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NEUTRAL FDE CANONICAL REPRESENTATIONS OF HYPERBOLIC SYSTEMS

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Dedicated to my long time friend and colleague, John Nohel,
on the occasion of his sixty-fifth birthday.

ABSTRACT. We show how a variety of linear control systems, basically of "hyperbolic type," can be transformed to neutral functional equations of various orders. Using this correspondence, which makes essential use of the spanning properties of sets of complex exponentials, we are able to explore the closed loop spectral assignment capabilities of a class of linear feedback mechanisms which may be applied in the original control system. We introduce a class of neutral equations of "negative order" and show that these serve as canonical forms for certain "deficient" hyperbolic systems, some of which arise quite naturally in applications.

1. Background on the control canonical form. The theory of spectral assignment for constant coefficient, finite dimensional controllable (cf. [**15**]) linear systems

$$\begin{gathered}\dot{x} = Ax + Bu, \quad x \in E^n,\ u \in E^m,\ m \le n, \\ A : n \times n,\ B : n \times m,\ \operatorname{rank} m,\end{gathered} \tag{1.01}$$

is based on the so-called *control canonical form* presented, e.g., in [**15**] and [**5**] and developed in its most complete form by Brunovsky in [**1**]. One shows that there are nonsingular $n \times n$ and $m \times m$ matrices P and Q, respectively, such that the transformations

$$x = Pw, \quad u = Qv$$

carry (1.01) into an equivalent system

$$\dot{w} = \hat{A}w + \hat{B}v$$

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wherein the diagonal blocks $\hat{A}_k$, $k = 1, 2, \ldots, m$, of $\hat{A}$, of dimensions $n_1, n_2, \ldots, n_m$, $n_1 + n_2 + \cdots + n_m = n$, are in rational canonical form, the rest of the entries of $\hat{A}$ are zero except for arbitrary entries in the n_1-th, $(n_1 + n_2)$-th, $\ldots$, and n-th rows, and

$$\hat{B} = (e_{n_1}, e_{n_1+n_2}, \ldots, e_n),$$

e_k being the k-th column of the $n \times n$ identity matrix. In this form it is very easy to select $m \times n$ feedback matrices $\hat{K}$ such that, with

$$v = \hat{K}w,$$

in the *closed loop* system,

$$\dot{w} = (\hat{A} + \hat{B}\hat{K})w,$$

$\hat{A} + \hat{B}\hat{K}$ has any specified set $\{\mu_1, \mu_2, \ldots, \mu_n\}$ of eigenvalues (there are some restrictions on the Jordan form of this matrix, however).

Our purpose in this article is to discuss a certain approach to the question of control canonical structure and eigenvalue specification for certain infinite dimensional systems whose principal component is a system of partial differential equations of hyperbolic type. Systems corresponding to a single scalar equation of this type are discussed in Section 2–Section 5 to follow.

For the moment we content ourselves with a very general system description. We let X be a complex, separable Hilbert space and consider a control system

$$\dot{x} = Ax + bu, \quad x \in X, \; u \in E^m, \tag{1.02}$$

where A generates a strongly continuous semigroup (generally a group in the class of systems considered in this paper) $S(t)$ on X, m is an integer and B is an *admissible input element* (cf. [**3, 21, 22**]). We will assume that the operator A has eigenvalues λ_j, $j \in J$, where J is a countable index set, and corresponding eigenvectors ϕ_j forming a Riesz basis for X. With restrictions one may also admit eigenvalues with finite multiplicity and corresponding generalized eigenvectors. Further restrictions on the forms of A and b will be introduced after we discuss some background material.

In [**14**] we studied a class of linear hyperbolic control systems of the form

$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} w \\ v \end{pmatrix} - A(x)\begin{pmatrix} w \\ v \end{pmatrix} = g(x)u(t), \quad 0 \le x \le 1, \ t \ge 0, \tag{1.03}$$

wherein w, v, u are scalar, $A(x)$ is a continuous 2×2 matrix and $g(x)$ is a two-dimensional vector function with entries in $L^2[0,1]$. The boundary conditions were assumed to have the forms

$$a_0 w(0,t) + b_0 v(0,t) = 0, \ a_1 w(1,t) + b_1 v(1,t) = 0 \tag{1.04}$$

with

$$\gamma = \frac{(a_0+b_0)(a_1-b_1)}{(a_0-b_0)(a_1+b_1)}$$

neither zero nor infinite, or the second of these might be replaced by

$$a_1 w(1,t) + b_1 v(1,t) = u(t)$$

and $g(x)$ by zero if boundary control was to be studied.

The simplicity of the development in that study arose from the fact that the eigenvalues of the operator appearing on the right-hand side of (1.03) with the boundary conditions (1.04) take the form

$$\gamma_j = \frac{1}{2}\log\gamma + j\pi i + O\left(\frac{1}{|j|}\right), \quad j \in J = \{j \mid -\infty < j < \infty\},$$

and, as a consequence (see [**4, 6, 8, 23**]), the exponentials $e^{\lambda_j t}$ form a Riesz basis for $L^2[0,2]$. Ultimately, every argument of that paper was based on that fact.

It is easy to find hyperbolic systems which do not fit this pattern involving a single wave equation and scalar control ($m = 1$), for example, the forced wave equation

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = g(x)u(t), \quad 0 \le x \le 1, \ t \ge 0, \tag{1.05}$$

with $g \in L^2[0,1]$ and nontrivial boundary conditions

$$a_0 w(0,t) + b_0\frac{\partial w}{\partial x}(0,t) = 0, \quad a_1 w(1,t) + b_1\frac{\partial w}{\partial x}(1,t) = 0. \tag{1.06}$$

With $v = \partial w / \partial t$, the corresponding first order form is

$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{\partial^2 w}{\partial x^2} \end{pmatrix} + \begin{pmatrix} 0 \\ g(x)u(t) \end{pmatrix} \equiv L \begin{pmatrix} w \\ v \end{pmatrix} + b(x)u(t).$$

If we take $b_0 = b_1 = 0$, the eigenvalues of the operator L are

$$\lambda_j = j\pi i, \quad j \in J = \{j \mid j = \pm 1, \pm 2, \dots\}.$$

The exponentials $e^{\lambda_j t}$ then miss being the standard Fourier basis in $L^2[0,2]$ for lack of the element $e^{0t} \equiv 1$; i.e., they are "deficient." On the other hand, if $b_0 \neq 0$, $b_1 \neq 0$, the eigenvalues take the form

$$\lambda_j = j\pi i + O\left(\frac{1}{|j|}\right), \quad j \in J = \{j \mid j = 0, \pm 1, \pm 2, \dots\} \tag{1.07}$$

together with an additional eigenvalue, which we will designate by σ (or, when $a_0 = a_1 = 0$, the eigenvalue λ_0 assumes multiplicity two), which cannot be included in the sequence (1.07) without disturbing the indicated asymptotic relationship between λ_j and $j\pi i$. In this case the set of exponentials $\{e^{\lambda_j t}\} \cup e^{\sigma t}$ is "excessive" and cannot be independent in $L^2[0,2]$ because $\{e^{\lambda_j t}\}$ is already a Riesz basis for that space.

The system (1.02) will be said to be of *hyperbolic type* if, in addition to the earlier stated hypothesis that the generalized eigenvectors ϕ_j, $j \in J$, of A form a Riesz basis for the state space X, there is a finite bound M on the multiplicity of any eigenvector λ_j, there exist positive numbers α, δ, and Δ, and a real number ρ such that the λ_j lie in a strip $|\operatorname{Re} \lambda_j - \rho| \leq \alpha$ of the complex plane and, for real numbers r_1, r_2 with $r_1 < r_2$, the number $N(r_1, r_2)$, of λ_j for which $r_1 \leq \operatorname{Im} \lambda_j \leq r_2$, counting multiplicity, satisfies

$$\delta(r_1 - r_2) \leq N(r_1, r_2) \leq \Delta(r_2 - r_1). \tag{1.08}$$

While this is a useful classification, it includes far too many systems to allow any precise description of the control canonical forms. Consequently, in later sections of this paper, we add some further restrictions to arrive at classes for which we can provide such a description.

We will say that a system of hyperbolic type is, further, of *scalar hyperbolic type* if, with μ_j the multiplicity of λ_j, the resolvent of A

has a pole of order μ_j at λ_j (i.e., the corresponding "Jordan block" has a full complement of superdiagonal 1's) and one of the following alternatives is valid:

(i) For some positive a (necessarily between $\pi\delta$ and $\pi\Delta$ (cf. [**6**]) the generalized exponentials $t^l e^{\lambda_j t}$, $j \in J$, $l = 0, 1, \ldots, \mu_j - 1$, form a Riesz basis in $L^2[-a, a]$;

(ii) Condition (i) is not valid but becomes valid if $m(\geq 1)$ of the simple λ_j are removed from the eigenvalue sequence or finitely many of the multiplicities μ_j are reduced by decrement which total m;

(iii) Condition (i) is not valid but becomes valid if $m(\geq 1)$ additional complex numbers $\sigma_1, \ldots, \sigma_m$ ARE adjoined to the eigenvalue sequence or finitely many of the multiplicities μ_j, as used in (i), are increased by increments which total m.

We will refer to these subclasses as being of *exact, augmented* and *deficient* scalar hyperbolic type, respectively. In the case of (ii) and (iii) we will refer to the system as having *excess of order* m and *deficiency of order* m, respectively. Examples of augmented and deficient systems corresponding to $m = 1$ have been presented in Section 3; examples of augmented systems of arbitrary excess m, consisting of a partial differential equation of "wave equation" type on a spatial interval $0 \leq x \leq 1$, coupled to linear finite dimensional systems at the boundary points, and example of deficient systems which arise out of the separation of variables process applied to the wave equation in a disc, have been given in [**13**], an earlier unpublished report which includes some of the material presented here.

2. Basis properties of the Sobolev spaces on an interval. Let $[a, b]$, $a < b$, be a closed interval of the real line. By $H^m[a, b]$, m a non-negative integer, we denote the space of complex valued functions $z : [a, b] \to \mathbf{C}$ which, together with their derivatives $z^{(k)}$, defined in the sense of the theory of distributions [**18**], lie in $L^2[a, b] \equiv H^0[a, b]$. While it is customary to endow these spaces with the inner product and norm

$$(2.01) \qquad (z, \hat{z})_m = \sum_{k=0}^{m} \int_a^b z^{(k)}(t)\overline{\hat{z}^{(k)}(t)}\, dt, \quad \|z\|_m = ((z, z)_m)^{1/2},$$

it is not convenient to use these in the present application. The inconvenience lies in the fact that the "coordinate" functions $z^{(k)}$, $k = 0, 1, 2, \dots, m$, cannot be independently specified, leading to some difficulties in, e.g., computing adjoints of operators. Rather, we here choose to represent the same member of $H^m[a, b]$ by

$$\left(z(c), z'(c), \dots, z^{(m-1)}(c), z^{(m)}(\cdot)\right) \in E^m \times L^2[a, b]. \tag{2.02}$$

Here c may be any point in $[a, b]$; different c give equivalent representations. Clearly, functions in $H^m[a, b]$ can be uniquely specified in this way and each of the elements in (2.02) can be selected independently to construct elements of that space via the algebraic and topological isomorphism $F : E^m \times L^2[a, b] \to H^m[a, b]$ defined by

$$F(\xi_0, \xi_1, \dots, \xi_{m-1}, \xi(\cdot))(t) = \sum_{k=0}^{m-1} \xi_k \frac{(t-c)^k}{k!} + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \xi(s)\, ds.$$

It is not hard to verify that the usual inner product and norm in $E^m \times L^2[a, b]$ is equivalent to the inner product and norm (2.01) under the mapping $z = F(\xi_0, \xi_1, \dots, \xi_{m-1}, \xi(\cdot))$.

Rather than identify the dual space $H^m[a, b]^*$ with $H^m[a, b]$ itself, we choose to follow a procedure now familiar from [**7**], e.g. The map $J : H^m[a, b] \to L^2[a, b]$ defined as the injection of the first space into the second is clearly continuous and one-to-one with dense range in $L^2[a, b]$. It follows that, for each $f \in L^2[a, b]$,

$$\langle z, f \rangle \equiv \int_a^b z(t) f(t)\, dt, \quad z \in H^m[a, b], \tag{2.03}$$

defines a continuous linear functional on $H^m[a, b]$. One defines

$$\|f\|_{-m} = \sup_{z \in H^m[a,b] - \{0\}} \left\{ \frac{|\langle z, f \rangle|}{\|z\|_m} \right\},$$

obtaining a norm on $L^2[a, b]$ for which, except in the case $m = 0$, that space is not complete. $D^m[a, b]$, the completion of $L^2[a, b]$ with respect to this norm can be shown [**9**] to be a representation of the dual space of $H^m[a, b]$, the bilinear functional $\langle z, f \rangle$ defined for

$z \in H^m[a,b]$, $f \in D^m[a,b]$ being the natural extension of $\langle z, f\rangle$ (cf. (2.03)) arising from this completion. Representing elements of $H^m[a,b]$ as in (2.02), one may identify $D^m[a,b]$ with a certain vector space of distributions f; namely, those having the form, for complex scalar coefficients $\varphi_0, \varphi_1, \dots, \varphi_{m-1}$,

$$f = \varphi_0 \delta_{\{c\}} + \varphi \delta'_{\{c\}} + \cdots + \varphi_{m-1} \delta^{(m-1)}_{\{c\}} + \varphi(\cdot),$$

wherein $\varphi \in L^2[a,b]$ and $\delta^{(k)}_{\{c\}}$ denotes the k-th order "Dirac" distribution with support consisting of the set $\{c\}$. Then, for $z \in H^m[a,b]$, $f \in D^m[a,b]$, $\langle z, f\rangle$ is equivalent to

$$\begin{aligned} \langle z, f\rangle_m &\equiv \sum_{k=0}^{m-1} \varphi_k z^{(k)}(c) + \int_a^b \varphi(s) z^{(m)}(s)\, ds \\ &= \sum_{k=0}^{m-1} \varphi_k \zeta_k + \int_a^b \varphi(s)\, ds. \end{aligned} \tag{2.04}$$

We proceed next to define the spaces $H^{-m}[a,b]$ for $m > 0$. The process begins with the specification of $D^{-m}[a,b]$:

$$D^{-m}[a,b] = \left\{ z \in H^m[a,b] \mid z^{(k)}(a) = z^{(k)}(b) = 0, k = 0, 1, 2, \dots, m-1 \right\}.$$

This space is also referred to in the literature as $H_0^m[a,b]$. The injection map $J : D^{-m}[a,b] \to L^2[a,b]$ is again continuous and one-to-one with dense range and we may use the same process as before to define $H^{-m}[a,b]$ as the dual of $D^{-m}[a,b]$ with respect to $L^2[a,b]$. From the fat that $D^{-m}[a,b]$ is a closed subspace of $H^m[a,b]$ we conclude that we may identify $H^{-m}[a,b]$ with a closed subspace of $D^m[a,b]$; in fact, it is the subspace complementary to the $2m$-dimensional subspace of $D^m[a,b]$ spanned by $\delta^{(k)}_{\{a\}}$, $\delta^{(k)}_{\{b\}}$, $k = 0, 1, 2, \dots, (m-1)$.

Our work on canonical forms for hyperbolic equations requires some background on Riesz bases for the spaces $H^m[a,b]$, $H^{-m}[a,b]$ consisting of generalized exponential functions. To this end, it is convenient to normalize the interval $[a,b]$ to $[-1,1]$ and take $c = 0$. We assume familiarity with the standard "nonharmonic Fourier series" results ([**4, 6, 8, 23**]) pertaining to such bases in the space $L^2[-1,1]$. The best known of these results can be summarized in the following way. If the sequence $\Lambda = \{\lambda_j \mid -\infty < j < \infty\}$ consists of distinct λ_j with the property, for some complex θ, that

$$\limsup_{|j|\to\infty} |\lambda_j - (\theta + j\pi i)| < \frac{\pi}{4}, \tag{2.05}$$

then

$$E(\Lambda) = \{p_j(t) \equiv e^{\lambda_j t} \mid \lambda_j \in \Lambda\} \tag{2.06}$$

is a Riesz basis in $L^2[-1,1]$; each $f \in L^2[-1,1]$ has a unique expansion,

$$f = \sum_{j=-\infty}^{\infty} f_j p_j, \tag{2.07}$$

convergent in that space, and there are positive numbers d, D, independent of f, such that

$$d^{-2}\|f\|^2 \le \sum_{j=-\infty}^{\infty} |f_j|^2 \le D^2 \|f\|^2. \tag{2.08}$$

It is known that the number $\pi/4$ in (2.05) is best possible within the given context, but there are many other types of sequences which yield Riesz bases (see, e.g., [**2, 20, 23**]). A very general result, presented in R. M. Young's book [**23**], is ideally suited to our purposes here. There it is show that if $\chi(i\lambda)$ is a function of "sine type" having growth like $e^{|\mathrm{Im}\,\lambda|}$ in the upper and lower half planes with separated zeros (i.e., there is a positive lower bound on the distance between zeros of $\chi(i\lambda)$) in a strip parallel to the real axis in the λ plane, then the functions $\{e^{\lambda_k t}\}$, where the λ_k are the zeros of $\chi(\lambda)$ itself (lying in a strip parallel to the imaginary axis) form a Riesz basis for $L^2[-1,1]$. For the purposes of this paper, the most significant application of this result concerns the case where we define $\chi(\lambda)$ in the form

$$\chi(\lambda) = e^{\lambda} + be^{-\lambda} + \int_{-1}^{1} e^{\lambda s}\, dv(s), \tag{2.09}$$

where $v(s)$ is a function of bounded variation on $(-\infty,\infty)$, assumed to be right continuous without loss of generality. Associated with such a function v and a subinterval $[a, b]$ we have the concept of the *total variation*, $V(v, [a, b])$, of v on that subinterval [**10**]. If we assume that

$$\lim_{\epsilon \to 0} V(v, [-1, -1 + \epsilon]) = \lim_{\epsilon \to 0} V(v, [1 - \epsilon, 1]) = 0 \tag{2.10}$$

and that $b \neq 0$, then $\chi(i\lambda)$ is a function of sine type with the indicated growth properties. Further restrictions allow one to assert that $\chi(\lambda)$ has separated zeros. For example, if there is a positive integer K such that

$$\frac{dv}{ds} = f(s) + \sum_{k=1}^{K-1} \varphi_k \delta_{\{-1+2(k/K)\}} \quad \text{(sum is absent if } K = 1\text{)}, \tag{2.11}$$

where $f \in L^2[-1, 1]$, $\delta_{\{r\}}$ is the delta distribution with support consisting of the single point r, and the φ_k are scalar coefficients such that the polynomial

$$\mu^K + \sum_{k=1}^{K-1} \varphi_k \mu^k + b$$

has distinct zeros, then the zeros of $\chi(\lambda)$ are separated and all but finitely many of them are of single multiplicity. If, in fact, all have single multiplicity and we call them λ_k, then the $e^{\lambda_k t}$ in such a case form a Riesz basis for $L^2[-1, 1]$.

It is also possible to introduce generalized exponentials into Riesz bases of exponential functions. The strongest result in this direction is contained in D. Ulrich's paper [**20**]. A much more elementary result is that if $\{p_j \mid j \in J\}$ is a Riesz basis for $L^2[-1, 1]$ consisting of exponential functions, then finitely many of them, say n "simple exponentials," $p_j(t)$, may be removed from $E(\Lambda)$ and r others, say $p_{j_1}, p_{j_2}, \ldots, p_{j_r}$ augmented to sets of generalized exponentials

$$\left\{ e^{\lambda_{j_\nu} t}, t^{e^{\lambda_{j_\nu} t}}, \ldots, (t^{\mu_\nu - 1}/(\mu_\nu - 1)!) e^{\lambda_{j_\nu} t} \right\}$$

with $\mu_1 + \mu_2 + \cdots + \mu_r = n$, and the Riesz basis property will remain intact. In this context we can remove the requirement that $\chi(\lambda)$ have distinct zeros in the example (2.09), (2.10), (2.11) discussed above.

In [**16**] these results are extended to the spaces $H^m[-1, 1]$ and $H^{-m}[-1, 1]$. We will state the extended results here for distinct exponentials with the understanding that those results may be modified to include generalized exponentials as noted above for $L^2[-1,1]$. In $H^m[-1,1]$ the extended result is

THEOREM 2.1. *Let J be a countable index set and $\Lambda = \{\lambda_j \mid j \in J\}$ a sequence of distinct complex numbers satisfying the conditions imposed on systems of hyperbolic type in the discussion accompanying* (1.02) *and such that $E(\Lambda)$, defined as in (2.06), forms a Riesz basis for $L^2[-1,1]$. Let $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_m\}$ be a set of distinct complex numbers not included in Λ, and let*

$$\mathcal{P}(\lambda) = \prod_{k=1}^{m} (\lambda - \sigma_k). \tag{2.12}$$

Then

$$E(\Lambda, \Sigma) = \left\{ e^{\lambda t} / \mathcal{P}(\lambda) \mid \lambda \in \Lambda \right\} \bigcup \left\{ e^{\sigma t} \mid \sigma \in \Sigma \right\} \tag{2.13}$$

forms a Riesz basis for $H^m[-1,1]$.

We will not repeat the proof here but we will indicate the basic idea, which is very simple. Given $z \in H^m[-1,1]$, $f = \mathcal{P}(D)z \in L^2[-1,1]$, where D denotes differentiation and $\mathcal{P}$ is the polynomial (2.12). We expand f in a series similar to (2.07) and let $\zeta_k = z^{(k)}(0)$, $k = 0, 1, 2, \dots, (m-1)$. Solving the initial value problems

$$\mathcal{P}(D)z = \sum_{j \in J} f_j p_j, \quad p_j(t) \equiv e^{\lambda_j t}, \quad j \in J,$$
$$z^{(k)}(0) = \zeta_k, \quad k = 0, 1, 2, \dots, m-1,$$

in a term by term fashion readily yields an expansion of z in terms of the functions in $E(\Lambda, \Sigma)$, as shown in (2.13).

We turn next to a comparable result for the spaces $H^{-m}[-1,1]$, which is also proved in [**16**] and bears the same relationship to deficient hyperbolic systems as the preceding theorem does to augmented hyperbolic systems, as will be made clear in later sections.

THEOREM 2.2. *Let Λ and $E(\Lambda)$ have the properties indicated in the preceding theorem. Let a new sequence be obtained from Λ by deletion* *of m of the λ in Λ, the deleted set being indicated by the symbol $\Sigma \equiv \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$. Then the functions $\mathcal{P}(\lambda)e^{\lambda t}$, $\lambda \in \Lambda - \Sigma$, which we will collectively refer to as $E(\Lambda - \Sigma)$, constitute a Riesz basis for the space $H^{-m}[-1,1]$.*

Again, we refer the reader to [**16**] for the complete proof, but we will indicate what the proof is based on. We define $\mathcal{P}(\lambda)$, $\mathcal{P}(D)$ as before (but with reference to deleted elements of Λ now) and we let $\mathcal{Q}(D)$ be the adjoint differential operator for $\mathcal{P}(D)$ relative to the bilinear form $\langle\ ,\ \rangle$ introduced in (2.03). We consider y solving

$$(\mathcal{Q}(D)y)(s) = f(s), \quad f \in L^2[-1,1], \\ y^{(k)}(-1) = 0, \quad k = 0,1,2,\ldots,m-1. \tag{2.14}$$

From the fact that the roots of $\mathcal{Q}$ are $-\sigma_j$, we can see that we have

$$y^{(k)}(1) = 0, \quad k = 0,1,2,\ldots,m-1,$$

and, hence, $y \in D^{-m}[-1,1]$, just in case

$$\int_{-1}^{1} e^{\sigma_j s} f(s)\,ds = 0, \quad j = 1,2,\ldots,m. \tag{2.15}$$

We recall that elements z in $H^{-m}[-1,1] \subset D^m[-1,1]$ are continuous linear functionals on $D^{-m}[-1,1]$. But, from (2.02), it is also clear that such functionals can be represented in the form

$$\ell(y) = \int_{-1}^{1} (s)(\mathcal{Q}(D)y)(s)\,ds = \int_{-1}^{t} \zeta(s)f(s)\,ds, \quad \zeta \in L^2[-1,1], \tag{2.16}$$

where $\mathcal{Q}$ is the adjoint operator for $\mathcal{P}$ introduced earlier.

The norm of z in $H^{-m}[-1,1]$ is equivalent to

$$\blacksquare\zeta\blacksquare \equiv \frac{\left|\int_{-1}^{1}\zeta(s)f(s)\,ds\right|}{\|y\|_{L^2[-1,1]}}. \tag{2.17}$$

Since $\zeta \in L^2[-1,1]$ and $E(\Lambda)$ is a Riesz basis for that space, we have the expansion

$$\zeta(t) = \sum_{j \in J} \zeta_j e^{\lambda_j t} \equiv \sum_{\lambda \in \Lambda} \zeta_\lambda e^{\lambda t},$$

convergent in $L^2[-1,1]$, $\{\zeta_j\} \in \ell^2(J)$ (equivalently, $\{\zeta_\lambda \in \ell^2(\Lambda)\}$). But, in view of (2.15) and (2.16),

$$\blacksquare \sum_{\lambda \in \Lambda - \Sigma} \zeta_\lambda e^{\lambda t} - \sum_{\lambda \in \Lambda} \zeta_\lambda e^{\lambda t} \blacksquare = 0. \tag{2.18}$$

Rewriting (2.16) in the form

$$\ell(y) = \int_{-1}^{1} (\mathcal{P}\,(D)\zeta(s)) y(s)\, ds,$$

we see that $z = \mathcal{P}\,(D)\zeta$ and, in the $H^{-m}[-1,1]$ norm, equivalent to $\blacksquare \zeta \blacksquare$,

$$z = \mathcal{P}\,(D)\zeta = \mathcal{P}\,(D) \sum_{\lambda \in \Lambda - \Sigma} \zeta_\lambda e^{\lambda t} = \sum_{\lambda \in \Lambda - \Sigma} \zeta_\lambda \mathcal{P}\,(\lambda) e^{\lambda t}.$$

Distributions in $H^{-m}[-1,1]$ may be thought of as m-th derivatives of functions in $L^2[-1,1]$, or as the result of applying an m-th order operator $\mathcal{P}\,(D)$ to such functions. Such operators $\mathcal{P}\,(D) : L^2[-1,1] \to H^{-m}[-1,1]$ are bounded and boundedly invertible on domain $\mathcal{D}$ in the space $L^2[-1,1]$ obtained by specifying the values of m continuous linear functionals on $L^2[-1,1]$ which are linearly independent over the m-dimensional kernel of $\mathcal{P}\,(D)$.

One may then proceed to identify $H^{-m}(-\infty,\infty)$ with distributions whose Fourier transforms have the form $(|\lambda|^m + 1)\varphi(\lambda)$, with $\varphi(\lambda)$ in $L^2(-\infty,\infty)$, $H^{-m}[-1,1]$ consisting of restrictions of elements of the space $H^{-m}(-\infty,\infty)$ to $[-1,1]$.

3. Scalar linear functional equations of neutral type and integral order. We will begin by studying *scalar linear neutral equations of non-negative integral order* m on (w.l.o.g.) the interval $[-1,1]$:

$$\begin{aligned} &\sum_{k=0}^{m} c_k \zeta^{(k)}(t+1) + \sum_{k=0}^{m} d_k \zeta^{(k)}(t-1) \\ &\qquad + \sum_{k=0}^{m} \int_{-1}^{1} \zeta^{(k)}(t+s)\, dv_k(s) = u(t), \end{aligned} \tag{3.01}$$

where u is the control term. The c_k, d_k are scalar coefficients with

$$c_m \neq 0, \quad d_m \neq 0,$$

and the v_k are functions of bounded variation on $[-1, 1]$ which may be assumed right continuous. The basic requirement for the initial value problem consisting of (3.01) and a specified initial state

$$\zeta(t+s)\Big|_{t=0} = \zeta_0(s), \quad \zeta_0 \in H^m[-1,1], \tag{3.02}$$

should constitute a well-posed problem is that v_m should satisfy the second part of condition (2.10) of the preceding section.

We have seen in the preceding section that there are various ways in which the elements of the space $H^m[-1,1]$ may be represented. There is comparable freedom in the expression of a given neutral equation (3.01). This leads us to adopt a standard form compatible with the way in which we have chosen to represent elements of $H^m[-1,1]$. First of all, each of the integrals

$$\int_{-1}^{1} \zeta^{(k)}(t+s)\, dv_k(s), \quad k = 0, 1, \ldots, m-1,$$

can be integrated by parts $m-k$ times to yield an integral of the form

$$\int_{-1}^{1} \zeta^{(m)}(t+s)\tilde{v}_k(s)\, ds, \quad \tilde{v}_k \in H^{m-k-1}[-1,1],$$

plus boundary terms involving $\zeta^{(j)}(t \pm 1)$, $j = k, k+1, \ldots, m-1$. As a result, (3.01) can be rewritten as

(3.03)

$$c_m\zeta^{(m)}(t+1) + d_m\zeta^{(m)}(t-1) + \sum_{k=0}^{m-1} \Big(\tilde{c}_k\zeta^{(k)}(t+1) + \tilde{d}_k\zeta^{(k)}(t-1)\Big) \\ + \int_{-1}^{1} \zeta^{(m)}(t+s)\, dv_m(s) + \int_{-1}^{1} \zeta^{(m)}(t+s)\tilde{v}(s)\, ds = u(t),$$

where

$$\tilde{v} = \sum_{k=0}^{m-1} \tilde{v}_k \in L^2[-1,1].$$

Next, selecting an arbitrary number $c \in [-1,1]$, we can write

$$\zeta^{(k)}(t+1) = \zeta^{(k)}(t+c) + \int_c^1 \frac{(1-s)^{m-k-1}}{(m-k-1)!}\zeta^{(m)}(t+s)\,ds,$$
$$k = 0,1,\dots,m-1,$$

and $\zeta^{(k)}(t-1)$ can be expressed similarly. Then (3.03) can be rewritten, redefining the coefficients c_k, as

$$\begin{aligned} \zeta^{(m)}(t+1) + c_m\zeta^{(m)}(t-1) + \sum_{k=0}^{m-1} c_k\zeta^{(k)}(t+c) \\ + \int_{-1}^1 \zeta^{(m)}(t+s)\,dv(s) = u(t), \end{aligned} \tag{3.04}$$

where we have absorbed the functions $\tilde{v}(s)$ and

$$\frac{(\pm 1-s)^{m-k-1}}{(m-k-1)!}, \quad k = 0,1,\dots,m-1,$$

into the function of bounded variation $v(s)$, which now satisfies the second part of (2.10) with v_m replaced by v.

If we let

$$z(t,s) = \zeta(t+s), \quad t \in (-\infty,\infty),\ s \in [-1,1],$$

it is clear that the functional equation (3.04) is equivalent to the partial differential equation

$$\frac{\partial z^m}{\partial t}(t,s) = \frac{\partial z^m}{\partial t}(t,s), \tag{3.05}$$

with the boundary conditions
(3.06)

$$z^m(t,1) + c_m z^m(t,-1) + \sum_{k=0}^{m-1} c_k z^k(t,c) + \int_{-1}^1 z^m(t,s)\,dv(s) = u(t)$$

and the differential equations

$$\frac{dz^k}{dt}(t,c) = z^{k+1}(t,c), \quad k = 0,1,\dots,m-1. \tag{3.07}$$

The form (3.05), (3.06), (3.07) of our system allows us to place it in the framework of strongly continuous semigroups in the Hilbert space $H^m[-1,1]$. Indeed, the system takes the form

$$\frac{d\hat{z}}{dt} = \mathcal{Q}\,\hat{z} + \hat{b}u, \quad \hat{z} \in H^m[-1,1],$$

where (with $z^0(t,c) \equiv z(t,c)$)

$$\hat{z}(t) = (z(t,c), z^1(t,c), \ldots, z^{m-1}(t,c), z^m(t,\cdot)),$$

$\mathcal{A}$ is the operator defined by

$$\mathcal{A}\,\hat{z} = \left(z^1(t,c), z^2(t,c), \ldots, z^m(t,c), \frac{\partial z^m}{\partial s}(t,\cdot)\right) \tag{3.08}$$

on the domain consisting of $\hat{z} \in H^{m+1}[-1,1]$ for which the homogeneous counterparts of the boundary condition (3.06) holds, and $\hat{b}$ is the admissible control input element (cf. [**3, 21, 23**]) such that

$$\langle \hat{y}, \hat{b} \rangle = y^m(1), \quad \hat{y} \in \mathcal{D}\,(\mathcal{A}^*) \in H^{m+1}[-1,1].$$

We complete this section with a discussion of *scalar neutral functional equations of negative order* $-m, m = 1, 2, 3, \ldots$. We have seen that scalar neutral equations of non-negative order m, with delay interval normalized to length 2, may be viewed as having as their basic state spaces the spaces $H^m[-1,1]$. We will see in the next section that their exponential eigenfunctions form a basis for those spaces, just as described in Theorem 2.1. Since we have a parallel for Theorem 2.1 for the spaces $H^{-m}[-1,1]$ in Theorem 2.2, it will not be surprising that there should be neutral functional equations with $H^{-m}[-1,1]$ as their basic state spaces. We define a scalar neutral equation of negative order $-m$ to be an equation of the form

$$\int_{-1}^{1} z(t,s)\, d\nu(s) = u(t), \tag{3.09}$$

where ν has support in $[-1,1]$, has an m-th derivative, in the distributional sense, which is a function of bounded variation on $(-\infty,\infty)$

(clearly, it continues to have support in $[-1,1]$), which we will take to be right continuous, and

$$\nu^m(-1) = \nu^m(-1+) \neq 0, \quad \nu^m(1-) \quad \text{exists and } \neq 0. \tag{3.10}$$

Of course, we need to demonstrate that equations of this sort give well-posed initial value problems when we specify an initial state in $H^{-m}[-1,1]$. The proof of this is an indirect one, but not complicated. We will give only a sketch here.

As a convolution equation we can write (3.09) in the form

$$N * z = u, \quad u \in L^2_{\text{loc}}(-\infty,\infty). \tag{3.11}$$

We consider the differential equation, written in convolution form,

$$\delta^{(m)}_{\{0\}}(0) * \zeta = w, \quad w \in H^{-m}_{\text{loc}}(-\infty,\infty). \tag{3.12}$$

Multiplying (3.12), in the convolution sense, on the left by N, gives

$$(N * \delta^{(m)}_{\{0\}}(0)) * \zeta = \left(\delta^{(m)}_{\{0\}}(0) * N\right) * \zeta = N * w. \tag{3.13}$$

Let us give an initial state $z_0 \in H^{-m}[-1,1]$ for the equation (3.11), and let $u \in H^2_{\text{loc}}(-\infty,\infty)$ as indicated. Then we solve the equation (as we may, nonuniquely, from the remarks at the end of Section 2)

$$\delta^{(m)}_{\{0\}}(0) * \zeta_0 = z_0 \tag{3.14}$$

to obtain an initial state $\zeta_0 \in L^2[-1,1]$ for (3.13). We also solve

$$N * w = u, \tag{3.15}$$

with the given $u \in L^2_{\text{loc}}(-\infty,\infty)$, to obtain $w \in H^{-m}(-\infty,\infty)$. Then (3.13) becomes

$$\left(\delta^{(m)}_{\{0\}}(0) * N\right) * \zeta = u, \tag{3.16}$$

a standard neutral equation of zero order which can be seen to have form

$$\zeta(t+1) + \gamma\zeta(t-1) + \int_{-2}^{2} \zeta(t+s)\, dg(s),$$

where g is a function of bounded variation having the property (3.10) (actually, it suffices to assume only the second part of (3.10) here) and $\gamma \neq 0$. Equation (3.13) with initial state ζ_0 determines a solution $\zeta \in L^2_{\mathrm{loc}}(-\infty, \infty)$ with exponential growth (or decay) determined by γ, g. Then $z = \delta^{(m)}_{\{0\}}(0) * \zeta \in H^{-m}(-\infty, \infty)$ is the desired solution of the original equation (3.11) with the given initial state $z_0 \in H^{-m}[-1, 1]$. Thus, existence, uniqueness, etc., for the negative order equation (3.09) are related to comparable questions for (3.13), which are very well studied in the literature.

4. Scalar neutral control systems. Our purpose in this section is to study the system (3.05), (3.06), (3.07) and the analogous system associated with the equation (3.09), as a control system, and to set forth its properties from this point of view in order to obtain the background we need for the main results of the next section. We consider the systems of non-negative order first. Accordingly, we suppose that we have

$$\frac{\partial z^m}{\partial t}(t,s) = \frac{\partial z^m}{\partial t}(t,s), \tag{4.01}$$

with the boundary conditions (specializing c in (3.05)–(3.07) to 0)

$$z^m(t+1) + c_m z^m(t,-1) + \sum_{k=0}^{m-1} c_k z^k(t,1) + \int_{-1}^{1} z^m(t,s)\,dv(s) = u(t), \tag{4.02}$$

and the differential equations

$$\frac{dz^k}{dt}(t,0) = z^{k+1}(t,0), \quad k = 0, 1, \dots, m-1. \tag{4.03}$$

THEOREM 4.1. *Let it be assumed that $c_m \neq 0$ and that $v(s)$ is a function of bounded variation satisfying the condition* (2.10) *of Section 1. Let σ_k, $k = 1, 2, \dots, m$, and λ_j, $j \in J$, be the (assumed separated) zeros of the characteristic function*

$$\chi(\lambda) = \lambda^m e^{\lambda} + c_m \lambda^m e^{-\lambda} + \sum_{k=0}^{m-1} c_k \lambda^k + \int_{-1}^{1} \lambda^m e^{\lambda s}\,dv(s), \tag{4.04}$$

Then the system (4.01), (4.02), (4.03) *can be equivalently represented as a system in* ℓ^2 *of the form*

$$\frac{d\tilde{z}_k}{dt} = \sigma_k \tilde{z}_k + \tilde{b}_k u, \quad k = 1, 2, \dots, m \tag{4.05}$$

(*these equations are vacuous if* $m = 0$*, of course*),

$$\frac{dz_j}{dt} = \lambda_j z_j + b + ju, \quad j \in J, \tag{4.06}$$

where the "control distribution coefficients" $\tilde{b}_k$*,* b_j *are bounded and bounded below.*

REMARK . The exponential functions $e^{\sigma_k s}, e^{\lambda_j s}$ corresponding to the zeros of $\chi(\lambda)$, where $\chi(i\lambda)$ factors into the product of a polynomial of degree m and a function of sine type, form a Riesz basis for the space $H^m[-1,1]$; equations (4.05), (4.06) constitute the expression of (4.01), (4.02), (4.03) in terms of that basis.

PROOF. Recall $\langle\ ,\ \rangle_m$, the bilinear product in $H^m[-1,1]$, as defined in (2.04), and consider a solution of (4.01), (4.02), (4.03), initially with $u(t) \equiv 0$ and initial state $z(0,s) = z_0(s)$ in the domain of the operator $\mathcal{A}$, as described in (3.08), so that solutions have all derivatives shown in the L^2 sense, at least. Letting $w(t,s)$ be a smooth function of t, s, we compute that

$$\begin{aligned} &\frac{d}{dt}\langle z(t,\cdot), w(t,\cdot)\rangle_m \\ &\quad = z(t,0)\frac{dw}{dt}(t,0) + \sum_{k=0}^{m-2} z^{(k+1)}(t,0)\left(w^{(k)}(t,0) + \frac{dw^{(k+1)}}{dt}(t,0)\right) \\ &\qquad + z^{(m)}(t,0)w^{(m-1)}(t,0) \\ &\qquad + \int_{-1}^{1}\left(\frac{\partial z^{(m)}}{\partial s}w^{(m)}(t,s) + z^{(m)}(t,s)\frac{\partial w^{(m)}}{\partial t}(t,s)\right) ds. \end{aligned} \tag{4.07}$$

Noting that, for $\sigma \in [-1,1]$,

$$z^{(m)}(t,0) = z^{(m)}(t,\sigma) - \int_0^\sigma \frac{\partial z^{(m)}}{\partial s}(t,s)\, ds,$$

we have

$$
(4.08)\quad \begin{aligned} z^{(m)}(t,0) &= \frac{1}{2}\int_{-1}^{1} z^{(m)}(t,s)\,ds - \frac{1}{2}\int_{-1}^{1}\int_{0}^{\sigma} \frac{\partial z^{(m)}}{\partial s}(t,s)\,ds\,d\sigma \\ &= \int_{-1}^{1} z^{(m)}(t,s)\,ds - \frac{1}{2}\int_{-1}^{1}(1-|s|)\frac{\partial z^{(m)}}{\partial s}(t,s)\,ds. \end{aligned}
$$

Substituting (4.08) into (4.07), integrating by parts and using the equation (4.01), we obtain

$$
\begin{aligned}
(4.09)\quad & \frac{d}{dt}\langle z(t,\cdot), w(t,\cdot)\rangle_m \\
&= z(t,0)\left(\frac{dw}{dt}(t,0) - w^{(m)}(t,1)\right) \\
&\quad + \sum_{k=0}^{m-2} z^{(k+1)}(t,0)\left(w^{(k)}(t,0) + \frac{dw^{(k+1)}}{dt}(t,0) - c_{k+1}w^{(m)}(t,1)\right) \\
&\quad - z^{(m)}(t,-1)\left(c_m w^{(m)}(t,1) + w^{(m)}(t,-1)\right) \\
&\quad + \int_{-1}^{1} z^{(m)}(t,s)\left(\frac{\partial w^{(m)}}{\partial t}(t,s) - -\frac{dw^{(m)}}{ds}(t,s)\right. \\
&\quad \left. - \frac{1}{2}\mathrm{sgn}\,(s) w^{(m-1)}(t,0) - v(s)w^{(m)}(t,1)\right) ds.
\end{aligned}
$$

The derivative (4.09) is identically zero for all solutions z of (4.01), (4.02), (4.03) just in case w satisfies the adjoint system

(4.10)
$$
\frac{\partial w^{(m)}}{\partial t}(t,s) = \frac{\partial w^{(m)}}{\partial s}(t,s) + \frac{1}{2}\mathrm{sgn}\,(s)w^{(m-1)}(t,0) - v(s)w^{(m)}(t,1),
$$

$$
(4.11)\qquad \frac{dw}{dt}(t,0) = w^{(m)}(t,1),
$$

$$
(4.12)\qquad \frac{dw^{(k)}}{dt}(t,0) = c_k w^{(k-1)}(t,0), \quad k = 1,2,\ldots,m-1,
$$

$$
(4.13)\qquad c_m w^{(m)}(t,1) + w^{(m)}(t,-1) = 0.
$$

Once this adjoint equation has been identified, it is a simple matter to verify from continuity considerations that the derivative (4.09) continues to be identically zero if z and w are solutions of the (homogeneous) original and adjoint equations, respectively, lying in the state space $H^m[-1,1]$ for each t. It is also straightforward to see that, when z satisfies (4.01), (4.02), (4.03) with $u(t)$ in general nonzero, we have

$$\frac{d}{dt}\langle z(t,\cdot), w(t,\cdot)\rangle_m = w^{(m)}(t,1)u(t). \tag{4.14}$$

Let us define (cf. (2.13))

$$\tilde{p}_k(t) = e^{\sigma_k t}, \quad k = 1,2,\dots,m, \tag{4.15}$$

$$p_j(t) = e^{\lambda_j t}/\mathcal{P}\,(\lambda_j), \quad j \in J. \tag{4.16}$$

Our assumptions, along with Theorem 2.1, guarantee that these exponential functions form a basis for $H^m[-1,1]$. Correspondingly, we have the dual Riesz basis of biorthogonal functions $\tilde{q}_k(t)$, $q_j(t)$. If we expand a solution z of (4.01), (4.02), (4.03) in terms of the functions (4.15), (4.16), viz.:

$$z(t,s) = \sum_{k=1}^{m} \tilde{z}_k(t)\tilde{p}_k(s) + \sum_{j\in J} z_j(t)p_j(s), \tag{4.17}$$

$$z^{(\ell)}(t,0) = \sum_{k=1}^{m} \sigma^\ell \tilde{z}_k(t) + \sum_{j\in J} \lambda^\ell z_j(t), \quad \ell = 0,1,\dots,m-1, \tag{4.18}$$

and successively use

$$w(t,s) = e^{-\sigma_k t}\tilde{q}_k(s), \quad k = 1,2,\dots,m,$$
$$w(t,s) = e^{-\lambda_j t}q_j(s), \quad j \in J;$$

in the bilinear product computation (4.14), we find that equations (4.05) and (4.06) are satisfied with

$$\tilde{b}_k = \tilde{q}_k^{(m)}(1), \quad k = 1,2,\dots,m, \tag{4.19}$$

$$b_j = q_j^{(m)}(1), \quad j \in J. \tag{4.20}$$

It should be noted that the biorthogonal functions $\tilde{q}_k$, q_j are eigenfunctions of the operator (cf. (3.08)) $\mathcal{A}^*$ ($-A^*$ is the generator of the adjoint semigroup and, thus, is described by equations (4.10)–(4.13)) whose domain is a subset of $H^{(m+1)}[-1,1]$. Consequently, the values appearing on the right-hand sides of (4.19) and (4.20) are well defined for k and j, as indicated.

Now we show that the $\tilde{b}_k$, b_j are bounded and bounded below (in particular, none of these are zero). For particular instances of the function of bounded variation v appearing in (4.02) it would be possible to do this by obtaining asymptotic formulae for the biorthogonal functions $\tilde{q}_k$, q_j, at least insofar as the asymptotic boundedness properties as j gets large are concerned. But this is a very unsatisfactory process because of length, if nothing else. We will use a controllability argument. Considering (4.05), (4.06) as a control system in ℓ^2, indexed by $k = 1, 2, \dots, m$ and $j \in J$, then states at time zero and at some later time, say T, are related by

$$\tilde{z}_k(T) = e^{\sigma_k T}\tilde{z}_k(0) + \tilde{b}_k \int_0^T e^{\sigma_k(T-t)}u(t)\,dt, \quad k = 1, 2, \dots, m,$$
$$z_j(T) = e^{\lambda_j T} z_j(0) + b_j \int_0^T e^{\lambda_j(T-t)}u(t)\,dt, \quad j \in J.$$

Results from [**3**] on *admissible input elements* along with density properties of the zeros of $\chi(\lambda)$ following from Rouche's theorem show that, given a state $z(0)$ in ℓ^2, we will have $z(t) \in \ell^2$, $t \in [0,T]$, for any $u \in L^2[0,T]$ just in case the $\tilde{b}_k$, b_j are uniformly bounded (clearly, this is only a restriction on the b_j). On the other hand, if we consider the problem of *exact controllability*, without loss of generality from the initial state $z(0) = 0$ to terminal states $z(T)$, we are led, as in [**11, 12**], e.g., to consider the moment problems

$$\tilde{b}_k \int_0^T e^{\sigma_k(T-t)}u(t)\,dt = \tilde{z}_k(T), \quad k = 1, 2, \dots, m, \tag{4.21}$$

$$b_j \int_0^T e^{\lambda_j(T-t)}u(t)\,dt = z_j(T), \quad j \in J. \tag{4.22}$$

From the fact that the exponentials $e^{\lambda_j t}$, $j \in J$, form a Riesz basis for $L^2[-1,1]$, and, hence, also for $L^2[0,2]$, it can be seen quite readily that the extended set, augmented by $e^{\sigma_k t}$, is *uniformly independent* in $L^2[-,2+\epsilon]$ for any $\epsilon > 0$, by which we mean that these functions form a Riesz basis for a closed subspace of that space. Then, expressing $u(t)$ in terms of functions biorthogonal to $e^{\sigma_k t}$, $e^{\lambda_j t}$, as in [**11, 12**], e.g., (these exist but are not unique; one would normally use those lying in the same closed subspace as is spanned by the exponentials in question, which are unique) one can see that it is possible to reach all states $z(T)$ in ℓ^2, $T = 2 + \epsilon > 2$, just in case the $\tilde{b}_k$, b_j are all nonzero and, in fact, uniformly bounded below.

Since (4.15), (4.16) is equivalent to the original system (4.01), (4.02), (4.03), in that the Riesz basis property of the exponentials involved shows that (4.17), (4.18) defines an isomorphism between ℓ^2 and $H^m[-1,1]$, the desired properties of the $\tilde{b}_k$, b_j in (4.15), (4.16) must follow if we can establish admissibility and exact controllability for (4.01), (4.02), (4.03), considered as a control system in the space $H^m[-1,1]$.

As far as admissibility is concerned, this is a standard result. Given an initial state $\zeta_0(s)$ in $H^m[-1,1]$, the property (2.10) of $v(s)$ allows one to establish, for $u(t)$ locally square integrable, the existence and uniqueness of a solution of (4.01), (4.02), (4.03), which may also be written in the form

$$\begin{aligned} \zeta^{(m)}(t+1) + c_m\zeta^{(m)}(t-1) + \sum_{k=0}^{m-1} c_k\zeta^{(k)}(t+1) \\ + \int_{-1}^{1} \zeta^{(m)}(t+s)\, dv(s) = u(t), \end{aligned} \tag{4.23}$$

for t in a sufficiently small interval $[0,\tau]$ by means of a very standard fixed point argument. The linearity of the equations allows one to see that τ may be taken independent of ζ_0 and, be repeated repetition of the process, solutions can be extended to any interval. There is no need for us to give further details here.

The controllability question for (4.01), (4.02), (4.03) is also a very easy one. One introduces a new input variable into (4.23) by means of a "feedback" relation

$$\begin{aligned} u(t) = c_m\zeta^{(m)}(t-1) + \sum_{k=0}^{m-1} c_k\zeta^{(k)}(t+1) \\ + \int_{-1}^{1} \zeta^{(m)}(t+s)\, dv(s) + f(t), \end{aligned} \tag{4.24}$$

thereby realizing a new, highly simplified, equation

$$\zeta^{(m)}(t+1) = f(t). \tag{4.25}$$

Considering first an interval $0 \leq t \leq \tau$, it is easy to see that if we take $f(t)$ to be a polynomial in t of degree $\leq n-1$ on this interval we can set up a one-to-one linear relationship between the coefficients of that polynomial and $\zeta^{(k)}(1+\tau)$, $k = 0, 1, \ldots, m-1$. Then, taking $f(t)$ to be an arbitrary element of $L^2[\tau, \tau+2]$, we can realize ζ as an arbitrary element of $H^m[\tau, \tau+2]$. Then we can determine $u(t)$ from $f(t)$ via (4.24) to recover the required control for (4.23). In the context of (4.01), (4.02), (4.03) this is precisely the statement that we can pass between arbitrary elements of $H^m[-1,1]$ using a control $u(t)$ in $L^2[0, 2+\tau]$, $\tau > 0$, assuming the initial state given at time $t = 0$. It is also easy to see that we can bound the norm of u in that space in terms of the norms of the initial and terminal states in $H^m[-1,1]$. For a neutral equation of order zero, control interval $[0, \tau+2]$ can be reduced to $[0,2]$.

Thus, we have admissibility and exact controllability for (4.01), (4.02), (4.03) and, therefore, as we have noted, for (4.05), (4.06), so that the $\tilde{b}_k$, b_j must be bounded above and below, thus completing the proof of the theorem.

THEOREM 4.2. *A system* (3.09) *of negative order* $-m$ *can be equivalently represented as a system in* ℓ^2 *of the form*

$$\frac{dz_j}{dt} = \lambda_j z_j + b_j u, \quad \lambda_j \in \Lambda - \Sigma, \tag{4.26}$$

where $E(\Lambda - \Sigma)$ *is a Riesz basis of exponentials for* $H^{-m}[-1,1]$, *as described in Section 2, and the "control distribution coefficients"* $\tilde{b}_k$, b_j *are bounded and bounded below.*

PROOF. As in Section 3, we replace the equation $N * z = u$ by $(\delta_{\{0\}}^{(m)}(0) * N) * \zeta = u$ and solve $\delta_{\{0\}}^{(m)} * \zeta_0 = z_0$ to obtain an initial state for the resulting zero order neutral equation. Given a terminal state $z_1 \in H^{-m}[-1,1]$, we solve $\delta_{\{0\}}^{(m)} * \zeta_1 = z_1$ to obtain a terminal state which we then see, from the discussion earlier, can be reached by employing a control $u \in L^2[0,2]$. Then $\zeta(t+2) = \zeta_1(t)$, $t \in [-1,1]$; applying the m-th derivative to ζ, ζ_1, obtains the trajectory z for the original equation of order $-m$ with the desired terminal state $z_1 \in H^{-m}[-1,1]$. Since $N * z = u$ can be written in the form (4.06) using only the $\lambda_j \in \Lambda - \Sigma$ by expanding z in terms of the exponentials $\mathcal{P}(\lambda_j)e^{\lambda_j}$ (cf. Theorem 2.2), the results on boundedness and boundedness below of the control input coefficients b_j follow from the same arguments as used above for positive order equations.

5. Transformation to control canonical form; spectral assignment. Let us start with a system of augmented hyperbolic type, as described in Section 1,

$$\dot{x} = Ax + gu, \quad x \in X, \tag{5.01}$$

wherein $g \in X$ or else is an admissible input element as described in [**3**], for example, and expanded on a little more in the paragraphs to follow. The operator A has eigenvalues $\sigma_1, \sigma_2, \dots, \sigma_m$ and λ_j, $j \in J$, as described earlier. Assume that the corresponding eigenvectors $\tilde{\varphi}_k$, $k = 1, 2, \dots, m$, φ_j, $j \in J$, form a Riesz basis for X, so that

$$x = \sum_{k=1}^{m} \tilde{x}_k \tilde{\varphi}_k + \sum_{j \in J} x_j \varphi_j, \quad x \in X, \tag{5.02}$$

$$g = \sum_{k=1}^{m} \tilde{g}_k \tilde{\varphi}_k + \sum_{j \in J} g_j \varphi_j, \tag{5.03}$$

the latter expansion to be understood as in [**3**] if g is not an element of X. We also make the further

ASSUMPTION 5.1. *The σ_k, $k = 1, 2, \dots, m$, and λ_j, $j \in J$, are separated and are the zeros of a function $\chi(\lambda)$ such that $\chi(i\lambda)$ is a function of sine type having the form* (4.04) *for coefficients $c_0, c_1, \dots, c_m$ and a function of bounded variation, v, having the property* (2.10).

REMARK . It is not easy to give an adequately general sufficient condition for Assumption 5.1 to be valid, mainly because it is difficult to characterize Laplace transforms of bounded measures dv. In most cases v and the coefficients c_k will be reasonably apparent from analysis of the system (5.01).

We then have the equivalent representation of (5.01) as a system in the space ℓ^2:

$$\frac{d\tilde{x}_k}{dt} = \sigma_k \tilde{x}_k + \tilde{g}_k u, \quad k = 1, 2, \dots, m, \tag{5.04}$$

$$\frac{dx_j}{dt} = \sigma_j x_j + g_j u, \quad j \in J. \tag{5.05}$$

If $b \in X$ we have, of course, $\sum_{j \in J} |g_j|^2 < \infty$, while if g is an admissible input element, the characterization in [**3, 21, 22**] shows that g_j, $j \in J$, must be bounded. In either case,

$$\tilde{g}_k = \langle g, \tilde{\psi}_k \rangle, \quad k = 1, 2, \dots, m, \quad g_j = \langle g, \psi_j \rangle, \quad j \in J, \tag{5.06}$$

where the $\tilde{\psi}_k$, ψ_j, are the elements of the dual Riesz basis biorthogonal to the $\tilde{\varphi}_k$, φ_j.

The main result which we wish to establish concerns spectral assignment in augmented or deficient hyperbolic systems by means of continuous linear, or otherwise *admissible* (cf. [**17**]) state feedback. We will again treat the augmented case, as described in the preceding paragraphs, first. *Continuous* linear state feedback means that we synthesize the control u in (5.01) by means of a *feedback* relation

$$u(t) = \langle x(t), f \rangle_X, \tag{5.07}$$

where, parallel to the more specialized $H^m[-1,1]$ situation described in Section 3, $\langle x(t), f \rangle_X = (x(t), \bar{f})_X$, where $(\ ,\ )_X$ is the inner product in the space X; thus, $\langle\ ,\ \rangle$ is linear in both x and f. In other cases of interest, f is an *admissible output element*, as described in [**3, 21, 22**], in a certain restricted class of linear functionals on X and $\langle\ ,\ \rangle$ denotes the linear functional relationship. In either case, we have an expansion (cf. (5.03), (5.06)), convergent in X when $f \in X$,

$$f = \sum_{k=1}^{m} \tilde{f}_k \tilde{\psi}_k + \sum_{j \in J} f_j \psi_j, \tag{5.08}$$

$$\tilde{f}_k = \langle \tilde{\varphi}_k, f \rangle_X, \quad k = 1, 2, \dots, m, \quad f_j = \langle \varphi_j, f \rangle_X, \quad j \in J. \tag{5.09}$$

Admissibility again requires that the $\tilde{f}_k$, f_j be uniformly bounded. Such an element f can be taken in a larger Hilbert space W with norm

$\| \quad \|_W$ in which X is densely and continuously embedded, each element of W being the limit, with respect to $\| \quad \|_W$, of a sequence $\{f_k\}$ in X and $\langle x, f\rangle_X = \lim_{k\to\infty}\langle x, f_k\rangle_X$, defined for x in a subspace V densely and continuously embedded in X with respect to which W is the dual relative to X (see [**7**], e.g.), so that we have the familiar inclusions

$$V \subset X \subset W.$$

It is further required that $\mathcal{D}(A) \subset V$ so that f, as a linear functional, is defined on a domain which includes that of A. The corresponding requirement for the admissibility of the input element g is that $g \in W$ and has domain including $\mathcal{D}(A^*)$, which is also required to be contained in V. The additional requirements imposed in [**3, 21, 22**] are covered by our assumptions on the spectral decomposition of A and the boundedness assumptions, imposed earlier, on the coefficients $\tilde{g}_k$, g_j, $\tilde{f}_k$, f_j in (5.02), (5.03), (5.08), (5.09).

In all cases described, combining (5.01) with (5.07) yields a *closed loop system*

$$\dot{x} = (A + g \otimes f)x \equiv \hat{A}x, \tag{5.10}$$

where $g \otimes f$ denotes the dyadic operator

$$g \otimes f \equiv \langle x, f\rangle_X g.$$

We should remark that, in general, further conditions must be met in order that $\hat{A}$ should generate a strongly continuous semigroup on X when f and g are admissible elements rather than elements of X (see [**17**], e.g.). These are automatically met in our present situation. Now, suppose we specify another set of complex numbers

$$\rho_1, \rho_2, \dots, \rho_m, \tau_j, \quad j \in J, \tag{5.11}$$

which are associated with coefficients $d_0, d_1, \dots, d_m$ and a function of bounded variation w, also satisfying the condition (2.10), in the same way as the original spectrum was associated with $c_0, c_1, \dots, c_m$ and the function of bounded variation v. We ask: under what circumstances can this sequence of complex numbers actually be realized as the spectrum of $\hat{A}$ in (5.10) by appropriate choice of f (assuming g given; given f the problem would be to determine g similarly).

THEOREM 5.2. *Let the control distribution coefficients* (5.02), (5.03) *for the system* (5.01) *all be nonzero. Then*:

(i) *By appropriate choice of the output element $f \in X$, one may realize as eigenvalues of the closed loop system any collection of numbers* (5.11) *for which*

$$\sum_{j \in J} \left| \frac{\tau_j - \lambda_j}{g_j} \right|^2 < \infty; \tag{5.12}$$

(ii) *If g is taken to be an admissible input element for which the control input coefficients g_j are bounded and bounded below, the condition* (5.12) *may be replaced by the weaker requirement that the differences $\tau_j - \lambda_j$, $j \in J$, should be square summable*;

(iii) *If, in addition to what is assumed under* (ii)*, f is permitted to be an admissible output element, then any spectrum* (5.11)*, subject to the conditions set forth there, can be realized in the closed loop system.*

PROOF. Our assumptions imply that we can find a system (4.01), (4.02), (4.03) with associated operator $\mathcal{A}$ described by (3.08) such that $\mathcal{A}$ has precisely the eigenvalues σ_k, $k = 1, 2, \ldots, m$, σ_j, $j \in J$. From the Riesz basis property of the associated exponentials (cf. Theorem 2.1)

$$\begin{aligned} \tilde{p}_k(t) &= e^{\sigma_k t}, \quad k = 1, 2, \ldots, m, \\ p_j(t) &= e^{\lambda_j t} / \mathcal{P}(\lambda_j), \quad j \in J, \end{aligned} \tag{5.13}$$

in $H^m[-1,1]$, we see that the linear operator $T_0 : H^m[-1,1] \to X$ given by

$$\begin{aligned} T_0(\tilde{p}_k) &= \tilde{\varphi}_k, \quad k = 1, 2, \ldots, m, \\ T_0(p_j) &= \varphi_j, \quad j \in J, \end{aligned} \tag{5.14}$$

is an isomorphism. Setting $x = T_0 y$, (5.01) is carried into

$$\dot{y} = T_0^{-1} A T_0 y + T_0^{-1} g u = \mathcal{A} y + T_0^{-1} g u, \tag{5.15}$$

which is in the form (4.01), (4.02), (4.03) except, in general, for the control term.

Let us first consider the case wherein $g, f \in X$. Then

$$T_0^{-1} g = \hat{g}(s) = \sum_{k=1}^{m} \tilde{g}_k \tilde{p}_k(s) + \sum_{j \in J} g_j p_j(s) \tag{5.16}$$

is convergent in $H^m[-1,1]$ and has a representation

$$\hat{g}(s) = (\hat{g}(0), \hat{g}'(0), \dots, \hat{g}^{(m-1)}(0), \hat{g}^{(m)}(\cdot)).$$

Taking account of the form of $\mathcal{A}$ as shown in (3.05), (3.06), (3.07), (5.15) becomes

$$\frac{\partial y^{(m)}}{\partial t}(t,s) = \frac{\partial y^{(m)}}{\partial x}(t,s) + \hat{g}^{(m)}(s) u(t), \tag{5.17}$$

(5.18)
$$\frac{dy^{(k)}}{dt}(t,0) = y^{(k+1)}(t,0) + \hat{g}^{(k)}(0) u(t), \quad k = 0, 1, \dots, m-1,$$

together with the homogeneous boundary condition
(5.19)
$$y^{(m)}(t,1) + c_m y^{(m)}(t,-1) + \sum_{k=0}^{m-1} c_k y^{(k)}(t,0) + \int_{-1}^{1} y^{(m)}(t,s) \hat{v}(s)\, ds = 0.$$

In the above process the feedback relation (5.07) transforms via

$$u(t) = \langle x(t), f \rangle_X = \langle T_0 y(t), f \rangle_X = \langle y(t), T_0^* f \rangle_{H^m[-1,1]}. \tag{5.20}$$

The element $f \in X$ is expressed via (5.08), (5.09) as a convergent series in the dual basis elements $\tilde{\psi}_k$, ψ_j, biorthogonal to the eigenvectors $\tilde{\varphi}_k$, $\tilde{\varphi}_j$ of A. Letting $\tilde{q}_k()$, $k = 1, 2, \dots, m$, $q_j(s)$, $j \in J$, be the dual Riesz basis for $H^m[-1,1]$ biorthogonal to the $\tilde{p}_k$, p_j, as shown in (5.13), we see easily that

$$\begin{aligned} T_0^* \tilde{\psi}_k &= \tilde{q}_k, \quad k = 1, 2, \dots, m, \\ T_0^* \psi_j &= q_j, \quad j \in J, \end{aligned} \tag{5.21}$$

so that

$$T_0^* f = \hat{f}(s) = \sum_{k=1}^{m} \tilde{f}_k \tilde{q}_k(s) + \sum_{j \in J} f_j q_j(s). \tag{5.22}$$

At this point the whole problem of spectral assignment has been transferred from (5.01), (5.07), to the system (5.17), (5.18), (5.19), since the fact that T_0 is an isomorphism guarantees that the eigenvalues of the corresponding closed loop systems will be identical.

Next we will construct a transformation which carries (5.17), (5.18), (5.19) into a system of the form (4.01), (4.02), (4.03), which serves as the *control canonical form* for the original system (5.01). We have seen that the control input coefficients for (4.01), (4.02), (4.03) are certain complex numbers $\tilde{b}_k$, b_j, bounded and bounded away from zero. We define a transformation T_1 on $H^m[-1,1]$ by specifying

$$T_1(\tilde{b}_k \tilde{p}_k) = \tilde{g}_k \tilde{p}_k, \quad k = 1, 2, \dots, m, \qquad T_1(b_j p_j) = g_j p_j, \quad j \in J. \tag{5.23}$$

Since the $\tilde{b}_k$, b_j are bounded and bounded away from zero, since the $\tilde{p}_k$, p_j form a Riesz basis for $H^m[-1,1]$, and since the $\tilde{g}_k$, g_j are square summable under our present assumptions, T_1 is bounded and one-to-one and has an inverse T_1^{-1} defined on its range. It is clear, however, that T_1^{-1} must be unbounded relative to the norm in $H^m[-1,1]$.

We transform (5.17), (5.18), (5.19) in the same way, setting

$$y = T_1 z. \tag{5.24}$$

Since T_1 commutes with $\mathcal{A}$, (5.15) transforms to

$$\dot{z} = \mathcal{A}\, z + T_1^{-1} T_0^{-1} g u. \tag{5.25}$$

But (5.14), (5.23) shows that

$$T_1^{-1} T_0^{-1} g = \sum_{k=1}^{m} \tilde{b}_k \tilde{p}_k + \sum_{j \in J} b_j p_j,$$

so that the control distribution coefficients are the same as for (4.01), (4.02), (4.03). We conclude that (5.24) transforms (5.17), (5.18), (5.19) over to (4.01), (4.02), (4.03).

The feedback relation (5.07), (5.20), transforms to

$$u(t) = \langle y(t), T_0^* f \rangle_{H^m[-1,1]} = \langle z(t), T_1^* T_0^* f \rangle_{H^m[-1,1]}. \tag{5.26}$$

Clearly,

$$T_1^* T_0^* f \equiv F(s) = \sum_{k=1}^{m} \frac{\tilde{g}_k \tilde{f}_k}{\tilde{b}_k} \tilde{q}_k(s) + \sum_{j \in J} \frac{g_j f_j}{b_j} q_j(s) \tag{5.27}$$
$$\equiv \sum_{k=1}^{m} \tilde{F}_k \tilde{q}_k(s) + \sum_{j \in J} F_j q_j(s).$$

It will be observed that the square summability of the f_j, together with the fact that the b_j are bounded and bounded below and (5.27), implies that

$$\sum_{j \in J} |F_j/g_j|^2 < \infty. \tag{5.28}$$

It is also clear that if F is given by the last expression in (5.27) and (5.28) obtains, then we can find f so that the first identity in holds.

The essential part of the proof lies in showing that, as we pass from the system (4.01), (4.02), (4.03) to the closed loop system, which we realize by adjoining the feedback relation (5.26), we may, in so doing realize any closed loop eigenvalues (5.11), as described and restricted there, which satisfy the additional condition (5.12). From the specification of these desired eigenvalues, we know that there is another system, in a form comparable to (4.01), (4.02), (4.03), which has these eigenvalues, namely,

$$\frac{\partial \zeta^{(m)}}{\partial t}(t,s) = \frac{\partial \zeta^{(m)}}{\partial s}(t,s), \quad -1 < s < 1, \ -\infty < t < \infty, \tag{5.29}$$

$$\frac{d\zeta^{(k)}}{dt}(t,0) = \zeta^{(k+1)}(t,0), \quad k = 0, 1, \dots, m-1, \tag{5.30}$$

(5.31)
$$\zeta^{(m)}(t,1) + d_m \zeta^{(m)}(t,-1) + \sum_{k=0}^{m-1} d_k \zeta^{(k)}(t,0) + \int_{-1}^{1} \zeta^{(m)}(t,s)\, dw(s) = 0.$$

If we make the designations

$$C = (c_0, c_1, \dots, c_{m-1}, dv), \quad D = (d_0, d_1, \dots, d_{m-1}, dw),$$

then (4.02) has the formal structure

$$z^{(m)}(t,1) + c_m z^{(m)}(t,-1) + \langle z, C\rangle_m = u(t) \tag{5.32}$$

while (5.31) has the comparable form

$$\zeta^{(m)}(t,1) + d_m \zeta^{(m)}(t,-1) + \langle \zeta, D\rangle_m = 0. \tag{5.33}$$

If we also write (cf. (5.27))

$$F = (F(0), F'(0), \dots, F^{(m-1)}(0), F^{(m)}, (\cdot)),$$

then the feedback relation (5.26) assumes the form

$$u(t) = \langle z, F\rangle_m. \tag{5.34}$$

The closed loop system obtained from substituting (5.34) into (5.32) is then

$$z^{(m)}(t,1) + c_m z^{(m)}(t,-1) + \langle z, C - F\rangle_m = 0. \tag{5.35}$$

Since there is a one-to-one correspondence between elements C, D, etc. and characteristic functions $\chi(\lambda)$ (cf. (4.04)), the systems (5.33) and (5.35) have the same spectrum if and only if they are, in fact, identical, which requires

$$d_m = c_m, \quad D = C - F. \tag{5.36}$$

In the case currently under consideration, $F^{(m)} \in L^2[-1,1]$. The coefficients $c_k - F^{(k)}(0)$ can be matched to d_k, $k = 0, 1, \dots, m-1$, without restriction, by appropriate choice of F. But since dv and dw are, in general, bounded measures, matching $C - F$ to D entails matching the measure $dv(s) - F^{(m)}(s)$ to $dw(s)$, which can be done only with some assumptions relating dv and dw, i.e., dv and dw may differ only by an absolutely continuous measure $\varphi(s)\,ds$ with $\varphi \in L^2[-1,1]$. There arises the question as to just when this is the case.

If we let

$$\mathcal{C} = \delta^{(m)}_{\{1\}} + c_m \delta^{(m)}_{\{-1\}} + C, \quad \mathcal{D} = \delta^{(m)}_{\{1\}} + c_m \delta^{(m)}_{\{-1\}} + D,$$

where $\delta^{(k)}$ denotes the k-th order Dirac distribution and $\{\quad\}$ indicates its support, then $\chi(\lambda)$, the characteristic function for (5.32), has the form

$$\chi(\lambda) = \langle e^{\lambda\cdot}, C\rangle,$$

while the characteristic function for the system (5.33) is

$$\Xi(\lambda) = \langle e^{\lambda\cdot}, \mathcal{D}\rangle.$$

Thus,

$$\chi(\lambda) - \Xi(\lambda) = \langle e^{\lambda\cdot}, \mathcal{C} - \mathcal{D}\rangle = \langle e^{\lambda\cdot}, C - D\rangle$$

and $D = C - F$ just in case

$$\chi(\lambda) - \Xi(\lambda) = \langle e^{\lambda\cdot}, F\rangle = \langle e^{\lambda\cdot}, F\rangle_m.$$

Representing F in the form (5.27), clearly (cf. (5.13)),

$$F_j = \langle e^{\lambda_j\cdot}/\mathcal{P}(\lambda_j), F\rangle_m, \quad j \in J. \tag{5.37}$$

On the other hand, the exponentials $e^{\rho_k t}$, $k = 1, 2, \dots, m$, $e^{\tau_j t}/\mathcal{R}(\tau_j)$, $j \in J$, also form a Riesz basis for $H^{(m)}[-1,1]$, with corresponding dual basis $\tilde{r}_k$, $k = 1, 2, \dots, m$, r_j, $j \in J$, and we may also write

$$\begin{aligned} F &\equiv \sum_{k=1}^{m} \tilde{\Phi}_k \tilde{r}_k(s) + \sum_{j\in J} \Phi_j r_j(s), \\ \Phi_j &= \langle e^{\lambda_j\cdot}/\mathcal{R}(\tau_j), F\rangle_m, \quad j \in J, \end{aligned} \tag{5.38}$$

where

$$\mathcal{R}(\lambda) = \prod_{k=1}^{m} (\lambda - \rho_k).$$

LEMMA 5.3. *There is a positive constant K such that, for $k = 0,, 1, \dots, m$,*

$$|\lambda_j^k e^{\lambda_j t}/\mathcal{P}(\lambda_j) - \tau_j^k e^{\tau_j t}/\mathcal{R}(\tau_j)| \le K|\lambda_j - \tau_j|, \quad j \in J,\ t \in [-1,1]. \tag{5.39}$$

PROOF. Let L_j be the straight line segment joining λ_j to τ_j in the complex plane. For $\lambda \in L_j$, we let

$$\mathcal{P}_j(\lambda) = \prod_{k=1}^{m} \left(\lambda - \left(\frac{\lambda - \lambda_j}{\tau_j - \lambda_j}\rho_k + \frac{\tau_j - \lambda}{\tau_j - \lambda_j}\sigma_k\right)\right)$$

if $\tau_j \neq \lambda_j$; otherwise, $\mathcal{P}_j(\lambda) \equiv \mathcal{P}(\lambda)$. Clearly,

$$\mathcal{P}_j(\lambda_j) = \mathcal{P}(\lambda_j), \quad \mathcal{P}_j(\tau_j) = \mathcal{R}(\tau_j).$$

Then

$$\lambda_j^k e^{\lambda_j t}/\mathcal{P}(\lambda_j) - \tau_j^k e^{\tau_j t}/\mathcal{R}(\tau_j) = \int_{\tau_j}^{\lambda_j} \frac{d}{d\lambda}\left(\frac{\lambda^k e^{\lambda t}}{\mathcal{P}_j(\lambda)}\right) d\lambda.$$

For $|\lambda_j|$ sufficiently large and $t \in [-1, 1]$, the functions $\lambda^k e^{\lambda t}\mathcal{P}_j(\lambda)$, $k = 0, 1, \ldots, m$, are uniformly bounded in a neighborhood of L_j whose radius is independent of j. Applying the Cauchy estimate for the derivative and performing the indicated integrations, we have (5.39). □

Continuation of Proof of Theorem 5.2. We have

$$\lambda_j^m e^{\lambda_j}/\mathcal{P}(\lambda_j) + c_m \lambda_j^m e^{-\lambda_j}/\mathcal{P}(\lambda_j) + \langle e^{\lambda_j \cdot}/\mathcal{P}(\lambda_j), C\rangle = 0, \tag{5.40}$$

$$\tau_j^m e^{\tau_j}/\mathcal{R}(\tau_j) + c_m \tau_j^m e^{-\tau_j}/\mathcal{R}(\tau_j) + \langle e^{\tau_j \cdot}/\mathcal{R}(\tau_j), D\rangle = 0. \tag{5.41}$$

From our earlier remark, $D = C - F$ just in case

$$\tau_j^m e^{\tau_j}/\mathcal{R}(\tau_j) + c_m \tau_j^m e^{-\tau_j}/\mathcal{R}(\tau_j) + \langle e^{\tau_j \cdot}/\mathcal{R}(\tau_j), C - F\rangle = 0. \tag{5.42}$$

Subtracting (5.40) from (5.42) gives

$$\begin{aligned}\langle e^{\tau_j \cdot}/\mathcal{R}(\tau_j), F\rangle &= \tau_j^m e^{\tau_j}/\mathcal{R}(\tau_j) + c_m \tau_j^m e^{-\tau_j}/\mathcal{R}(\tau_j) + \langle e^{\tau_j \cdot}/\mathcal{R}(\tau_j), C\rangle \\ &\quad - \lambda_j^m e^{\lambda_j}/\mathcal{P}(\lambda_j) - c_m \lambda_j^m e^{-\lambda_j}/\mathcal{P}(\lambda_j) - \langle e^{\lambda_j \cdot}/\mathcal{P}(\lambda_j), C\rangle.\end{aligned} \tag{5.43}$$

Taking account of the form of C and applying the estimate (5.39), we have

$$\Phi_j = \langle e^{\tau_j \cdot}/\mathcal{R}(\tau_j), F\rangle = O(|\lambda_j - \tau_j|).$$

Assuming that F lies in $H^m[-1, 1]$, use of the estimate (5.39) again shows that

$$\begin{aligned} F_j &= \langle e^{\lambda_j \cdot}/\mathcal{P}(\lambda_j), F\rangle \\ &= \langle e^{\tau_j \cdot}/\mathcal{R}(\tau_j), F\rangle + \left\langle \left(e^{\lambda_j \cdot}/\mathcal{P}(\lambda_j) - e^{\tau_j \cdot}/\mathcal{R}(\tau_j)\right), F\right\rangle \\ &= O(|\lambda_j - \tau_j|),\end{aligned} \tag{5.44}$$

and we conclude that if (5.12) is true, then (5.28) is true also. Then, as remarked earlier, we can achieve the relationship (5.27) between f and F with $f \in X$ and we have completed the proof of the first part of the theorem.

We remark that the results so far proved can be obtained by a different procedure developed by Sun in [**19**] and, in the zero order case, were obtained by the present author in [**14**] using a similar procedure to that just outlined.

The proof of part (ii) is really no different from that given above, the only (agreeable) difference being that the transformation T_1 in (5.23) is boundedly invertible.

The proof of (iii) is trivial in the functional equation context; in fact, we do not need to set $c_m = d_m$ as assumed earlier in (5.36). Given C, D, c_m and d_m, we simply set $F = c_m \delta^{(m)}_{\{-1\}} + C - d_m \delta^{(m)}_{\{-1\}} - D$. The only question is whether f, the output element indirectly found, via F and (5.27), for the original system, is admissible. From the results in [**3**] (dualized from input to output), [**21**] and [**22**, with the density results (cf. discussion preceding (4.21)) for the spectra of Theorem 4.1 and (5.11), it is sufficient to show that the coefficients f_j, and, hence, in view of (5.27) and the assumption that the g_j are bounded below, the F_j are bounded. From (5.40)–(5.44),

$$F_j = \langle e^{\lambda_j \cdot}/\mathcal{P}\,(\lambda_j), C - D\rangle_m.$$

Now

$$\langle e^{\lambda_j \cdot}/\mathcal{P}\,(\lambda_j), C\rangle_m = \left(c_m \lambda_j^m e^{-\lambda_j} + \sum_{k=1}^{m-1} c_k \lambda_j^k + \int_{-1}^{1} e^{\lambda_j s}\, dv(s) \right) /\mathcal{P}\,(\lambda_j)$$

is bounded because $\mathcal{P}$ has degree m and v is a function of bounded variation. The same argument applies with C replaced by D and c_m by d_m, and we see that the F_j, hence the f_j, are bounded. □

Part (iii) of Theorem 5.2 almost says that we can pass from the spectrum of Theorem 4.1 to (5.11) if the differences $\tau_j - \lambda_j$ are just bounded. It does not quite say that, and there is good reason for it; the Riesz basis property of exponentials in $H^m[-1, 1]$, as described in

Section 2, is not, in general, preserved by bounded perturbations of the λ_j. The classical example is given by Levinson in [**6**], where he shows that the exponential set $\left\{e^{\pi i[k+1/2+\operatorname{sgn}(k+1/2)/4]} \mid -\infty < k < \infty\right\}$ is not a Riesz basis for $L^2[-1,1]$.

THEOREM 5.4. *The results of* Theorem 5.2 *remain valid for systems* (5.01) *of deficient hyperbolic type.*

SKETCH OF THE PROOF. Mappings T_0 and T_1 carrying system (5.01) into a system corresponding to a neutral equation of negative order are constructed in much the same way as they are in Theorem 5.2; there is no need to repeat the process. It is sufficient to consider the details of passing from a neutral system of order $-m$ to another system of the same type by means of linear feedback control synthesis.

Suppose we are given such a system, say

$$N * z = u, \quad u \in L^2_{\mathrm{loc}}(-\infty,\infty), \tag{5.45}$$

and a second, homogeneous, system of the same (negative) order:

$$M * z = 0. \tag{5.46}$$

We use a procedure similar to that used at the end of Section 3. Let $\mathcal{P}(\lambda)$ be a polynomial of degree m with distinct zeros $\sigma_1, \sigma_2, \ldots, \sigma_m$ different from any of the zeros of the characteristic functions $\mathcal{N}(\lambda)$ of (5.45) or the characteristic function $\mathcal{M}(\lambda)$ for (5.46). Multiplying both equations, in the convolution sense, on the right by $\mathcal{P}(\delta'_{\{0\}})$ yields

$$(\mathcal{P}(\delta'_{\{0\}}) * N) * \zeta = \mathcal{P}(\delta'_{\{0\}}) * u \equiv w, \tag{5.47}$$

$$(\mathcal{P}(\delta'_{\{0\}}) * M)\zeta = 0. \tag{5.48}$$

Since M and N have the form indicated in (3.09), the equations (5.47) and (5.48) are standard scalar neutral equations of order zero; they correspond to systems consisting of (4.01) with $m = 0$ and respective boundary conditions

$$z(t,1) + cz(t,-1) + \int_{-1}^{1} z(t,s)\, d\nu(s) = w(s)$$

and

$$z(t,1) + dz(t,-1) + \int_{-1}^{1} z(t,s)\, d\mu(s) = 0,$$

where ν and μ are functions of bounded variation as described in (2.10). Writing (5.47) and (5.48) as (cf. (5.32), (5.33))

$$z(t,1) + \langle z, c\delta_{\{-1\}} + C\rangle = w, \tag{5.49}$$

$$z(t,1) + \langle z, d\delta_{\{-1\}} + D\rangle = 0, \tag{5.50}$$

a feedback relation

$$w(t) = \langle z(t,\cdot), e\delta_{\{-1\}} + E\rangle \tag{5.51}$$

carries (5.49) into (5.50) just in case $e = c - d$ and $E = C - D$. Since $\mathcal{P}(\lambda)$ is a common polynomial factor of $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$, if we determine e and E in this way, $\mathcal{P}(\lambda)$ will be a factor of $\mathcal{E}(\lambda)$, the characteristic function of $e\delta_{\{-1\}} + E$. Consequently,

$$e\delta_{\{-1\}} + E = \mathcal{P}(\delta'_{(0)})F,$$

and the feedback relation

$$u(t) = \langle z(t,\cdot), F\rangle \tag{5.52}$$

then carries (5.45) into (5.46). The results of Theorem 5.2 apply to (5.49), (5.50), and (5.51), the only special circumstance being that the zeros $\sigma_1, \sigma_2, \ldots, \sigma_m$ are left invariant. All three parts of Theorem 5.2 will then have counterparts in Theorem 5.4. With a condition of the form (5.12), which requires $c = d$, we can take $e = 0$ and E an element of $H^0[-1,1] = l^2[-1,1]$, and then F will be an element of $H^{-m}[-1,1]$, the state space for the systems (5.45), (5.46). This gives counterparts of parts (i) and (ii). The counterpart of (iii) allows c and d to be different. In this case, comparing with the originally given form (3.09) for neutral equations of negative order, the feedback relation F will take the form

$$u(t) = \int_{-1}^{1} z(t,s)\, d\varphi(s),$$

where φ satisfies the conditions imposed on ν (with $m = 0$) in (3.10), except that $\neq$ in the second condition of (3.09) for ν is replaced by $=$ in the corresponding conditions on φ.

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