 = e^{-|x|} * u^2 - 2\delta * u^2 = u - 2u^2.$$

The ordinary differential equation (O.D.E. for short) $u'' = u - 2u^2$ admits as the integral $u'^2 = u^2 - (4/3)u^3 + C$. For $C = 0$, we can effectively construct a positive solution u which is also symmetrically decreasing and such that $\lim_{x\to+\infty} u(x) = 0$.

Suppose that $u \leq 3/4$ and $v^2 = 1 - (4/3)u$. Then we obtain $(dv/dx)^2 = (1-v^2)^2/4$, and using the equation $dx/dv = 2/(1-v^2)$ we

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get $x = 2\operatorname{arctanh} v$. Hence $u = (3/4)(1 - \tanh^2(x/2))$ is a nontrivial solution which solves the given problem.

Now, instead of $k(x) = e^{-|x|}$, consider $k(x) = e^{-|x|} * e^{-2|x|}$. Then the above calculation gives rise to an explicit fourth order O.D.E., but it is not clear whether this equation has a nontrivial solution. Even more, if we take $k(x) = e^{-x^2}$, we lose the fact that we can transform the equation $k * u^2 = u$ into an equivalent O.D.E., even if we know an explicit solution.

Let $k \geq 0$, $k(-x) = k(x)$, $k \in L^{3/2}(\mathbf{R}) \cap L^3(\mathbf{R})$. We consider the following problem:

($\mathcal{Q}$) Find a nontrivial $u \in L^3(\mathbf{R})$ such that $k * u^2 = u$, $u \geq 0$, $u(\pm\infty) = 0$.

The motivation of this note is the following: we use problem ($\mathcal{Q}$) as a model for a more general Hammerstein equation,

$$u(t) = \int g(t,s)h(s,u(s))\,ds,$$

in order to exhibit the techniques we utilize to solve it. We refer the reader to [**1**] for details.

The idea is to transform this problem into a maximization problem:

$$(\mathcal{P}) \qquad M := \operatorname{Max}\left\{\int_{\mathbf{R}} k * u^2 \cdot u^2; \int_{\mathbf{R}} u^3 \leq 1\right\}$$

where $\int_{\mathbf{R}} g$ stands for $\int_{\mathbf{R}} g(t)\,dt$.

The solution of this problem solves the Euler equation $k * u^2 = \lambda u$. We don't know if such a function exists. If we take a maximizing sequence $\{u_n\}$ for ($\mathcal{P}$), then the limit may be 0 because ($\mathcal{P}$) is invariant under translations. So we break up the problem into two parts.

(1) Find an approximate problem and an approximate nontrivial solution of this problem.

(2) Prove the limit of this approximate solution to be a nontrivial solution.

By doing this, we lose the fact that the nontrivial solution maximizes $(\mathcal{P})$.

2. The approximate problem. We consider in $L^3([-n,n])$ the approximate maximization problem $(\mathcal{P}_n)$:

$$(\mathcal{P}_n)\quad \mathrm{Sup}\left\{\int_{-\infty}^{+\infty} k * u^2 \cdot u^2 : |u|_3 = 1,\ u \geq 0,\ \text{support } u \subset [-n,n]\right\},$$

For this problem, we get the compactness property,

Lemma 1. *The operator* $T : L^{3/2}([-n,n]) \to L^3(\mathbf{R})$ *defined by setting*

$$Tv(x) := \int_{-\infty}^{+\infty} k(x-t)v(t)\,dt$$

is compact.

Proof. We know that $k \in L^{3/2}(\mathbf{R})$ is the limit in norm $L^{3/2}$ of a sequence $\{k_n\}$ of C^∞ functions of compact supports. Set $T_n v := k_n * v$. For $||v||_{3/2} = 1$, we have

$$||Tv - T_n v||_3 = ||(k_n - k) * v||_3 \leq ||k_n - k||_{\frac{3}{2}} \cdot ||v||_{\frac{3}{2}} = ||k_n - k||_{\frac{3}{2}},$$

and, therefore, T_n tends to T in the operator norm. Hence, it suffices to show that T_n is compact.

Suppose k is C^∞ with compact support. Then $k * v$ has its support included in some interval $[-a,a]$, $a > 0$. Let us now approach k in the L^∞-norm by a sequence of trigonometric polynomials with period $2a$:

$$k(t) = \lim_{n\to+\infty} \sum_{m=-n}^{m=n} c_m e^{\frac{im\pi t}{a}} \quad \text{if } t \in [-a,a],\ k(t) := 0 \text{ otherwise.}$$

This yields,

$$k * v(t) = \lim_{n\to\infty} \sum_{m=-n}^{m=n} c_m \left(\int_{-n}^{n} e^{-\frac{is\pi}{a}} v(s)\, ds \right) e^{\frac{im\pi t}{a}} \quad \text{if } t \in [-a, a]$$

and $k * v(t) = 0$ otherwise.

Then T_n clearly has a finite rank and, therefore, is a compact operator. It forces T to be compact also, as desired. □

Lemma 2. *Problem* $(\mathcal{P}_n)$ *is well defined.*

Proof. By virtue of Hölder's inequality we have

$$\left| \int_{-\infty}^{+\infty} k * u^2 \cdot u^2 \right| \le |k * u^2|_3 \cdot |u^2|_{\frac{3}{2}}.$$

Since $|k * u^2|_3 \le |k|_{3/2} \cdot |u^2|_{3/2}$, we get that $\int_{-\infty}^{+\infty} k * u^2 \cdot u^2 \le |k|_{3/2} |u|_3^4$. If $|u|_3 = 1$ this yields $\int_{-\infty}^{+\infty} k * u^2 \cdot u^2 \le |k|_{3/2}$, and the proof is complete. □

Lemma 3. *The supremum is a maximum.*

Proof. Let $\{u_m\}$ be a sequence such that $\int_{-\infty}^{+\infty} k * u_m^2 u_m^2$ tends to

$$M_n := \operatorname{Sup} \left\{ \int_{-\infty}^{+\infty} k * u^2 \cdot u^2 : |u|_3 = 1,\ u \ge 0 \text{ and support } u \subset [-n, n] \right\}.$$

Since $|u_m|_3 = 1$, on relabeling if necessary we may suppose that $\{u_m\}$ tends weakly to u (we shall use the symbol $\rightharpoonup$ for the weak convergence) in $L^3([-n, n])$. In particular, there exists some v such that $u_m^2 \rightharpoonup v$ in $L^{3/2}([-n, n])$. Since T is compact from $L^{3/2}([-n, n])$ to $L^3(\mathbf{R})$, we may also suppose that $k * u_m^2 \to k * v$ in $L^3(\mathbf{R})$. Using the duality between $L^{3/2}(\mathbf{R})$ and $L^3(\mathbf{R})$, we get

$$\int_{-\infty}^{+\infty} k * u_m^2 \cdot u_m^2 \to M_n = \int_{-\infty}^{+\infty} k * v \cdot v.$$

Since the norm is weakly lower semicontinuous, we have

$$|v|_{\frac{3}{2}} \leq \lim_{k\to+\infty} |u_m^2|_{\frac{3}{2}} = \lim_{k\to+\infty} |u_m|_3^2 = 1,$$

and since

$$M_n = \int_{-\infty}^{\infty} k * v \cdot v \leq M_n \cdot |v|_{3/2}, \quad \text{with } M_n \neq 0,$$

we necessarily have $|v|_{3/2} = 1$. It follows that $u_m^2 \to v$ in $L^{3/2}(\mathbf{R})$ and $|u_m^2|_{3/2} \to |v|$. Using the Kadec-Klee property [**4**] of the norm in $L^{3/2}(\mathbf{R})$, we derive that $|u_m^2 - v|_{3/2}$ tends to 0 and, therefore, that $u_m^2(x) \to v(x)$ (a.e.). Since $|u_m - u|_3 \to 0$, we necessarily have $u^2(x) = v(x)$ (a.e.). Therefore, $M_n = \int_{-\infty}^{\infty} k * u^2 \cdot u^2$ with $|u|_3 = 1$, $u \geq 0$ and support $u \subset [-n, n]$, so that the supremum is attained, as claimed. □

Lemma 4. *Let $u_n \in L^3(\mathbf{R})$ be such that $M_n = \int_{-\infty}^{\infty} k * u_n^2 \cdot u_n^2$. Then one has*

$$u_n > 0 \quad \text{(a.e.) } on\ [-n, n].$$

Proof. Indeed, let us set $A := u_n^{-1}(0) \cap [-n, n]$. Let us show that A has zero measure. In order to do so, set $v_\varepsilon := u_n + \varepsilon \mathbf{1}_A$, where $\mathbf{1}_A$ stands for the characteristic function of A. Then $v_\varepsilon^2 = u_n^2 + \varepsilon^2 \mathbf{1}_A$, and therefore, $\psi(\varepsilon)$ given by

$$\psi(\varepsilon) := \int_{-\infty}^{+\infty} k * v_\varepsilon^2 \cdot v_\varepsilon^2 \times \left(\int_{-\infty}^{+\infty} |v_\varepsilon|^3 \right)^{-\frac{4}{3}}$$

$$\text{satisfies } \psi(\varepsilon) \geq M_n + 2\varepsilon^2 \int_A k * u_n^2.$$

Hence, we get $\psi(\varepsilon) > \psi(0)$, and thus a contradiction. □

Lemma 5. *Every solution u_n of $(\mathcal{P}_n)$ satisfies the equation*

$$k * u_n^2 = M_n u_n \quad (a.e.)\ on\ [-n, n].$$

Proof. Indeed, since u_n solves the maximization problem $(\mathcal{P}_n)$, the inequality

$$\int_{-\infty}^{+\infty} k * (u_n + \lambda\varphi)^2 \cdot (u_n + \lambda\varphi)^2 \le M_n \bigg(\int_{-\infty}^{+\infty} |u_n + \lambda\varphi|^3 \bigg)^{\frac{4}{3}}$$

is satisfied for each λ and $\varphi \in L^3([-n, n])$. From the preceding inequality we also have

$$\int_{-\infty}^{+\infty} k*(u_n+\lambda\varphi)^2\cdot(u_n+\lambda\varphi)^2 - \int_{-\infty}^{+\infty} k*u_n^2\cdot u_n^2 \le M_n[|u_n+\lambda\varphi|_3^4 - |u_n|_3^4].$$

Divide this last inequality by $\lambda > 0$, and let λ tend to 0. Then after a calculation we obtain

$$\int_{-\infty}^{+\infty} k * u_n^2 \cdot u_n\varphi \le M_n \int_{-\infty}^{+\infty} u_n^2 \cdot u_n\varphi$$

and therefore

$$\text{(1)} \quad \int_{-\infty}^{\infty} k * u_n^2 \cdot u_n\varphi = M_n \int_{-\infty}^{\infty} u_n^2 u_n\varphi, \quad \text{for each } \varphi \in L^3([-n, n]).$$

From (1) we get

$$\text{(2)} \qquad (k * u_n^2 - M_n u_n^2)u_n = 0 \quad \text{(a.e.) on } [-n, n].$$

Lemma 4 combined with (2) then gives

$$\text{(3)} \qquad k * u_n^2 = M_n u_n \quad \text{(a.e.) on } [-n, n]$$

and the desired result. $\square$

3. The main result.

Theorem 6. *The equation $k * u^2 = \lambda u$ has a nontrivial solution.*

Proof. We first show the existence of a solution. Since the sequence $\{M_n\}$ is increasing and is bounded from above, it admits a limit denoted by M_∞. Since $|u_n|_3 = 1$, on relabeling if necessary we may always suppose that $u_n \rightharpoonup u$ in $L^3(\mathbf{R})$ and, moreover, that $u_n^2 \rightharpoonup v$ in $L^{3/2}(\mathbf{R})$. As a result, since $k \in L^3(\mathbf{R})$ it follows that, for each x, $\lim_{n\to+\infty}(k*u_n^2)(x) = (k*v)(x)$. Furthermore, since M_n tends to M_∞, by virtue of (3) we have $\lim_{n\to+\infty} u_n(x) = M_\infty^{-1} \cdot (k*v)(x)$ for each $x \in \mathbf{R}$. Hence, we get $M_\infty u = k*v$ and $u_n \rightharpoonup u$ in $L^3(\mathbf{R})$. Therefore, $u^2 = v$ and $k * u^2 = M_\infty u$.

It remains to show that u is nontrivial. By reductio ad absurdum, suppose $u = 0$. We have $k * u_n^2 \to 0$ everywhere, and in particular, $u_n(0) \to 0$. We derive a contradiction by using an argument of Hardy-Littlewood-Pólya [**3**]:

If f, g, h are continuous positive functions of compact support, then

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)g(y)h(-x-y)\,dx\,dy \\ \leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f^*(x)g^*(y)h^*(-x-y)\,dx\,dy,$$

where f^*, g^*, h^* are the rearrangement of f, g, h in symmetrical decreasing order:

$$f^*(-x) = f^*(x) \quad \textit{and} \quad f^*(x) = \mu^{-1}(2x) \\ \text{with } \mu(y) := \text{meas}\{x : f(x) \geq y\}.$$

We observe that f^* decreases symmetrically on each side of the origin where it generally has an infinite cusp. By virtue of the Hardy-Littlewood-Pólya inequality we may always suppose that u_n is positive symmetric and decreasing. Hence, $|u_n|_\infty = |u_n(0)|$ tends to 0. Using the interpolation

$$|u_n|_{3+\varepsilon} \leq |u_n|_3^{\frac{3}{3+\varepsilon}} \cdot |u_n|_\infty^{1-(3/3+\varepsilon)} \leq |u_n|_\infty^{1-\frac{3}{3+\varepsilon}}$$

we get $|u_n|_p \to 0$ for each $p > 3$. We also remark that

$$||u_n^2 * u_n^2||_{3/2} \leq ||u_n||_3^4 = 1$$

and

$$||u_n^2 * u_n^2||_\infty \leq ||u_n||_3^2 \cdot ||u_n||_6^2.$$

This forces, $u_n^2 * u_n^2$ tends weakly to zero in $L^3(\mathbf{R})$. Since,

$$M_n = \int_{-\infty}^{+\infty} k * u_n^2 \cdot u_n^2 = \int_{-\infty}^{+\infty} k \cdot u_n^2 * u_n^2$$

by passing to the limit we get $M\infty = 0$ and a contradiction. □

4. Remarks.

(1) $u_1 = (1/\lambda)u$ satisfies $k * u_1^2 = u_1$.

(2) We proved that the problem has a nontrivial symmetric solution.

(3) The same method can be applied to an equation of the form

$$k * \varphi(u) = u$$

with, for example, $\varphi(t) = t^s$, $1 < s < \infty$.

(4) A problem which remains open is to prove the uniqueness of the nontrivial solution (up to a translation). We can show by a shooting argument that $e^{-|x|} * u^{-2|x|} * u^2 = u$ has a unique nontrivial solution. So we conjecture the unicity for the general problem.

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