JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 18, Number 2, Summer 2006

ASYMPTOTIC BEHAVIOR OF GENERALIZED CONVOLUTIONS: AN ALGEBRAIC APPROACH

BERND SILBERMANN AND OLGA ZABRODA

Dedicated to Professor Kendall Atkinson on the occasion
of his sixty-fifth birthday

ABSTRACT. We study the structure of the C^*-algebra generated by sequences of generalized convolutions (also called variable Toeplitz matrices) and derive results on the asymptotic behavior of the spectra of elements belonging to this algebra. For instance, we prove the analog of the Se Legue/Szegö and Avram/Parter theorems.

1. Introduction. Given a sequence $\{A_N\}_{N\in\mathbf{Z}^+}$, $\mathbf{Z}^+ := \mathbf{N}\cup\{0\}$, of quadratic $(N+1)\times(N+1)$ matrices constituted by some rule, one of the important questions is about the asymptotic behavior of the eigenvalues $\lambda_0^{(N)},\dots,\lambda_N^{(N)}$ of the matrices A_N.

For example, if $A_N = (\hat{a}_{n-k})_{n,k=0}^N$ are Toeplitz matrices given by the Fourier coefficients of an L^∞-function a, defined on the complex unit circle $\mathbf{T}$, then

$$\frac{1}{N+1}\operatorname{tr} f(T_N(a)) = \frac{1}{N+1}\sum_{j=0}^{N} f\left(\lambda_j^{(N)}\right) \longrightarrow \frac{1}{2\pi}\int_0^{2\pi} f\left(a\left(e^{i\theta}\right)\right)\, d\theta,$$

where $\operatorname{tr} B$ is the trace of the (quadratic) matrix B, f analytic on some open set $\Omega\in\mathbf{C}$ containing the convex hull of $R(a)$, the essential range of a. This result is one of the versions of Szegö's first limit theorem. A discussion of the topic, including a refinement of Se Legue's approach and the Avram/Parter theorem, can be found in [**1**, Sections 5.4–5.7].

Similar problems were studied for various generalizations of familiar Toeplitz matrices, especially for locally Toeplitz matrices, [**5, 7–9, 12**], and generalized convolutions (variable Toeplitz matrices), [**2, 6, 10, 11**].

Received by the editors on August 3, 2005, and in revised form on October 5, 2005.

In this paper, the case of generalized convolutions is taken up again. The significant difference to previous work is the consideration of sequences belonging to the C^*-algebra of all sequences generated by "smooth" generating functions. We give a full description of this algebra and derive the mentioned generalizations. Furthermore, results concerning the convergence of condition numbers, ε-pseudo spectra and sets of singular values can be obtained by help of the theory developed in [**5**]. Because this can easily be done, we omit the details.

Given a continuous (complex-valued) function a on the set $[0,1] \times [0,1] \times \mathbf{T}$, assign to a the sequence $\{A_N(a)\}_{N \in \mathbf{N}}$ of variable Toeplitz matrices

$$A_N(a) := \left(\hat{a}_{n-k}\left(\frac{n}{N}, \frac{k}{N}\right)\right)_{n,k=0,\dots,N}$$

where $\hat{a}_n(x,y)$ are the Fourier-coefficients of a, see Section 1. It can be shown that there exists a continuous function $b \in C([0,1] \times [0,1] \times \mathbf{T})$ such that the sequence $\{A_N(b)\}$ is not bounded with respect to spectral norm (S. Grudsky, private communication). So we have to impose some conditions on the generating functions; this will be done in Section 2, where also some important strong limits are computed. Sections 3–6 are devoted to some relations which are important to analyze the structure of the C^*-algebra under consideration. The papers [**4**] and [**11**] are of special importance for us because some technical equipment is taken from there. The most important ingredient is however the C^*-technique introduced by Roch and one of the authors, see for instance [**5**].

1. Preliminary notations. Let us introduce the following notation: $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ are the sets of natural, integer, real and complex numbers, respectively, $\mathbf{Z}^+ = \mathbf{N} \cup \{0\}$;

$C(M)$ is the space of bounded continuous complex-valued functions, defined on a Hausdorff compact space M, with the standard norm

$$||f(z)||_{C(M)} = \sup_{z \in M} |f(z)|;$$

$l_q(\mathbf{Z}^+)$, where $q \geqslant 1$, is the standard Banach space of complex sequences $X = \{X_n\}_{n \in \mathbf{Z}^+}$ with the norm

$$||X|| = \left(\sum_{n \in \mathbf{Z}^+} |X_n|^q\right)^{1/q} < \infty;$$

P_N for $N \in \mathbf{N}$ is the projector in $l_2(\mathbf{Z}^+)$, acting as

$$P_N : \{X_0, X_1, \dots, X_N, X_{N+1}, \dots\} \longmapsto \{X_0, X_1, \dots, X_N, 0, 0, \dots\};$$

$\mathbf{T}$ is the unit circle in the complex plane;

μ is the Lebesque-measure on $\mathbf{T}$.

For a function $a(x,y,t) \in C([0,1] \times [0,1] \times \mathbf{T})$, let us denote by $\hat{a}_n(x,y)$ its Fourier coefficients, defined by

$$\hat{a}_n(x,y) = \frac{1}{2\pi} \int_{\mathbf{T}} a(x,y,t) t^{-n} \, d\mu, \quad x, y \in [0,1], \quad n \in \mathbf{Z}.$$

Further, let us denote by $A_N(a) = A_N(a(x,y,t))$ the linear operator acting in the space $\operatorname{Im} P_N$ which has the following matrix

$$\left(\hat{a}_{n-k} \left(\frac{n}{N}, \frac{k}{N} \right) \right)_{n,k=0,\dots,N}$$

with respect to the canonical basis of $\operatorname{Im} P_N$. We shall identify operators in $\operatorname{Im} P_N$ with matrices in this natural way.

2. The strong convergence of operators $A_N(a)$, $A_N^*(a)$, $W_N A_N(a) W_N$ and $W_N A_N^*(a) W_N$ in the case of the smooth generating function. For $b \in C(\mathbf{T})$ we designate by $T(b) : l^2(\mathbf{Z}^+) \to l^2(\mathbf{Z}^+)$ the Toeplitz operator which has the following (infinite) matrix

$$\left(\hat{b}_{n-k} \right)_{n,k \in \mathbf{Z}^+}.$$

It is well known that

$$||T(b)|| = ||b||_{C(\mathbf{T})}.$$

Let $C_t^\infty([0,1] \times [0,1] \times \mathbf{T})$ be the subspace of $C([0,1] \times [0,1] \times \mathbf{T})$, consisting of all functions, which are infinitely differentiable in the variable $t \in \mathbf{T}$ with derivatives from $C([0,1] \times [0,1] \times \mathbf{T})$.

Let us note one important detail. If the function $a(x,y,t)$ belongs to $C_t^\infty([0,1] \times [0,1] \times \mathbf{T})$, then (this can be shown using its continuity) the sum of the series of its Fourier coefficients satisfies the Weierstrass' criterion of uniform convergence:

$$\sum_{n\in\mathbf{Z}} \sup_{x,y\in[0,1]} |\hat{a}_n(x,y)| < \infty. \tag{2.1}$$

Proposition 2.1. *Let* $a(x,y,t) \in C_t^\infty([0,1]\times[0,1]\times\mathbf{T})$.

Then the operator $A_N(a)$ *converges strongly to the operator* $T(a(0,0,t))$ *and the operator* $A_N^*(a)$ *converges strongly to the operator* $T^*(a(0,0,t))$*, respectively.*

Proof. Let $\mathcal{M} = \sum_{n\in\mathbf{Z}} \sup_{x,y\in[0,1]} |\hat{a}_n(x,y)| < \infty$.

It is known, see [**1**, Lemma 2.22], that the operator $P_N T(a(0,0,t))P_N$ converges strongly to $T(a(0,0,t))$ for $N\to\infty$. Therefore, it is enough to show that

$$||A_N(a)X - P_N T(a(0,0,t))P_N X|| \underset{N\to\infty}{\longrightarrow} 0$$

for $X = \{X_n\}_{n\in\mathbf{Z}^+} \in l_2(\mathbf{Z}^+)$.

Let us fix an arbitrary $\varepsilon > 0$. Then for $N \in \mathbf{N}$ large enough we obtain:

$$\begin{aligned}
&||A_N(a)X - P_N T(a(0,0,t))P_N X||^2 \\
&\quad = \sum_{n=0}^{N} \left| \sum_{k=0}^{N} \left(\hat{a}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right) - \hat{a}_{n-k}(0,0)\right) X_k \right|^2 \\
&\quad \leqslant \sum_{n=0}^{N} \left(\sum_{k=0}^{N} \left| \hat{a}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right) - \hat{a}_{n-k}(0,0)\right| |X_k| \right)^2 \\
&\quad \leqslant \sum_{n=0}^{N} \left\{ \sum_{k=0}^{N} \left| \hat{a}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right) - \hat{a}_{n-k}(0,0)\right| \right\} \\
&\qquad \times \left\{ \sum_{k=0}^{N} \left| \hat{a}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right) - \hat{a}_{n-k}(0,0)\right| |X_k|^2 \right\} \\
&\quad \leqslant 2\mathcal{M} \sum_{n=0}^{N} \sum_{k=0}^{N} \left| \hat{a}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right) - \hat{a}_{n-k}(0,0)\right| |X_k|^2
\end{aligned}$$

$$= 2\mathcal{M} \sum_{0\leqslant n\leqslant \beta\sqrt{N}} \sum_{0\leqslant k\leqslant \alpha\sqrt{N}} \left|\hat{a}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right) - \hat{a}_{n-k}(0,0)\right| |X_k|^2$$
$$+ 2\mathcal{M} \sum_{n=0}^{N} \sum_{\alpha\sqrt{N}<k\leqslant N} \left|\hat{a}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right) - \hat{a}_{n-k}(0,0)\right| |X_k|^2$$
$$+ 2\mathcal{M} \sum_{\beta\sqrt{N}<n\leqslant N} \sum_{0\leqslant k\leqslant \alpha\sqrt{N}} \left|\hat{a}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right) - \hat{a}_{n-k}(0,0)\right| |X_k|^2,$$

where α and β are arbitrary positive real numbers.

Since the Fourier coefficients of the function $a(x,y,t)$ are continuous in variables $x, y \in [0,1]$, there exists such a $\delta > 0$ that

$$\sum_{n\in\mathbf{Z}} \sup_{x,y\in[0,\delta]} |\hat{a}_n(x,y) - \hat{a}_n(0,0)| < \frac{1}{3}\,\frac{\varepsilon^2}{2\mathcal{M}||X||^2}.$$

Let us fix further α and β so that $0 < \alpha < \beta < \delta$.

So long as $X = \{X_n\}_{n\in\mathbf{Z}^+} \in l_2(\mathbf{Z}^+)$, there exists such $N_0 \in \mathbf{N}$, that the following inequalities hold for all $N > N_0$:

$$\frac{\beta}{\sqrt{N}} < \delta,$$

$$\sum_{|k|>\alpha\sqrt{N}} |X_k|^2 < \frac{1}{3}\,\frac{\varepsilon^2}{4\mathcal{M}^2}$$

and

$$\sum_{|n|>(\beta-\alpha)\sqrt{N}} \sup_{x,y\in[0,1]} |a_n(x,y)| < \frac{1}{3}\,\frac{\varepsilon^2}{4\mathcal{M}||X||^2}.$$

Then, for such α, β and N_0, we obtain the following inequality:

$$||A_N(a)X - P_N T(a(0,0,t))P_N X||^2$$
$$\leqslant 2\mathcal{M} \sum_{k\in\mathbf{Z}} \left\{ \sum_{n\in\mathbf{Z}} \sup_{x,y\in[0,\delta]} |\hat{a}_n\,(x,y) - \hat{a}_n(0,0)| \right\} |X_k|^2$$
$$+ 2\mathcal{M} \sum_{|k|>\alpha\sqrt{N}} \left\{ \sum_{n\in\mathbf{Z}} \sup_{x,y\in[0,1]} |\hat{a}_n(x,y) - \hat{a}_n(0,0)| \right\} |X_k|^2$$

$$
\begin{aligned}
&+ 2\mathcal{M} \sum_{k\in\mathbf{Z}} \Big\{ \sum_{|n|>(\beta-\alpha)\sqrt{N}} \sup_{x,y\in[0,1]} |\hat{a}_n(x,y) - \hat{a}_n(0,0)| \Big\} |X_k|^2 \\
&< \frac{1}{3}\,\varepsilon^2 + \frac{1}{3}\,\varepsilon^2 + \frac{1}{3}\,\varepsilon^2 = \varepsilon^2,
\end{aligned}
$$

which proves the proposition.

Analogous reasoning holds for the conjugate operator. □

We denote by W_N the operator of inversion from $l_2(\mathbf{Z}^+)$ into $\operatorname{Im} P_N$ which act as follows:

$$W_N : \{X_0, X_1, \dots, X_N, X_{N+1}, \dots\} \longmapsto \{X_N, X_{N-1}, \dots, X_0, 0, 0, \dots\};$$

Proposition 2.2. *Let $a(x,y,t) \in C_t^\infty([0,1] \times [0,1] \times \mathbf{T})$.*

Then the operator $W_N A_N(a) W_N$ converges strongly to the operator $T\left(a\left(1,1,t^{-1}\right)\right)$ and the operator $W_N A_N^*(a) W_N$ converges to the operator $T^*\left(a\left(1,1,t^{-1}\right)\right)$, accordingly.

Proof. It can be shown easily that

$$W_N A_N(a) W_N = A_N\left(a\left(1-x, 1-y, t^{-1}\right)\right).$$

Taking into account Proposition 2.1, we obtain what we need. The reasoning for the conjugated operator is analogous. □

3. The structure of the operator $A_N(a)$ in the case of the smooth generating function. The operator $H(b) : l^2(\mathbf{Z}^+) \to l^2(\mathbf{Z}^+)$ with $b(t) \in C(\mathbf{T})$ which has the following matrix

$$\left(\hat{b}_{n+k+1}\right)_{n,k\in\mathbf{Z}^+}$$

is called the Hankel operator, $H_N(b) = P_N H(b) P_N$. It is known that $H(b)$ is compact.

The operator $H_N(a) = H_N(a(x,y,t))$ with $a(x,y,t) \in C([0,1] \times [0,1] \times \mathbf{T})$ acting in the space $\operatorname{Im} P_N$, which has the following matrix representation

$$\left(\hat{a}_{n+k+1}\left(\frac{n}{N}, \frac{k}{N}\right)\right)_{n,k=0,\dots,N},$$

is called the truncated generalized Hankel operator.

Further we need some additional notations.

For $a(x,y,t) \in C([0,1] \times [0,1] \times \mathbf{T})$ and $N \in \mathbf{N}$ let us denote by $A_N(a(x,x,t))$, $A_N(a(y,y,t))$, $A_N(a(y,x,t))$, $H_N(a(x,x,t))$ and $H_N(a(y,y,t))$ linear operators acting in the space $\operatorname{Im} P_N$ which have the following matrices:

$$\left(\hat{a}_{n-k}\left(\frac{n}{N},\frac{n}{N}\right)\right)_{n,k=0,\dots,N},$$

$$\left(\hat{a}_{n-k}\left(\frac{k}{N},\frac{k}{N}\right)\right)_{n,k=0,\dots,N},$$

$$\left(\hat{a}_{n-k}\left(\frac{k}{N},\frac{n}{N}\right)\right)_{n,k=0,\dots,N},$$

$$\left(\hat{a}_{n+k+1}\left(\frac{n}{N},\frac{n}{N}\right)\right)_{n,k=0,\dots,N},$$

$$\left(\hat{a}_{n+k+1}\left(\frac{k}{N},\frac{k}{N}\right)\right)_{n,k=0,\dots,N},$$

respectively.

In these notations, for $a(x,y,t) \in C([0,1] \times [0,1] \times \mathbf{T})$, $b(x,y,t) \in C([0,1]\times[0,1]\times\mathbf{T})$ and $N \in \mathbf{N}$ we obtain for example that the operator $A_N\left(a(x,x,t)b(y,y,t)\right)$ has the following matrix representation:

$$\left(\hat{c}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right)\right)_{n,k=0,\dots,N},$$
$$\text{where}\quad \hat{c}_n\left(x,y\right) = \frac{1}{2\pi}\int_{\mathbf{T}} a\left(x,x,t\right) b\left(y,y,t\right) t^n \, d\mu, \quad n \in \mathbf{N}.$$

For $a(x,y,t) \in C([0,1] \times [0,1] \times \mathbf{T})$ we define also $\tilde{a}(x,y,t) = a(x,y,t^{-1})$.

Let the functions $a(x,y,t)$ and $b(x,y,t)$ be infinitely differentiable in the variable t. Using the result obtained in [**2**] (the equality (2.3)) and infinite differentiability of symbols, the following equality can be shown:

$$
\begin{aligned}
(3.1)\quad & A_N\,(a(x,x,t))\,A_N\,(b(y,y,t)) \\
&= A_N\,(a(x,x,t)b(y,y,t)) - H_N\,(a(x,x,t))\,H_N(\tilde{b}(y,y,t)) \\
&\quad - W_N H_N\,(\tilde{a}(x,x,t))\,H_N\,(b(y,y,t))\,W_N + o(1),
\end{aligned}
$$

where $o(1)$ is an operator depending on N, the norm of which tends to zero as $N \to \infty$.

Lemma 3.1. *Suppose that for the function $a(x,y,t)$ the condition* (2.1) *is fulfilled. Then*

$$
\begin{aligned}
||A_N(a(x,y,t)) - A_N(a(x,x,t))|| &\underset{N\to\infty}{\longrightarrow} 0, \\
||A_N(a(x,y,t)) - A_N(a(y,y,t))|| &\underset{N\to\infty}{\longrightarrow} 0, \\
||A_N(a(x,y,t)) - A_N(a(y,x,t))|| &\underset{N\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Proof. We prove the first statement. Let us fix an arbitrary $\varepsilon > 0$.

Let $\mathcal{M} = \sum_{n\in\mathbf{Z}} \sup_{x,y\in[0,1]} |\hat{a}_n(x,y)|$ and $\mathcal{R}(m) = \sum_{|n|>m} \sup_{x,y\in[0,1]} \times |\hat{a}_n(x,y)|$, $m \in \mathbf{N}$.

The following estimation can be obtained analogously to the proof of Proposition 2.1:

$$
\begin{aligned}
&||A_N(a(x,y,t)) - A_N(a(x,x,t))||^2 \\
&\quad \leqslant 2\mathcal{M} \sum_{n=0}^{N} \sum_{\substack{0\leqslant k\leqslant N \\ |n-k|\leqslant m}} \left| \hat{a}_{n-k}\left(\frac{n}{N}, \frac{k}{N}\right) - \hat{a}_{n-k}\left(\frac{n}{N}, \frac{n}{N}\right) \right| |X_k|^2 \\
&\qquad + 2\mathcal{M} \sum_{n=0}^{N} \sum_{\substack{0\leqslant k\leqslant N \\ |n-k|> m}} \left| \hat{a}_{n-k}\left(\frac{n}{N}, \frac{k}{N}\right) - \hat{a}_{n-k}\left(\frac{n}{N}, \frac{n}{N}\right) \right| |X_k|^2,
\end{aligned}
$$

where $m \in \mathbf{N}$.

Since the Fourier coefficients of the function $a(x,y,t)$ are continuous in variables $x, y \in [0,1]$, there exists such a $\delta > 0$, that

$$\sum_{n\in\mathbf{Z}} \sup_{\substack{x_1,x_2,y_1,y_2\in[0,1]\\ |x_1-x_2|<\delta,\ |y_1-y_2|<\delta}} |\hat{a}_n(x_1,y_1) - \hat{a}_n(x_2,y_2)| < \frac{1}{2}\,\frac{\varepsilon^2}{2\mathcal{M}}.$$

Let us fix $m \in \mathbf{N}$ so, that $\mathcal{R}(m) < 1/2(\varepsilon^2/2\mathcal{M})$.

Further, there is such an $N_0 \in \mathbf{N}$ that the inequality

$$\frac{m}{N} < \delta$$

holds for any $N > N_0$.

Then, for chosen m and N_0, we obtain:

$$\begin{aligned}
&||A_N(a(x,y,t))X - A_N(a(x,x,t))X||^2\\
&\quad\leqslant 2\mathcal{M} \sum_{\substack{0\leqslant k\leqslant N\\ |n-k|\leqslant m}} \left\{ \sum_{n=0}^{N} \left| \hat{a}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right) - \hat{a}_{n-k}\left(\frac{n}{N},\frac{n}{N}\right)\right| \right\} |X_k|^2\\
&\qquad + 2\mathcal{M} \sum_{\substack{0\leqslant k\leqslant N\\ |n-k|> m}} \left\{ \sum_{n=0}^{N} \left| \hat{a}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right) - \hat{a}_{n-k}\left(\frac{n}{N},\frac{n}{N}\right)\right| \right\} |X_k|^2\\
&\quad< \frac{1}{2}\,\varepsilon^2 + \frac{1}{2}\,\varepsilon^2 = \varepsilon^2,
\end{aligned}$$

which proves the convergence.

The other two statements can be proved analogously. □

Lemma 3.2. *Let the condition* (2.1) *hold for the functions* $a(x,y,t)$ *and* $b(x,y,t)$. *Then*

$$||A_N(a(x,x,t)[b(y,y,t) - b(x,x,t)])|| \underset{N\to\infty}{\longrightarrow} 0.$$

Proof. For convenience, let $c(x,y,t) = a(x,x,t)[b(y,y,t) - b(x,x,t)]$.

Then the matrix of the operator $A_N(a(x,x,t)[b(y,y,t)-b(x,x,t)]) = A_N(c(x,y,t))$ has the form:

$$\left(\hat{c}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right)\right)_{n,k=0,\dots,N},$$

where $\quad \hat{c}_n\,(x,y) = \displaystyle\int_\beta Ta(x,x,t)[b(y,y,t)-b(x,x,t)]t^{-n}\,d\mu, \quad n \in \mathbf{N}.$

It is also clear that, if the functions $a(x,y,t)$ and $b(x,y,t)$ satisfy the condition (2.1), then the function $c(x,y,t)$ possesses this property too.

Let us fix some $\varepsilon > 0$. Let us denote $\mathcal{C} = \sum_{n\in\mathbf{Z}} \sup_{x,y\in[0,1]} |\hat{c}_n(x,y)| < \infty$ and $\mathcal{R}(m) = \sum_{|n|>m} \sup_{x,y\in[0,1]} |\hat{c}_n(x,y)|$, $m \in \mathbf{N}$.

By analogy with the previous proofs, we can show that for $m \in \mathbf{N}$ and $X = \{X_n\}_{n\in\mathbf{Z}} \in l_2$ with $\|X\| = 1$, the following inequality is correct:

$$\begin{aligned} ||A_N(c(x,y,t))||^2 & \\ \leqslant 2\mathcal{C} \sum_{n=0}^{N} & \sum_{\substack{0\leqslant k\leqslant N\\ |n-k|\leqslant m}} \left|\hat{c}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right)\right| |X_k|^2 \\ & + 2\mathcal{C} \sum_{n=0}^{N} \sum_{\substack{0\leqslant k\leqslant N\\ |n-k|> m}} \left|\hat{c}_{n-k}\left(\frac{n}{N},\frac{k}{N}\right)\right| |X_k|^2 \quad \text{for} \quad m \in \mathbf{N}. \end{aligned}$$

We choose $m \in \mathbf{N}$ so that $\mathcal{R}(m) < 1/2(\varepsilon^2/2\mathcal{C})$.

Taking into account the continuity of the function $b(x,y,t)$, there exists such a $\delta > 0$, that

$$\sum_{n\in\mathbf{Z}} \sup_{\substack{x,y\in[0,1]\\ |x-y|<\delta}} |\hat{c}_n(x,y)| < \frac{1}{2}\,\frac{\varepsilon^2}{2\mathcal{C}}.$$

Let us further fix $N_0 \in \mathbf{N}$ so that $m/N < \delta$ for all $N > N_0$. For fixed m and N_0, we obtain

$$||A_N(c(x,y,t))||^2 < \frac{1}{2}\,\varepsilon^2 + \frac{1}{2}\,\varepsilon^2 = \varepsilon^2. \qquad \square$$

The following statement can be proved by analogy to previous Lemmas 3.1 and 3.2.

Lemma 3.3. *Suppose the condition* (2.1) *is fulfilled for the function* $a(x,y,t)$. *Then*

$$||H_N(a(x,x,t)) - H(a(0,0,t))|| \underset{N\to\infty}{\longrightarrow} 0,$$
$$||H_N(a(y,y,t)) - H(a(0,0,t))|| \underset{N\to\infty}{\longrightarrow} 0.$$

We denote by $\mathcal{K}$ the space of all compact operators acting in l_2.

Lemma 3.4. *From the relation*

$$A_N(a(x,x,t)) = A_N(b(x,x,t)) + P_N K P_N + W_N L W_N + B_N, \tag{3.2}$$

where $a(x,y,t), b(x,y,t) \in C_t^\infty([0,1]\times[0,1]\times\mathbf{T})$, $K, L \in \mathcal{K}$, $||B_N|| \underset{N\to\infty}{\to} 0$, *it follows that* $K = 0$, $L = 0$, $B_N = 0$ *for any* $N \in \mathbf{N}$ *and* $b(x,x,t) = a(x,x,t)$.

Remark. It is easy to show, that the operator $W_N L W_N$ converges strongly to zero. Indeed, the sequence $W_N X$ for an arbitrary $X = \{X_n\}_{n\in\mathbf{Z}} \in l_2$ converges weakly to zero. Then because the operator L is compact, the sequence $W_N L W_N X$ converges to zero in the norm of space l_2, which is equivalent to the strong convergence of the operator.

Analogously, the operator $P_N K P_N$ converges strongly to K (even in norm).

Proof. Passing in (3.2) to the strong limit when $N \to \infty$, we obtain

$$T(a(0,0,t)) = T(b(0,0,t)) + K.$$

A Toeplitz operator is compact if and only if it is the zero operator. Therefore $K = T(a(0,0,t) - b(0,0,t)) = 0$.

Multiplying the relation (3.2) from left and right by the operator W_N (taking into account that $W_N W_N = P_N$) and passing to the strong limit, we get

$$T(\tilde{a}(1,1,t)) = T(\tilde{b}(1,1,t)) + L,$$

and thus $L = T(\tilde{a}(1,1,t) - \tilde{b}(1,1,t)) = 0$.

Therefore $A_N(a(x,x,t)) = A_N(b(x,x,t)) + B_N$ and

$$B_N = A_N(c(x,x,t)), \quad \text{where} \quad c(x,x,t) = a(x,x,t) - b(x,x,t).$$

Suppose, that the equality $a(x,x,t) = b(x,x,t)$ does not hold. Then $c(x,x,t) = 0$ is not identical to zero. This means, that $\hat{c}_{n_0}(x_0,x_0) \neq 0$ for some $x_0 \in [0,1]$ and $n_0 \in \mathbf{Z}$.

The numbers n/N, where $n = 1,\dots,N$, $N \in \mathbf{N}$, form an everywhere dense set for $N \to \infty$. So we can find a sequence n_i/N_i, $n_i \leqslant N_i$, converging to x_0 for $i \to \infty$. Thus for any $i \in \mathbf{N}$ we can choose also $n_i \geqslant n_0$.

Let us consider now the matrix $A_N(c(x,x,t))$. Its element on the crossing of the n_ith line and the k_ith column, where $k_i = n_i - n_0$, is equal to $\hat{c}_{n_0}(n_i/N_i, n_i/N_i)$ and

$$\hat{c}_{n_0}\left(\frac{n_i}{N_i}, \frac{n_i}{N_i}\right) \underset{i\to\infty}{\longrightarrow} \hat{c}_{n_0}(x_0,x_0) \neq 0.$$

But this contradicts the fact that $||A_{N_i}(c(x,x,t))|| \underset{i\to\infty}{\to} 0$. Therefore the assumption is not true and $a(x,x,t) = b(x,x,t)$. From this it follows that $B_N = 0$ for any $N \in \mathbf{N}$. $\square$

Obviously, $A_N^*(a(x,y,t)) = A_N\left(\overline{a(y,x,t)}\right)$.

The correctness of the following proposition follows from the relation (3.1) and has already proved Lemmas 3.1, 3.2, 3.3 and 3.4.

Proposition 3.1. *Let* $a(x,y,t), b(x,y,t) \in C_t^\infty([0,1]\times[0,1]\times\mathbf{T})$. *Then the two following representations hold*:

$$A_N(a(x,y,t))\,A_N(b(x,y,t)) = A_N(a(x,x,t)b(x,x,t)) + P_N K_1 P_N + W_N L_1 W_N + o(1), \tag{3.3}$$

$$A_N(a(x,y,t))\,A_N^*(b(x,y,t)) = A_N\left(a(x,x,t)\overline{b(x,x,t)}\right) + P_N K_2 P_N + W_N L_2 W_N + o(1), \tag{3.4}$$

where $K_1, K_2, L_1, L_2 \in \mathcal{K}$ are nonzero operators, $o(1)$ denotes an operator with the norm converging to zero. Moreover, these representations are unique.

4. The stability of an operator sequence. The algebras $\mathcal{F}$, $\mathcal{F}/\mathcal{N}$ and $\mathcal{F}/\mathcal{I}$. In this section we make review of some results contained in [**1**], for example, which we will use later.

Let $\{B_N\}_{N\in\mathbf{N}}$ be some sequence of operators $B_N \in \operatorname{End}(\operatorname{Im} P_N)$. This sequence is called stable, if there exists such $N_0 \in \mathbf{N}$ that for all $N > N_0$ the operator B_N is invertible and

$$\sup_{N>N_0} ||B_N^{-1}|| < \infty.$$

By $\mathcal{F}$ we denote the set of all operator sequences $\{B_N\}_{N\in\mathbf{Z}}$, $B_N \in \operatorname{End}(\operatorname{Im} P_N)$ with the following properties:

1) $\sup_{N\in\mathbf{N}} ||B_N|| < \infty$,

2) there exist two operators $B, \widetilde{B} \in \operatorname{End}(l_2(\mathbf{Z}^+))$ such that the following strong convergences hold (the asterisk refers to the adjoint operator):

$$B_N \longrightarrow B, \quad B_N^* \longrightarrow B^*, \quad W_N B_N W_N \longrightarrow \widetilde{B}, \quad W_N B_N^* W_N \longrightarrow \widetilde{B}^*.$$

The algebraic operations in $\mathcal{F}$ are defined by

$$\begin{gathered}\{B_N\} + \{C_N\} = \{B_N + C_N\},\ \{B_N\}, \{C_N\} \in \mathcal{F},\\ \lambda\{B_N\} = \{\lambda B_N\},\ \{B_N\} \in \mathcal{F},\ \lambda \in \mathbf{C},\\ \{B_N\}\{C_N\} = \{B_N C_N\},\ \ \{B_N\}, \{C_N\} \in \mathcal{F},\\ \{B_N\}^* = \{B_N^*\}.\end{gathered}$$

It is not difficult to see that $\mathcal{F}$ with the norm

$$||\{B_N\}|| = \sup_{N\in\mathbf{N}} ||B_N|| < \infty$$

is a C*-algebra, that is, a Banach algebra with the property

$$||\{B_N\}||^2 = ||\{B_N^* B_N\}||.$$

We denote by $\mathcal{N}$ the subset of $\mathcal{F}$ consisting of all sequences $\{M_N\}_{N\in\mathbf{Z}}$ with $||M_N|| \underset{N\to\infty}{\to} 0$.

Let us denote now by $\mathcal{I}$ the subset of the algebra $\mathcal{F}$ consisting of all sequences of the form

$$\{P_N K P_N + W_N L W_N + M_N\} \quad \text{where} \quad K, L \in \mathcal{K},\ ||M_N|| \underset{n\to\infty}{\to} 0.$$

One can also check, see [**1**], that $\mathcal{N}$ and $\mathcal{I}$ are closed two-sided ideals in $\mathcal{F}$. Therefore we can consider the factor-algebras $\mathcal{F}/\mathcal{N}$ and $\mathcal{F}/\mathcal{I}$ (which are C*-algebras too).

The next statement about the connection between the stability and the invertibility in the factor-algebras $\mathcal{F}/\mathcal{N}$ and $\mathcal{F}/\mathcal{I}$ was proved in [**1**].

Proposition 4.1. *Let $\{B_N\} \in \mathcal{F}$; let B and $\widetilde{B}$ be the strong limits of the operators B_N and $W_N B_N W_N$. Then the following statements are equivalent*:

1) *the sequence $\{B_n\}$ is stable,*

2) *the element $\{B_N\} + \mathcal{N}$ is invertible in $\mathcal{F}/\mathcal{N}$,*

3) *B and $\widetilde{B}$ are invertible and the element $\{B_N\} + \mathcal{I}$ is invertible in the factor-algebra $\mathcal{F}/\mathcal{I}$.*

Remark. It is easy to show that, for $a(x, y, t) \in C_t^\infty([0,1]\times[0,1]\times\mathbf{T})$, the estimation

$$\sup_{n\in\mathbf{N}} ||A_N(a)|| \leqslant \sum_{n\in\mathbf{Z}} \sup_{x,y\in[0,1]} |\hat{a}_n(x,y)| \tag{4.1}$$

holds. Therefore, by Propositions 2.1 and 2.2, the sequence $\{A_N(a)\}$ belongs to $\mathcal{F}$.

Corollary 4.1. *Let $a(x, y, t) \in C_t^\infty([0,1] \times [0,1] \times \mathbf{T})$. Then the following statements are equivalent*:

1) *the sequence $\{A_N(a)\}$ is stable,*

2) *the element $\{A_N(a)\} + \mathcal{N}$ is invertible in $\mathcal{F}/\mathcal{N}$,*

3) *the operators $T(a(0,0,t))$ and $T\left(a\left(1,1,t^{-1}\right)\right)$ are invertible and the element $\{A_N(a)\} + \mathcal{I}$ is invertible in $\mathcal{F}/\mathcal{I}$.*

Proof. This result follows directly from Propositions 2.1, 2.2, 2.3 and 4.1. □

5. The norm of $\{A_N(a)\} + \mathcal{I}$ in the algebra $\mathcal{F}/\mathcal{I}$ in case of a smooth generating function. We consider first an important lemma.

Lemma 5.1. *Let $a \in C_t^\infty([0,1] \times [0,1] \times \mathbf{T})$. The coset $\{A_N(a)\} + \mathcal{I}$ is invertible in $\mathcal{F}/\mathcal{I}$ if and only if the function $a(x,x,t)$ does not vanish on $[0,1] \times \mathbf{T}$.*

Proof. The proof of this statement is based on the local principle of Allan-Douglas, see [**1**, Theorem 2.29 and Section 2.7].

The sufficiency is obvious and follows directly from Proposition 3.1. We prove the necessity.

Let $\{A_N(a)\} + \mathcal{I}$ be invertible in $\mathcal{F}/\mathcal{I}$. Suppose that there exist $x_0 \in [0,1]$ and $t_0 \in \mathbf{T}$ such that $a(x_0, x_0, t_0) = 0$. Let us consider a particular case, when the function a has the form

$$a(x,y,t) = \sum_{n=-m}^{m} \hat{a}_n(x,y)t^n, \quad m \in \mathbf{N}.$$

From the simple representation

$$a(x,y,t) = \sum_{n=-m}^{m} \hat{a}_n(x,y)[(t-t_0)+t_0]^n,$$

it follows that the local representatives of the coset $\{A_N(a)\} + \mathcal{I}$ at $\mathcal{I}_{t_0}$ is the coset $\{A_N(a(x,x,t_0))\} + \mathcal{I} + \mathcal{I}_{t_0}$, where $\mathcal{I}_{t_0}$ is a closed two-sided ideal in $\mathcal{F}/\mathcal{I}$ which has been defined in [**1**, Section 2.7]. The matrix of $A_N(a(x,x,t_0))$ is diagonal.

Since the numbers n/N, $n = 0,1,\dots,N$ are dense in [0,1] as $N \to \infty$, then it is not hard to show, that the local representative $\{A_N(a(x,x,t_0))\} + \mathcal{I} + \mathcal{I}_{t_0}$ is not invertible in $(\mathcal{F}/\mathcal{I})/\mathcal{I}_{t_0}$. Therefore, according to the local principle of Allan-Douglas, the coset $\{A_N(a)\}+\mathcal{I}$ is not invertible in $\mathcal{F}/\mathcal{I}$ and we get a contradiction.

The general case, if $a(x,y,t) = \sum_{n\in\mathbf{Z}} \hat{a}_n(x,y)t^n$, can be easily obtained from one considered above by passing to the limit as $n \to \infty$. $\square$

It is well known that, for an element b of a C*-algebra the following equality holds:

$$||b||^2 = \sup\{\lambda : \lambda \in \text{sp } (b^*b)\},$$

where sp (b^*b) is the spectrum of the element b^*b.

For $\{B_N\} + \mathcal{I} \in \mathcal{F}/\mathcal{I}$ we denote by $\text{sp}_{\mathcal{F}/\mathcal{I}}(\{B_N\} + \mathcal{I})$ the spectrum of the coset $\{B_N\} + \mathcal{I}$ in the algebra $\mathcal{F}/\mathcal{I}$. According to the remark above, we have

$$||\{B_N\} + \mathcal{I}||^2_{\mathcal{F}/\mathcal{I}} = \sup\left\{\lambda : \lambda \in \text{sp}_{\mathcal{F}/\mathcal{I}}(\{B_N\} + \mathcal{I})(\{B_N\} + \mathcal{I})^*\right\}.$$

Let us consider further the sequence $\{A_N(a)\} + \mathcal{I}$ in the case when $a(x,y,t) \in C_t^\infty([0,1] \times [0,1] \times \mathbf{T})$.

By Proposition 3.1 (the representation (3.4)), we get

$$\begin{aligned}(\{A_N(a)\} + \mathcal{I})(\{A_N(a)\} + \mathcal{I})^* &= \{A_N(a)A_N^*(a)\} + \mathcal{I}\\ &= \{A_N(a(x,x,t)\overline{a(x,x,t)})\} + \mathcal{I}.\end{aligned}$$

So we have to find the supremum of the spectrum points of the sequence $\{A_N(a(x,x,t)\overline{a(x,x,t)})\} + \mathcal{I}$ in the algebra $\mathcal{F}/\mathcal{I}$.

Let us denote by I_N the identity operator in $\operatorname{Im} P_N$. The number $\lambda_0 \in \mathbf{C}$ is in the spectrum of the coset $\{A_N(a(x,x,t)\overline{a(x,x,t)})\} + \mathcal{I}$ if and only if the sequence

$$\{A_N(a(x,x,t)\overline{a(x,x,t)}) - \lambda_0 I_N\} + \mathcal{I} = \{A_N(a(x,x,t)\overline{a(x,x,t)} - \lambda_0)\} + \mathcal{I}$$

is not invertible in $\mathcal{F}/\mathcal{I}$.

As it follows from Lemma 5.1, this is possible if and only if $\lambda_0 = (a(x_0,x_0,t_0)\overline{a(x_0,x_0,t_0)}) = |a(x_0,x_0,t_0)|^2$ for some $x_0 \in [0,1]$, $t_0 \in \mathbf{T}$.

The spectral radius is equal to $\sup_{x,y\in[0,1],t\in\mathbf{T}} |a(x,x,t)|^2$. Therefore,

$$||\{A_N(a)\} + \mathcal{I}||^2_{\mathcal{F}/\mathcal{I}} = \sup_{x\in[0,1],t\in\mathbf{T}} |a(x,x,t)|^2.$$

Proposition 5.1. *If* $a(x,y,t) \in C_t^\infty([0,1]\times[0,1]\times\mathbf{T})$*, then*

$$||\{A_N(a)\}+\mathcal{I}||_{\mathcal{F}/\mathcal{I}} = ||a(x,x,t)||_{C([0,1]\times\mathbf{T})}.$$

Let $\mathcal{A}\subset\mathcal{F}$ be the smallest C^*-algebra containing all sequences $\{A_N(a)\}$ with smooth generating functions. Notice, that $\mathcal{I}\subset\mathcal{A}$, see [**1**, Lemma 2.21] and [**7**, paragraph 4.2.1].

Proposition 5.2. *The* C^**-algebras* $\mathcal{A}/\mathcal{I}$ *and* $C([0,1]\times\mathbf{T})$ *are isomorphic.*

Proof. Proposition 3.1 shows that $\mathcal{A}/\mathcal{I}$ is a commutative C^*-algebra. Gelfand theory claims that $\mathcal{A}/\mathcal{I}$ is isometrically isomorphic to $C(M)$, where M is the space of maximal ideals of $\mathcal{A}/\mathcal{I}$.

We immediately prove that $\mathcal{A}/\mathcal{I}$ is isometrically isomorphic to $C([0,1]\times\mathbf{T})$. For, introduce the (non-closed) subalgebra

$$\mathcal{A}_0 := \left\{\sum_{i=1}^{k}\prod_{j=1}^{l}\left(\{A_N(a_{ij})\}+\mathcal{I}\right),\ a_{ij}\text{ smooth},\ k,l\in\mathbf{N}\right\}.$$

Consider the map $\varphi:\mathcal{A}_0\mapsto C([0,1]\times\mathbf{T})$ defined by

$$\sum_{i=1}^{k}\prod_{j=1}^{l}\left(\{A_N(a_{ij})\}+\mathcal{I}\right)\longmapsto\sum_{i=1}^{k}\prod_{j=1}^{l}a_{ij}.$$

Because of $\sum_{i=1}^{k}\prod_{j=1}^{l}\left(\{A_N(a_{ij})\}+\mathcal{I}\right)=\left\{A_N\left(\sum_{i=1}^{k}\prod_{j=1}^{l}a_{ij}\right)\right\}+\mathcal{I}$ and Lemma 3.4, this map is correctly defined and is clearly an isometric homomorphism which can be extended by continuity onto the whole of $\mathcal{A}$. By the Stone-Weierstrass theorem, the image of this extension coincides with $C([0,1]\times\mathbf{T})$. The space of maximal ideals of $\mathcal{A}/\mathcal{I}$ is therefore homeomorphic to $C([0,1]\times\mathbf{T})$ and we are done. □

As a corollary we obtain

Proposition 5.3. *For any element* $\{A_N\}+\mathcal{I} \in \mathcal{A}/\mathcal{I}$ *and the function* $a(x,t) = \varphi(\{A_N\}+\mathcal{I}) \in C([0,1]\times\mathbf{T})$ *it holds*:

$$||\{A_N\}+\mathcal{I}||_{\mathcal{A}/\mathcal{I}} = ||a(x,t)||_{C([0,1]\times\mathbf{T})}.$$

Remark. The function $a(x,t) = \varphi(\{A_N\}+\mathcal{I})$ will be called the symbol of the sequence $\{A_N\} \in \mathcal{A}$.

6. The norm of $\{A_N(a)\}+\mathcal{N}$ in the algebra $\mathcal{F}/\mathcal{N}$ in case of a smooth generating function. Notice that Proposition 4.1 remains true if the algebra $\mathcal{F}$ is replaced by $\mathcal{A}$. The reason is very simple: C^*-algebras are inverse closed, that is, in the case at hand the spectrum of an element a in $\mathcal{A}$ coincides with the spectrum of a considered in $\mathcal{F}$.

Assign to an element $\{A_N\} \in \mathcal{A}$ the triple $(A, \widetilde{A}, \{A_N\}+\mathcal{I})$, where A and $\widetilde{A}$ are strong limits of A_N and $W_N A_N W_N$, respectively. Then Proposition 4.1 tells us that $\{A_N\} \in \mathcal{A}$ is stable if and only if $(A, \widetilde{A}, \{A_N\}+\mathcal{I})$ is invertible in the C^*-algebra $\mathcal{B}$ constituted by all elements $(B, \widetilde{B}, \{B_N\}+\mathcal{I})$, $\{B_N\} \in \mathcal{A}$ (the algebraic operations are componentwise defined, $||(B, \widetilde{B}, \{B_N\}+\mathcal{I})|| := \max\{||B||, ||\widetilde{B}||, ||\{B_N\}+\mathcal{I}||)\}$). It is a feature of C^*-algebras that then $\mathcal{A}/\mathcal{N}$ and $\mathcal{B}$ are isometrically isomorphic. This means especially, that (a is smooth)

$$\begin{aligned} ||\{A_N\}+\mathcal{N}||_{\mathcal{A}/\mathcal{N}} &= ||\{A_N\}+\mathcal{N}||_{\mathcal{F}/\mathcal{N}} \\ &= \max\left\{||A||, ||\widetilde{A}||, ||\{A_N\}+\mathcal{I}||_{\mathcal{A}/\mathcal{I}}\right\}. \end{aligned}$$

Proposition 6.1. *Let* $\{A_N\} \in \mathcal{A}$. *Then*

$$||\{A_N\}+\mathcal{N}||_{\mathcal{A}/\mathcal{N}} = \max\{||A||, ||\widetilde{A}||, ||\{A_N\}+\mathcal{I}||_{\mathcal{A}/\mathcal{I}}\},$$

where A *and* $\widetilde{A}$ *are the strong limits of operators* A_N *and* $W_N A_N W_N$ *as* $N \to \infty$.

Corollary 6.1. *Let* $a(x,y,t) \in C_t^\infty([0,1]\times[0,1]\times\mathbf{T})$. *Then*

$$||\{A_N(a)\}+\mathcal{N}||_{\mathcal{F}/\mathcal{N}} = ||a(x,x,t)||_{C([0,1]\times\mathbf{T})}.$$

Proof. This statement follows from Propositions 2.1, 2.2, 5.3, 6.1 and the following well-known equalities:

$$||T(a(0,0,t))|| = ||a(0,0,t)||_{C(\mathbf{T})},$$
$$||T(\tilde{a}(1,1,t))|| = ||a(1,1,t^{-1})||_{C(\mathbf{T})} = ||a(1,1,t)||_{C(\mathbf{T})}. \qquad \square$$

7. A Szegö-type theorem for the sequence of self-adjoint operators. The trace of an $N \times N$-matrix A, that is, the sum of its diagonal elements, will be denoted by $\operatorname{tr} A$.

Lemma 7.1. *Let $a \in C_t^\infty([0,1]\times[0,1]\times\mathbf{T})$, p a complex polynomial. Then*

$$\lim_{N\to\infty} \frac{1}{N+1} \operatorname{tr} p\left(A_N(a)\right) = \frac{1}{2\pi} \int_0^1 \int_{\mathbf{T}} p(a(x,x,t))\, d\mu\, dx.$$

Proof. Obviously, it is enough to prove this statement for polynomials of the form $p(z) = z^k$, $k \in \mathbf{N}$.

By Proposition 3.4, we obtain

$$p\left(A_N(a)\right) = A_N^k(a) = A_N\left(a^k\right) + P_N K P_N + W_N L W_N + B_N,$$

where $K, L \in \mathcal{K}$, $||B_N|| \to 0$ for $N \to \infty$.

Since the operators K and L are compact, it is clear that

$$\lim_{N\to\infty} \frac{1}{N+1} \operatorname{tr} P_N K P_N = 0, \quad \lim_{N\to\infty} \frac{1}{N+1} \operatorname{tr} W_N L W_N = 0.$$

Analogously, if $\{B_N\}$ is the operator sequence satisfying the condition $||B_N|| \underset{N\to\infty}{\to} 0$, then

$$\lim_{N\to\infty} \frac{1}{N+1} \operatorname{tr} B_N = 0.$$

Finally we have to consider the operator $A_N(a^k)$.

The trace of operator is equal to the sum of diagonal elements of the matrix of this operator. In this case

$$\begin{aligned}\frac{1}{N+1}\operatorname{tr} A_N(a^k) &= \frac{1}{N+1}\sum_{n=0}^{N}\widehat{a^k}_0\left(\frac{n}{N},\frac{n}{N}\right)\\ &= \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{\mathbf{T}} a^k\left(\frac{n}{N},\frac{n}{N},t\right)d\mu.\end{aligned}$$

Further, the right-hand side of the last expression is a Riemann sum for the continuous function

$$g(x) = \frac{1}{2\pi}\int_{\mathbf{T}} a^k(x,x,t)\,d\mu.$$

Hence,

$$\frac{1}{N+1}\operatorname{tr} A_N(a^k) \underset{N\to\infty}{\longrightarrow} \frac{1}{2\pi}\int_0^1\int_{\mathbf{T}} a^k(x,x,t)\,d\mu\,dx,$$

which proves the statement. □

Theorem 7.1. *Let* $\{A_N\}\in\mathcal{A}$*, and let* $a(x,t)\in C([0,1]\times\mathbf{T})$ *be the symbol of* $\{A_N\}+\mathcal{I}$*. Let also* p *be a complex polynomial. Then*

$$\lim_{N\to\infty}\frac{1}{N+1}\operatorname{tr} p(A_N) = \frac{1}{2\pi}\int_0^1\int_{\mathbf{T}} p(a(x,t))\,d\mu\,dx.$$

Proof. We prove the statement for $p(z)=z^k$, $k\in\mathbf{N}$.

Let us fix $\varepsilon>0$.

Since $a(x,t)\in C([0,1]\times\mathbf{T})$, we can find such a sequence of functions $a_i(x,y,t)\in C_t^\infty([0,1]\times[0,1]\times\mathbf{T})$, $i\in\mathbf{N}$, that

$$||a_i(x,x,t)-a(x,t)||_{C([0,1]\times\mathbf{T})} \underset{i\to\infty}{\longrightarrow} 0.$$

It is clear that the sequence $\{A_N^k(a_i)\}+\mathcal{I}$ approximates $\{A_N^k\}+\mathcal{I}$, i.e.,

$$||\{A_N^k(a_i)\}-\{A_N^k\}+\mathcal{I}||_{\mathcal{A}/\mathcal{I}} \underset{i\to\infty}{\longrightarrow} 0.$$

By the standard definition of the norm in a factor-algebra,

$$\inf_{\{B_N\}\in\mathcal{I}} ||\{A_N^k(a_i)\} - \{A_N^k\} + \{B_N\}|| \underset{i\to\infty}{\longrightarrow} 0,$$

therefore

$$\inf_{\{B_N\}\in\mathcal{I}} \sup_N ||A_N^k(a_i) - A_N^k + B_N|| \underset{i\to\infty}{\longrightarrow} 0.$$

So there is i_0' such that for all $i > i_0'$ the inequality

$$\inf_{\{B_N\}\in\mathcal{I}} \sup_N ||A_N^k(a_i) - A_N^k + B_N|| < \frac{\varepsilon}{6}$$

holds.

Then there exists a sequence $\{B_N'\} \in \mathcal{I}$ such that

$$\sup_N ||A_N^k(a_i) - A_N^k + B_N'|| < \frac{\varepsilon}{3}. \tag{7.1}$$

Thus,

$$\begin{aligned}
&\left|\frac{1}{N+1}\operatorname{tr} A_N^k - \frac{1}{2\pi}\int_0^1\int_{\mathbf{T}} a^k(x,t)\,d\mu\,dx\right| \\
&\quad\leqslant \left|\frac{1}{N+1}\operatorname{tr} A_N^k - \frac{1}{N+1}\operatorname{tr}\left(A_N^k(a_i) - B_N'\right)\right| \\
&\qquad + \left|\frac{1}{N+1}\operatorname{tr}\left(A_N^k(a_i) - B_N'\right) - \frac{1}{2\pi}\int_0^1\int_{\mathbf{T}} a_i^k(x,x,t)\,d\mu\,dx\right| \\
&\qquad + \left|\frac{1}{2\pi}\int_0^1\int_{\mathbf{T}} a_i^k(x,x,t)\,d\mu\,dx - \frac{1}{2\pi}\int_0^1\int_{\mathbf{T}} a^k(x,t)d\mu\,dx\right|.
\end{aligned}$$

Taking into account the inequality (7.1) and the fact that

$$A_N^k - A_N^k(a_i) + B_N' = P_N(A_N^k - A_N^k(a_i) + B_N')P_N,$$

we obtain for every $N \in \mathbf{N}$:

$$\begin{aligned}
&\left|\frac{1}{N+1}\operatorname{tr} A_N^k - \frac{1}{N+1}\operatorname{tr}\left(A_N^k(a_i) - B_N'\right)\right| \\
&\qquad = \frac{1}{N+1}\left|\operatorname{tr}\left(A_N^k - A_N^k(a_i) + B_N'\right)\right| \\
&\qquad \leqslant \frac{1}{N+1}\left|\left|P_N\left(A_N^k - A_N^k(a_i) + B_N'\right)\right|\right|_{\mathrm{tr}} \\
&\qquad \leqslant \frac{1}{N+1}||P_N||_2^2\left|\left|A_N^k - A_N^k(a_i) + B_N'\right|\right| < \frac{\varepsilon}{3},
\end{aligned}$$

where $||\cdot||_{\mathrm{tr}}$ is the trace norm and $||\cdot||_2$ the Hilbert-Schmidt norm ($||P_N||_2 = \sqrt{N+1}$).

Since

$$\frac{1}{N+1}\operatorname{tr} B'_N \underset{N\to\infty}{\longrightarrow} 0,$$

then, by Lemma 7.1, there exists $N_0 \in \mathbf{N}$ such that for all $N > N_0$

$$\left|\frac{1}{N+1}\operatorname{tr} A_N^k(a_i) - \frac{1}{2\pi}\int_0^1\int_{\mathbf{T}} a_i^k(x,x,t)\,d\mu\,dx\right| < \frac{\varepsilon}{3}.$$

In addition, because of the convergence of the functions a_i to a, we find $i_0 > i_0'$ such that for all $i > i_0$

$$\left|\frac{1}{2\pi}\int_0^1\int_{\mathbf{T}} a_i^k(x,x,t)\,d\mu\,dx - \frac{1}{2\pi}\int_0^1\int_{\mathbf{T}} a^k(x,t)\,d\mu\,dx\right| < \frac{\varepsilon}{3}.$$

Summarizing everything said above, we obtain that

$$\left|\frac{1}{N+1}\operatorname{tr} A_N^k - \frac{1}{2\pi}\int_0^1\int_{\mathbf{T}} a^k(x,t)\,d\mu\,dx\right| < \varepsilon$$

for $N > N_0$. This proves the statement. □

Theorem 7.2. *Let $\{A_N\} \in \mathcal{A}$ be the sequence of self-adjoint operators, and let the function $a(x,t) \in C([0,1]\times\mathbf{T})$ be the symbol of $\{A_N\}$. Let further $\Delta = ||\{A_N\}+\mathcal{N}||_{\mathcal{A}/\mathcal{N}}$, and let the real function f be continuous on the segment $[-\Delta,\Delta]$. Then*

$$\lim_{N\to\infty}\frac{1}{N+1}\operatorname{tr} f(A_N) = \frac{1}{2\pi}\int_0^1\int_{\mathbf{T}} f(a(x,t))\,d\mu\,dx.$$

Remark. It follows from Propositions 5.3 and 6.1 that

$$\Delta = \max\{||A||, ||\widetilde{A}||, ||\{A_N\}+\mathcal{I}||_{\mathcal{A}/\mathcal{I}}\},$$

where A and $\widetilde{A}$ are the strong limits of the operators A_N and $W_N A_N W_N$ for $N\to\infty$. And thus $\Delta \geqslant ||a(x,t)||_{C([0,1]\times\mathbf{T})}$.

Proof. Let us fix an arbitrary $\varepsilon > 0$. Since $f(x)$ is continuous on the segment $[-\Delta, \Delta]$, then there exists such a polynomial $p(x)$, that

$$\sup_{x\in[-\Delta,\Delta]} |p(x) - f(x)| < \frac{\varepsilon}{3}.$$

By Theorem 7.1, we find such $N_0(\in \mathbf{N})$, that for all $N > N_0$

$$\left|\frac{1}{N+1} \operatorname{tr} p(A_N) - \frac{1}{2\pi} \int_0^1 \int_{\mathbf{T}} p(a(x,t))\, d\mu\, dx\right| < \frac{\varepsilon}{3}.$$

Now consider the following estimation:

$$\begin{aligned}
&\left|\frac{1}{N+1} \operatorname{tr} f(A_N) - \frac{1}{2\pi} \int_0^1 \int_{\mathbf{T}} f(a(x,t))\, d\mu\, dx\right| \\
&\qquad \leqslant \left|\frac{1}{N} \operatorname{tr} f(A_N) - \frac{1}{N+1} \operatorname{tr} p(A_N)\right| \\
&\qquad\quad + \left|\frac{1}{N+1} \operatorname{tr} p(A_N) - \frac{1}{2\pi} \int_0^1 \int_{\mathbf{T}} p(a(x,t))\, d\mu\, dx\right| \\
&\qquad\quad + \left|\frac{1}{2\pi} \int_0^1 \int_{\mathbf{T}} p(a(x,t))\, d\mu\, dx - \frac{1}{2\pi} \int_0^1 \int_{\mathbf{T}} f(a(x,t))\, d\mu\, dx\right| \\
&\qquad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus, for every $\varepsilon > 0$, there exists such an $N_0 \in \mathbf{N}$ that, for all $N > N_0$

$$\left|\frac{1}{N+1} \operatorname{tr} f(A_N) - \frac{1}{2\pi} \int_0^1 \int_{\mathbf{T}} f\left(a(x,t)\right)\, d\mu\, dx\right| < \varepsilon,$$

which proves the statement. $\square$

Remark. Putting $A_N = B_N^* B_N$, $\{B_N\} \in \mathcal{A}$, one gets the announced generalization of the Avram/Parter theorem, for instance.

8. A Szegö-type theorem for arbitrary sequences. Theorem 7.1 allows to take as test functions arbitrarily given real and continuous functions f with compact support. If one is interested in arbitrary sequences in $\mathcal{A}$, the test functions must have more specific properties. One of the reasons is the definition of $f(A_N)$, $N \in \mathbf{N}$, which is ensured if f is analytic on the spectrum of A_N for any N. By s-lim of a sequence we understand as usual the strong limit.

Theorem 8.1. *Let $\{A_N\} \in \mathcal{A}$, and let $a \in C([0,1] \times \mathbf{T})$ be the symbol of $\{A_N\}$. If f is holomorphic on the open set $\Omega \subset \mathbf{C}$ and*

$$\mathcal{U} := \mathrm{sp}\,(s - \lim A_N) \cup \mathrm{sp}\,(s - \lim W_N A_N W_N) \cup R(a) \subset \Omega,$$

then

$$\frac{1}{N+1}\,\mathrm{tr}\, f(A_N) = \frac{1}{N+1}\sum_{i=0}^{N} f\left(\lambda_i^{(N)}\right) \underset{N\to\infty}{\longrightarrow} \frac{1}{2\pi}\int_0^1 \int_{\mathbf{T}} f(a(x,t))\, d\mu\, dx,$$

where $\lambda_0^{(N)}, \dots, \lambda_N^{(N)}$ are the eigenvalues of A_N.

Proof. First we prove the result for the particular case $\{A_N\} \in \mathcal{A}$, $A_N = A_N(a) + J_N$, where $a \in C_t^\infty([0,1] \times \mathbf{T})$, $\{J_N\} \in \mathcal{I}$. Let $D \subset \mathbf{C}$ be an open set such that $\mathcal{U} \subset D \subset \Omega$ and $\overline{D} \subset \Omega$. For $f = g_\lambda$, $g_\lambda(z) = 1/(z-\lambda)$ and $\lambda \notin \overline{D}$, we obviously have by Proposition 4.1 that the sequence $\{A_N\} - \lambda\{P_N\}$ is stable. Hence, $(\{A_N\} - \lambda\{P_N\}) + \mathcal{N}$ is invertible in $\mathcal{A}/\mathcal{N}$ and the inverse is given by $g_\lambda(\{A_N\} + \mathcal{N}) = \{A_N((a-\lambda)^{-1}) + P_N K P_N + W_n L P_N\} + \mathcal{N}$, where $K, L \in \mathcal{K}$ are uniquely defined.

For N large enough, say for $N \geqslant N_0$, we have $\mathrm{sp}\, A_N \subset \overline{D}$ (if this would not be true then using Theorem 3.19 in [**5**] one easily gets a contradiction) and for those N the function $g_\lambda(A_N)$ is well-defined and there is a representant $\{E_N\}$ of $g_\lambda(\{A_N\} + \mathcal{N})$ such that $E_N = g_\lambda(A_N)$ for $N \geqslant N_0$. Thus we obtain

$$\frac{1}{N+1}\,\mathrm{tr}\, g_\lambda(A_N) = \frac{1}{N+1}\sum_{i=0}^{N} g\left(\lambda_i^{(N)}\right) \to \frac{1}{2\pi}\int_0^1 \int_{\mathbf{T}} g_\lambda(a)\, d\mu\, dx$$

because of $1/N+1\,\mathrm{tr}\,J_N \to 0$ for any sequence $\{J_N\} \subset \mathcal{I}$. Now we use that any function f holomorphic on Ω can be on $\overline{D}$ approximated as closely as desired (in the supnorm) by functions of the form

$$\sum_{j=1}^{\gamma} c_j g_{\lambda_j}, \quad \lambda_j \notin \overline{D}$$

where c_j are complex numbers, see [**4**] for instance, and for such holomorphic functions (in the right-hand side) the result holds. By approximation the result is clearly true for any function f holomorphic on Ω. Using the upper continuity of the spectrum in a unital Banach algebra and the argumentation of the proof of Theorem 7.2, the result follows in the general case again by approximation. $\square$

9. Acknowledgments and remarks. The authors are grateful to one of the reviewers. He (she) suggested an interesting generalization of Theorem 7.2 concerning sequences of variable Toeplitz matrices. First of all he (she) pointed out that sequences of variable Toeplitz matrices studied in this paper are generalized locally Toeplitz (GLT) sequences (the Definition can be found in [**8, 9, 12**]) and therefore the underlying theory of these sequences applies. Moreover, (unbounded) sequences of variable Toeplitz matrices can be studied. One set of conditions (proposed by the reviewer) is the following:

1) $a(x,y,t) \in L^1(\mathbf{T})$ for every fixed $x, y \in [0,1]$;

2) for any fixed n, the Fourier coefficient $\hat{a}_n(x,y)$ is Riemann integrable;

3) for every $\varepsilon > 0$ and for every $x, y \in [0,1]$, there exists a trigonometric polynomial in s, $a_\varepsilon(x,y,t)$, for which $a(x,y,t) - a_\varepsilon(x,y,t)$ has the norm in $L^1(\mathbf{T})$ bounded by ε.

Then the following theorem is valid:

Theorem 9.1. *Let the function a defined on $[0,1] \times [0,1] \times \mathbf{T}$ satisfy the conditions* 1)–3). *Then the sequence $\{A_N\}$ of the operators, whose (n,k) entry is given by the expression $\hat{a}_{n-k}(n/N, k/N)$, is a GLT sequence with symbol $\kappa(x,t) = a(x,x,t)$, $x \in [0,1]$, $t \in \mathbf{T}$.*

A consequence of this theorem is the following asymptotic for singular values of A_N:

$$\frac{1}{N+1}\operatorname{tr} f\left((A_N^* A_N)^{1/2}\right) \underset{N\to\infty}{\longrightarrow} \frac{1}{2\pi}\int_0^1 \int_{\mathbf{T}} f(\kappa(x,t))\, d\mu\, dx,$$

where f is continuous on a segment containing the singular values of A_N for all N large enough.

Moreover, if the matrices of A_N are Hermitian, then

$$\frac{1}{N+1}\operatorname{tr} f(A_N) \underset{N\to\infty}{\longrightarrow} \frac{1}{2\pi}\int_0^1 \int_{\mathbf{T}} f(\kappa(x,t))\, d\mu\, dx,$$

where f is continuous on a segment containing the eigenvalues of A_N for all N large enough.

Remark. As mentioned before, there are continuous functions defined on $[0,1]\times[0,1]\times\mathbf{T}$ which do not produce a bounded sequence of variable Toeplitz matrices. But a continuous function on $[0,1]\times[0,1]\times\mathbf{T}$ is obviously subject to the conditions 1)–3). On the other hand, for any continuous function b on $[0,1]\times[0,1]\times\mathbf{T}$ there is a sequence $\{A_N\}\in\mathcal{A}$ the symbol of which is b_0: $b_0(x,t)=b(x,x,t)$.

Question. Is any sequence $\{A_N\}\in\mathcal{A}$ a generalized locally Toeplitz sequence?

Appendix

10. Reviewer's proof of Theorem 9.1.

Step 1. Let $a(x,y,t)=\alpha(x,y)t^m$ with Riemann integrable $\alpha(x,y)$ and m fixed integer (independent of N). Then

$$A_N(a) = D_N(\alpha)T_N(t^m),$$
$$(D_N(\alpha))_{n\,k} = \alpha\left(\frac{n}{N}, \frac{n}{N}-\frac{m}{N}\right) = \alpha\left(\frac{n}{N}, \frac{n}{N}+o(1)\right).$$

$\{D_N(\alpha)\}$ is a basic GLT sequence with symbol $\alpha(x,x)$ by definition since it is the sampling matrix of the Riemann integrable function

$\alpha(x, x)$, see [**8**, Definition 2.3]. $\{T_N(t^m)\}$ is also a basic GLT sequence with symbol t^m since, by [**8**, Theorem 5.2], every Toeplitz sequence with $L^1(\mathbf{T})$ symbol is a GLT sequence with the same symbol. The GLT class is an algebra and therefore $\{A_N(a) = D_N(\alpha)T_N(t^m)\}$ is also a GLT sequence with symbol $a(x, x, t) = \alpha(x, x)t^m$.

Step 2. Let $a(x, y, t)$ be a trigonometric polynomial in the variable t. Therefore $a(x, y, t)$ can be written as a finite linear combination of monomials as those considered in Step 1, where the parameters m are independent of N. Consequently,

$$A_N(a) = \sum_m A_N(\alpha_m(x, y)t^m),$$

because $A_N(\cdot)$ is a linear (matrix-valued) operator. In other words, our sequence $\{A_N(a)\}$ is a finite linear combination of GLT sequences with symbols $\alpha_m(x, y)t^m$. In conclusion, since the GLT class is an algebra, see [**9**], $\{A_N(a)\}$ belongs to the GLT class as well with symbol $a(x, x, t) = \sum_m \alpha_m(x, y)t^m$.

Step 3. Let $a(x, y, t)$ satisfy the conditions 1)–3). For $\varepsilon > 0$, consider the approximating polynomials $a_\varepsilon(x, y, t)$ as in condition 3). Then the following facts are immediate or have already been proved:

- $\{A_N(a_\varepsilon)\}$ is a GLT sequence with symbol $a_\varepsilon(x, x, t)$ by Step 2.
- $a_\varepsilon(x, x, t)$ converges in measure to $a(x, x, t)$ on $[0, 1] \times \mathbf{T}$ by the third assumption.

Moreover, by the third assumption, we have that $|\hat{a}_n(x, y) - \widehat{a_\varepsilon}_n(x, y)| \leqslant \varepsilon$ and therefore the Frobenius norm of $A_N(a) - A_N(a_\varepsilon)$ is bounded by εN. By exploiting a standard singular value decomposition of $A_N(a) - A_N(a_\varepsilon)$, the fact that Frobenius norm is bounded by εN, directly implies that $\{\{A_N(a_\varepsilon) : \varepsilon > 0\}\}$ is an approximating class of sequences (a.c.s.) for $\{A_N(a)\}$ (for the notion of a.c.s., see [**8**, Definition 2.2]). Therefore putting together the latter information we have:

- $\{A_N(a_\varepsilon)\}$ is a GLT sequence with symbol $a_\varepsilon(x, x, t)$.
- $a_\varepsilon(x, x, t)$ converges in measure to $a(x, x, t)$ on $[0, 1] \times \mathbf{T}$.
- $\{\{A_N(a_\varepsilon) : \varepsilon > 0\}\}$ is an a.c.s. for $\{A_N(a)\}$.

These three facts, by definition of the GLT sequence (see Definition 2.3 in [**8**]), tell us that $\{A_N(a)\}$ is a GLT sequence with symbol $a(x, x, t)$.

REFERENCES

1. A. Böttcher and B. Silbermann, *Introduction to large truncated Toeplitz matrices*, Springer, New York, 1999.

2. T. Ehrhardt and B. Shao, *Asymptotic behavior of variable-coefficient Toeplitz determinants*, J. Math. Anal. Appl. **7** (2001), 71–92.

3. D. Fasino and S. Serra-Capizzano, *From Toeplitz matrix sequences to zero distribution of orthogonal polynomials*, Contemp. Math., vol. 323, Amer. Math. Soc., Providence, 2003, pp. 329–339.

4. D. Geyer, *Vorlesung über Approximation im Komplex*, Birkhäuser Verlag, Basel, 1980.

5. R. Hagen, S. Roch and B. Silbermann, *C^*-algebras and numerical analysis*, Marcel Dekker, Inc., New York, 2001.

6. M. Kac, W.L. Murdock and G. Szegö, *On the eigenvalues of certain Hermitian forms*, J. Rational Mech. Anal. **2** (1953), 767–800.

7. A. Kuijlaars and S. Serra-Capizzano, *Asymptotic zero distribution of orthogonal polynomials with discontinuously varying recurrence coefficients*, J. Approx. Theory **113** (2001), 142–155.

8. S. Serra-Capizzano, *Generalized locally Toeplitz sequences: spectral analysis and applications to discretized partial differential equations*, Linear Algebra Appl. **366** (2003), 371–402.

9. ———, *The GLT class as a generalized Fourier analysis and applications*, Technical Report, SCCM-05-07, 2005.

10. B. Shao, *A trace formula for variable-coefficient Toeplitz matrices with symbols of bounded variation*, J. Math. Anal. Appl. **222** (1998), 505–546.

11. I.B. Simonenko and O.N. Zabroda, *Asymptotic invertibility and the collective asymptotic spectral behavior of generalized one-dimensional discrete convolutions*, Funct. Anal. Appl. **38** , No. 1, January (2004), 65–66.

12. P. Tilli, *Locally Toeplitz sequences: spectral properties and applications*, Linear Algebra Appl. **278** (1998), 91–120.

Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany
E-mail address: silbermn@mathematik.tu-chemnitz.de

Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany
E-mail address: olga.zabroda@s2003.tu-chemnitz.de