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ON THE KONTOROVICH-LEBEDEV TRANSFORMATION

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ABSTRACT. Further developments of the results on the Kontorovich-Lebedev integral transformation are given. In particular, properties of the boundedness, compactness in $L_{\nu,p}$, $1 \le p \le \infty$, $\nu < 1$, are established. The Bochner type representation theorem is proved. An example of the Fredholm integral equation associated with the Kontorovich-Lebedev operator is considered.

1. Introduction and auxiliary results. In this paper we investigate mapping properties of the Kontorovich-Lebedev operator [**3**], [**5**], [**11**]

$$K_{ir}[f] = \sqrt{\frac{2}{\pi}} \int_0^\infty K_{i\tau}(x) f(x)\, dx, \quad \tau \in \mathbf{R}_+, \tag{1.1}$$

which is associated with the Macdonald function $K_{i\tau}$ as the kernel [**1**] in its natural domain of definition $f \in L^0 \equiv L_1(\mathbf{R}_+; K_0(x)\,dx)$, i.e.,

$$L^0 := \Big\{ f : \int_0^\infty K_0(x) |f(x)|\, dx < \infty \Big\}. \tag{1.2}$$

In particular, it contains all spaces $L^\alpha \equiv L_1(\mathbf{R}_+; K_0(\alpha x)\,dx)$, $0 < \alpha \le 1$ and $L_{\nu,p}(\mathbf{R}_+)$, $\nu < 1$, $1 \le p \le \infty$, with the norms

$$\|f\|_{L^\alpha} = \int_0^\infty K_0(\alpha x) |f(x)|\, dx < \infty, \tag{1.3}$$

$$\begin{gathered} \|f\|_{\nu,p} = \left(\int_0^\infty x^{\nu p - 1} |f(x)|^p\, dx \right)^{\frac{1}{p}} < \infty, \\ \|f\|_{\nu,\infty} = \operatorname{ess\,sup}_{x \ge 0} |x^\nu f(x)| < \infty. \end{gathered} \tag{1.4}$$

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When $\nu = \frac{1}{p}$ we obtain the usual norm in L_p denoted by $\| \ \|_p$. The Macdonald function can be represented by the Fourier integral [**5**]

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh u} \cos(\tau u)\, du. \tag{1.5}$$

Therefore, for $x > 0$, $\tau \in \mathbf{R}$, it is real-valued and even function with respect to the index $i\tau$. Furthermore, via the analytic properties of the integrand in (1.5), the latter integral can be extended to the strip $\delta \in [0, \pi/2)$ in the upper half-plan (cf. in [**9**]), i.e.,

$$K_{i\tau}(x) = \frac{1}{2} \int_{i\delta-\infty}^{i\delta+\infty} e^{-x\cosh\beta + i\tau\beta}\, d\beta. \tag{1.6}$$

This gives us for each $x > 0$ an immediate uniform estimate

$$|K_{i\tau}(x)| \le e^{-\delta|\tau|} K_0(x\cos\delta), \quad 0 \le \delta \le \delta_0 < \frac{\pi}{2}. \tag{1.7}$$

We note that the Macdonald function is the modified Bessel function of the second kind $K_\mu(z)$ (in our case $\mu = i\tau$), which satisfies the differential equation

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - (z^2 + \mu^2) u = 0.$$

It has the asymptotic behavior [**1**]

$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} [1 + O(1/z)], \quad z \to \infty, \tag{1.8}$$

and near the origin

$$z^\mu K_\mu(z) = 2^{\mu-1}\Gamma(\mu) + o(1), \quad z \to 0, \tag{1.9}$$

$$K_0(z) = -\log z + O(1), \quad z \to 0. \tag{1.10}$$

By using relation (2.16.51.8) in [**4**, Vol. 2], we obtain the useful formula

$$\begin{aligned} &\int_0^\infty \tau \sinh((\pi - \delta)\tau) K_{i\tau}(x) K_{i\tau}(y)\, d\tau \\ &= \frac{\pi}{2} xy \sin\delta \, \frac{K_1((x^2 + y^2 - 2xy\cos\delta)^{\frac{1}{2}})}{(x^2 + y^2 - 2xy\cos\delta)^{\frac{1}{2}}}, \\ &\qquad x, y > 0, \quad 0 < \delta \le \pi. \end{aligned} \tag{1.11}$$

In the next section we establish mapping properties of the Kontorovich-Lebedev operator (1.1) in the spaces L^α and $L_{\nu,p}$ (see also [**8**], [**10**]). Mapping properties for distributions were considered in [**9**].

2. Kontorovich-Lebedev transform. In order to study boundedness properties of the operator (1.1) we prove its composition representation in terms of the Fourier and Laplace operators, which are defined respectively by

$$(\mathcal{F}f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixt} f(t)\,dt, \tag{2.1}$$

$$(\mathcal{L}f)(x) = \int_0^{\infty} e^{-xt} f(t)\,dt. \tag{2.2}$$

Lemma 1. *Let $\nu < 1$, $1 \le p \le \infty$, $q = p/(p-1)$ and $0 < \alpha \le 1$. Then the embedding*

$$L_{\nu,p}(\mathbf{R}_+) \subseteq L^\alpha \subseteq L^0 \tag{2.3}$$

is true and

$$\|f\|_{L^0} \le \|f\|_{L^\alpha} \le \frac{\sqrt{\pi}}{2}(\alpha q)^{\nu-1}\frac{\Gamma((1-\nu)/2)\Gamma^{1/q}(q(1-\nu))}{\Gamma(1-(\nu/2))}\,\|f\|_{\nu,p}, \quad 1 < p \le \infty, \tag{2.4}$$

$$\|f\|_{L^0} \le \|f\|_{L^\alpha} \le \alpha^{\nu-1}\|f\|_{\nu,1}\sup_{x\ge 0}[K_0(x)x^{1-\nu}]. \tag{2.5}$$

Proof. Letting in (1.5) $\tau = 0$ we easily find that $K_0(x)$ is steadily decreasing function and invoking (1.3) and (1.4) with the Hölder inequality we obtain

$$\|f\|_{L^0} \le \|f\|_{L^\alpha} \le \left(\int_0^\infty K_0^q(\alpha x)x^{(1-\nu)q-1}\,dx\right)^{1/q}\|f\|_{\nu,p}.$$

Hence, applying the generalized Minkowski inequality we derive

$$\begin{aligned}
\left(\int_0^\infty K_0^q(\alpha x)x^{(1-\nu)q-1}\,dx\right)^{1/q}& \\
&= \left(\int_0^\infty x^{(1-\nu)q-1}\left(\int_0^\infty e^{-\alpha x\cosh u}\,du\right)^q dx\right)^{1/q}\\
&\le \int_0^\infty du\left(\int_0^\infty x^{(1-\nu)q-1}e^{-\alpha q x\cosh u}\,dx\right)^{1/q}\\
&= (\alpha q)^{\nu-1}\,\Gamma^{1/q}(q(1-\nu))\int_0^\infty \frac{du}{\cosh^{1-\nu} u}\\
&= \frac{\sqrt{\pi}}{2}\,(\alpha q)^{\nu-1}\,\frac{\Gamma((1-\nu)/2)\Gamma^{1/q}(q(1-\nu))}{\Gamma(1-(\nu/2))}.
\end{aligned}$$

Meanwhile, for $f \in L_{\nu,1}(\mathbf{R}_+)$ we easily verify that

$$\int_0^\infty K_0(\alpha x)|f(x)|\,dx \le \alpha^{\nu-1}\|f\|_{\nu,1}\sup_{x\ge 0}[K_0(x)x^{1-\nu}] < \infty$$

and complete the proof of Lemma 1.

Lemma 2. *The Kontorovich-Lebedev transformation* (1.1) *is a bounded operator from* L^α, $0<\alpha\le 1$, *into the space of bounded continuous functions vanishing at infinity. Besides, the following composition representation in terms of operators* (2.1) *and* (2.2) *takes place*

$$K_{i\tau}[f] = (\mathcal{F}(\mathcal{L}f)(\cosh x))(\tau). \tag{2.6}$$

Proof. Substituting (1.6) with $\delta = 0$ in (1.1) we have the estimate

$$\begin{aligned}
|K_{i\tau}[f]| &\le \frac{1}{\sqrt{2\pi}}\int_0^\infty |f(x)|\,dx\int_{-\infty}^\infty e^{-x\cosh\beta}\,d\beta\\
&\le \frac{1}{\sqrt{2\pi}}\int_0^\infty |f(x)|\,dx\int_{-\infty}^\infty e^{-\alpha x\cosh\beta}\,d\beta\\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty K_0(\alpha x)|f(x)|\,dx = \sqrt{\frac{2}{\pi}}\,\|f\|_{L^\alpha}.
\end{aligned}$$

Hence, in view of Fubini's theorem, we can invert the order of integration in the corresponding iterated integral and arrive at the composition (2.6). It converges uniformly with respect to $\tau \geq 0$ and therefore $K_{i\tau}[f]$ is a bounded continuous function. Further, under conditions of Lemma 2 and the latter estimate we have that $(\mathcal{L}f)(\cosh x) \in L_1(\mathbf{R})$. Hence $K_{i\tau}[f] \to 0$ when $\tau \to \infty$ as an immediate consequence of the Riemann-Lebesgue lemma. Lemma 2 is proved.

Corollary 1. *Operator* $K_{i\tau}[f] : L_{\nu,p}(\mathbf{R}_+) \to L_p(\mathbf{R}_+)$, $p \geq 2$, $\nu < 1$ *is bounded and*

$$\|K_{i\tau}[f]\|_p \leq \sqrt{\frac{2}{\pi}} \left(\frac{q}{2}\right)^{\nu-1} \frac{\pi^{\frac{1}{p}}}{2^{3/q}} \left[\Gamma\left(\frac{q}{2}(1-\nu)\right)\right]^{2/q} \|f\|_{\nu,p}, \qquad q = \frac{p}{p-1}. \tag{2.7}$$

In particular, for $p = 2$, $\nu = \frac{1}{2}$, *we get*

$$\|K_{i\tau}[f]\|_2 \leq \frac{\sqrt{\pi}}{2} \|f\|_2. \tag{2.8}$$

Proof. Indeed, by appealing to the composition (2.6) and L_p-theorem for the Fourier transform [**6**, Theorem 74], we find

$$\|K_{i\tau}[f]\|_p \leq (2\pi)^{(\frac{1}{p})-(\frac{1}{2})} \left(\int_0^\infty |(\mathcal{L}f)(\cosh x)|^q \, dx\right)^{1/q}, \qquad q = \frac{p}{p-1}. \tag{2.9}$$

Hence by the generalized Minkowski and Hölder inequalities with formula (2.16.2.2) from [**4**, Volume 2], we obtain

$$
\begin{aligned}
&(2\pi)^{(\frac{1}{p})-(\frac{1}{2})}\left(\int_0^\infty |(\mathcal{L}f)(\cosh u)|^q \, du\right)^{1/q}\\
&\quad \le (2\pi)^{(\frac{1}{p})-(\frac{1}{2})} \int_0^\infty |f(x)| \left(\int_0^\infty e^{-qx\cosh u}\, du\right)^{1/q} dx\\
&\quad = (2\pi)^{(\frac{1}{p})-(\frac{1}{2})} \int_0^\infty |f(x)| K_0^{1/q}(qx)\, dx\\
&\quad \le (2\pi)^{(\frac{1}{p})-(\frac{1}{2})} \left(\int_0^\infty K_0(qx) x^{(1-\nu)q-1}\, dx\right)^{1/q} \|f\|_{\nu,p}\\
&\quad = \sqrt{\frac{2}{\pi}}\left(\frac{q}{2}\right)^{\nu-1} \frac{\pi^{\frac{1}{p}}}{2^{3/q}} \left[\Gamma\left(\frac{q}{2}(1-\nu)\right)\right]^{2/q} \|f\|_{\nu,p}.
\end{aligned}
$$

Consequently combining with (2.9) we have (2.7) and (2.8). Corollary 1 is proved.

Furthermore, we will establish now that the Kontorovich-Lebedev transform (1.1) is a completely continuous operator.

Lemma 3. *The transformation $K_{i\tau}[f] : L_{\nu,p}(\mathbf{R}_+) \to L_q(\mathbf{R}_+)$, $1 < p \le 2$, $\nu < 1$, $q = p/(p-1)$ is a completely continuous operator.*

Proof. Indeed, this can be done by approximation of the Macdonald function $K_{i\tau}(x)$ in terms of kernels of finite rank. However, first we verify the following integral condition

$$
\int_0^\infty \int_0^\infty \left|\sqrt{\frac{2}{\pi}}\, K_{i\tau}(x)\right|^q x^{(1-\nu)q-1}\, d\tau\, dx < \infty. \tag{2.10}
$$

We treat double integral (2.10) by using again Fourier integrals (1.5), (1.6) and Theorem 74 from [**6**], see (2.9). Thus, we have

$$
\begin{aligned}
&\left(\int_0^\infty \int_0^\infty \left|\sqrt{\frac{2}{\pi}}\, K_{i\tau}(x)\right|^q x^{(1-\nu)q-1}\, d\tau\, dx\right)^{1/q}\\
&\le (2\pi)^{(1/q)-(\frac{1}{2})} \left(\int_0^\infty K_0^{1/(p-1)}(px) x^{(1-\nu)q-1}\, dx\right)^{1/q}\\
&= 2^{(\frac{1}{2})-(2/p)} \pi^{1/(2q)} q^{\nu-1} \frac{[\Gamma(q(1-\nu))]^{1/q} [\Gamma((p/2)(1-\nu))]^{1/p}}{\left[\Gamma\left(\frac{p(1-\nu)+1}{2}\right)\right]^{1/p}} < \infty.
\end{aligned}
$$

In particular, by taking the values of integrals (2.16.52.6) and (2.16.2.1) from [**4**, Volume 2] for the Hilbert-Schmidt norm, $p = q = 2$, $\nu = \frac{1}{2}$, of $K_{i\tau}[f]$, we find

$$\int_0^\infty \int_0^\infty \left| \sqrt{\frac{2}{\pi}}\, K_{i\tau}(x) \right|^2 d\tau\, dx = \frac{\pi}{4}. \tag{2.11}$$

Therefore, in the Lebesgue space with the norm,

$$\left(\int_0^\infty \int_0^\infty |h(\tau, x)|^q\, x^{(1-\nu)q-1}\, d\tau\, dx \right)^{1/q} \tag{2.12}$$

there is a sequence $\{K_n(\tau, x)\}$ of continuous in $\mathbf{R}_+^2$ and finite kernels, which converges to $\sqrt{2/\pi}\, K_{i\tau}(x)$ in the space (2.12), namely,

$$\int_0^\infty \int_0^\infty \left| \sqrt{\frac{2}{\pi}}\, K_{i\tau}(x) - K_n(\tau, x) \right|^q x^{(1-\nu)q-1}\, d\tau\, dx \to 0, \quad n \to \infty. \tag{2.13}$$

Denoting by K_n integral operators with kernels $K_n(\tau, x)$ we use the Hölder inequality to obtain

$$\begin{aligned} &|K_{i\tau}[f] - K_n f(\tau)|^q \\ &\quad = \left| \int_0^\infty \left[\sqrt{\frac{2}{\pi}}\, K_{i\tau}(x) - K_n(\tau, x) \right] f(x)\, dx \right|^q \\ &\quad \le \int_0^\infty \left| \sqrt{\frac{2}{\pi}}\, K_{i\tau}(x) - K_n(\tau, x) \right|^q x^{(1-\nu)q-1}\, dx \left(\int_0^\infty |f(x)|^p x^{\nu p-1}\, dx \right)^{q/p}. \end{aligned}$$

Hence,

$$\begin{aligned} &\|K_{i\tau}[f] - K_n f\|_q^q \\ &\qquad = \int_0^\infty \left| K_{i\tau}[f] - K_n f(\tau) \right|^q d\tau \\ &\qquad \le \|f\|_{\nu,p}^q \int_0^\infty \int_0^\infty \left| \sqrt{\frac{2}{\pi}}\, K_{i\tau}(x) - K_n(\tau, x) \right|^q x^{(1-\nu)q-1}\, dx\, d\tau. \end{aligned}$$

Consequently, the norm of difference

$$\|K_{i\tau} - K_n\|_q^q \le \int_0^\infty \int_0^\infty \left| \sqrt{\frac{2}{\pi}}\, K_{i\tau}(x) - K_n(\tau, x) \right|^q x^{(1-\nu)q-1}\, dx\, d\tau \to 0,$$

$$n \to \infty$$

and the Kontorovich-Lebedev operator (1.1) is the L_q-limit of the sequence $\{K_n\}$ of completely continuous integral operators with continuous finite kernels. Thus $K_{i\tau}[f]$ is a completely continuous operator. Lemma 3 is proved.

Let us introduce for all $x > 0$ the following regularization operator (cf. in [**2**], [**9**])

$$(I_\varepsilon g)(x) = \sqrt{\frac{2}{\pi}} \frac{1}{\pi x} \int_0^\infty \tau \sinh((\pi - \varepsilon)\tau) K_{i\tau}(x) g(\tau)\, d\tau, \tag{2.14}$$

where $\varepsilon \in (0, \pi)$. We will show that (2.14) gives in some sense an inversion of (1.1) in $L_{\nu,p}(\mathbf{R}_+)$ space when $\varepsilon \to 0$.

Lemma 4. *If* $g(\tau) = K_{i\tau}[f]$, $f(y) \in L_{\nu,p}(\mathbf{R}_+)$, $\nu < 1$, $1 \le p \le \infty$, *then*

$$\begin{aligned}(I_\varepsilon g)(x) &\equiv (KL_\varepsilon g)(x) \\ &= \frac{\sin\varepsilon}{\pi} \int_0^\infty \frac{K_1((x^2 + y^2 - 2xy\cos\varepsilon)^{\frac12})}{(x^2 + y^2 - 2xy\cos\varepsilon)^{\frac12}}\, y f(y)\, dy, \qquad (2.15) \\ & \qquad x > 0.\end{aligned}$$

Proof. Substituting $g(\tau)$ in (2.14) and inverting the order of integration we use relation (1.11) and immediately arrive at the representation (2.15). The motivation of the change of the order of integration is due to Fubini's theorem and the following estimate (see (1.7))

$$\begin{aligned}|(KL_\varepsilon f)(x)| \le \frac{2K_0(x\cos\delta_1)}{\pi^2 x} &\int_0^\infty \tau \sinh((\pi - \varepsilon)\tau) e^{-(\delta_1+\delta_2)\tau}\, d\tau \\ &\cdot \left(\int_0^\infty K_0^q(y\cos\delta_2) y^{(1-\nu)q-1}\, dy \right)^{1/q} \|f\|_{\nu,p} < \infty,\end{aligned}$$

where $\delta_i \in [0, \pi/2)$, $i = 1, 2$, $\delta_1 + \delta_2 + \varepsilon > \pi$, $\nu < 1$, $1 < p < \infty$. Since in view of the asymptotic behavior of the Macdonald function (1.8), (1.9) and (1.10), we have (cf. (2.4) and (2.5))

$$\int_0^\infty K_0(y\cos\delta_2)|f(y)|\,dy \le (\cos\delta_2)^{\nu-1}\|f\|_{\nu,1}\sup_{y\ge 0}[K_0(y)y^{1-\nu}] < \infty,$$

$$\int_0^\infty K_0(y\cos\delta_2)|f(y)|\,dy \le 2^{-\nu-1}(\cos\delta_2)^{\nu-1}\,\Gamma^2\left(\frac{1-\nu}{2}\right)\|f\|_{\nu,\infty} < \infty,$$

it follows that representation (2.15) is also true when $p = 1, \infty$, respectively. Lemma 4 is proved.

We are ready to prove now the Bochner type representation theorem for the regularization operator (2.15).

Theorem. *Let* $f \in L_{\nu,p}(\mathbf{R}_+)$, $0 < \nu < 1$, $1 \le p < \infty$. *Then*

$$f(x) = \lim_{\varepsilon\to 0}(KL_\varepsilon f)(x), \tag{2.16}$$

where the limit is with respect to the norm (1.4). *Besides, the limit* (2.16) *exists for almost all* $x > 0$.

Proof. Making substitution $y = x(\cos\varepsilon + t\sin\varepsilon)$ in the integral (2.15), we deduce

$$(KL_\varepsilon f)(x) = \frac{x\sin\varepsilon}{\pi}\int_{-\cot\varepsilon}^\infty \frac{K_1(x\sin\varepsilon\sqrt{t^2+1})}{\sqrt{t^2+1}} \cdot f(x(\cos\varepsilon + t\sin\varepsilon))(\cos\varepsilon + t\sin\varepsilon)\,dt. \tag{2.17}$$

Hence, due to the generalized Minkowski inequality and elementary inequality for the Macdonald function $xK_1(x) \le 1$, $x \ge 0$, cf. (1.5), (1.8) and (1.9), we estimate the $L_{\nu,p}$-norm, $0 < \nu < 1$, $1 \le p < \infty$, of the integral (2.17) as follows

$$\begin{aligned}\|KL_\varepsilon f\|_{\nu,p} &\le \frac{1}{\pi}\int_{-\cot\varepsilon}^\infty \frac{dt}{t^2+1}\,\|f(x(\cos\varepsilon + t\sin\varepsilon))(\cos\varepsilon + t\sin\varepsilon)\|_{\nu,p} \\ &= \frac{\|f\|_{\nu,p}}{\pi}\int_{-\cot\varepsilon}^\infty \frac{(\cos\varepsilon + t\sin\varepsilon)^{1-\nu}\,dt}{t^2+1} \\ &\le \frac{\|f\|_{\nu,p}}{\pi}\int_{-\infty}^\infty \frac{(1+|t|)^{1-\nu}\,dt}{t^2+1} \le \frac{4}{\pi}\left(1+\frac{1}{\nu}\right)\|f\|_{\nu,p}.\end{aligned}$$

Consequently, by using the identity

$$\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}=1-\frac{\varepsilon}{\pi}$$

and denoting by

$$R(x,t,\varepsilon)=x\sin\varepsilon\sqrt{t^2+1}\,K_1\big(x\sin\varepsilon\sqrt{t^2+1}\,\big) \tag{2.18}$$

we find that

$$\begin{aligned}
&\|KL_\varepsilon f-f\|_{\nu,p}\\
&\quad\le\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\Big\|f(x(\cos\varepsilon+t\sin\varepsilon))(\cos\varepsilon+t\sin\varepsilon)R(x,t,\varepsilon)\\
&\qquad\qquad\qquad\qquad\qquad\qquad-\left(1-\frac{\varepsilon}{\pi}\right)^{-1}f(x)\Big\|_{\nu,p}\\
&\quad\le\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\Big\|\Big[f(x(\cos\varepsilon+t\sin\varepsilon))(\cos\varepsilon+t\sin\varepsilon)\\
&\qquad\qquad\qquad\qquad\qquad-\left(1-\frac{\varepsilon}{\pi}\right)^{-1}f(x)\Big]R(x,t,\varepsilon)\Big\|_{\nu,p}\\
&\qquad+\frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\|f(x)[R(x,t,\varepsilon)-1]\|_{\nu,p}\\
&\quad=I_1(\varepsilon)+I_2(\varepsilon).
\end{aligned}$$

But, since (see [**1**], [**2**])

$$\frac{d}{dx}\left[xK_1(x)\right]=-\,xK_0(x),$$

and $xK_1(x)\to 1$, $x\to 0$, we obtain the following representation

$$R(x,t,\varepsilon)-1=-\int_0^{x\sin\varepsilon(t^2+1)^{\frac{1}{2}}}y\,K_0(y)\,dy.$$

Hence, appealing again to the generalized Minkowski inequality, we deduce

$$
\begin{aligned}
I_2(\varepsilon) &= \frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1} \\
&\quad\cdot\bigg(\int_0^\infty x^{\nu p-1}\bigg(\int_0^{x\sin\varepsilon(t^2+1)^{\frac{1}{2}}} yK_0(y)\,dy\bigg)^p |f(x)|^p\,dx\bigg)^{\frac{1}{p}} \\
&\le \frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\int_0^\infty yK_0(y) \\
&\quad\cdot\bigg(\int_{y/(\sin\varepsilon(t^2+1)^{\frac{1}{2}})}^{\infty} x^{\nu p-1}|f(x)|^p\,dx\bigg)^{\frac{1}{p}}dy \\
&\le \frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}dt\int_0^\infty uK_0(u\sqrt{t^2+1})\bigg(\int_{\frac{u}{\sin\varepsilon}}^{\infty} x^{\nu p-1}|f(x)|^p\,dx\bigg)^{\frac{1}{p}}du \\
&= \frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}dt\bigg(\int_0^{\sqrt{\varepsilon}}+\int_{\sqrt{\varepsilon}}^{\infty}\bigg)uK_0(u\sqrt{t^2+1}) \\
&\quad\cdot\bigg(\int_{\frac{u}{\sin\varepsilon}}^{\infty} x^{\nu p-1}|f(x)|^p\,dx\bigg)^{\frac{1}{p}}du \\
&\le \frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}dt\int_0^{\sqrt{\varepsilon}} uK_0(u\sqrt{t^2+1})\bigg(\int_{\frac{u}{\sin\varepsilon}}^{\infty} x^{\nu p-1}|f(x)|^p\,dx\bigg)^{\frac{1}{p}}du \\
&\quad+\frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\int_0^\infty uK_0(u)\,du\cdot\bigg(\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} x^{\nu p-1}|f(x)|^p\,dx\bigg)^{\frac{1}{p}} \\
&\le \frac{\varepsilon^{\nu/2}\|f\|_{\nu,p}}{\pi-\varepsilon}\int_{-\infty}^{\infty}(t^2+1)^{(\nu/2)-1}\,dt \\
&\quad\cdot\int_0^\infty u^{1-\nu}K_0(u)\,du+\frac{\pi}{\pi-\varepsilon}\bigg(\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} x^{\nu p-1}|f(x)|^p\,dx\bigg)^{\frac{1}{p}} \\
&= \frac{\pi}{\pi-\varepsilon}\bigg(\varepsilon^{\nu/2}\Gamma(1-\nu)\|f\|_{\nu,p}+\bigg(\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} x^{\nu p-1}|f(x)|^p\,dx\bigg)^{\frac{1}{p}}\bigg)\to 0, \\
&\quad \varepsilon\to 0.
\end{aligned}
$$

Concerning the integral I_1 we first approximate $f \in L_{\nu,p}(\mathbf{R}_+)$ by a smooth function φ with a compact support on $\mathbf{R}_+$. This implies that there exists a function $\varphi \in C_0^1(\mathbf{R}_+)$ such that $\|f-\varphi\|_{\nu,p} \le \varepsilon$. Hence, since the kernel (2.18), $R(x,t,\varepsilon) \le 1$, then in view of the representation,

$$
\begin{aligned}
&\varphi(x(\cos\varepsilon + t\sin\varepsilon))(\cos\varepsilon + t\sin\varepsilon) - \varphi(x) \\
&\quad = \int_1^{\cos\varepsilon + t\sin\varepsilon} \frac{d}{dy}\left[y\varphi(xy)\right] dy \\
&\quad = \int_1^{\cos\varepsilon + t\sin\varepsilon} \left[\varphi(xy) + xy\varphi'(xy)\right] dy.
\end{aligned}
\tag{2.19}
$$

In a similar manner, we have

(2.20)

$$
\begin{aligned}
I_1(\varepsilon) \le{}& \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1}\, \|[f(x(\cos\varepsilon + t\sin\varepsilon)) \\
&\qquad - \varphi(x(\cos\varepsilon + t\sin\varepsilon))](\cos\varepsilon + t\sin\varepsilon)\|_{\nu,p} \\
&+ \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1} \left\|\varphi(x(\cos\varepsilon + t\sin\varepsilon))(\cos\varepsilon + t\sin\varepsilon)\right. \\
&\qquad \left. - \left(1 - \frac{\varepsilon}{\pi}\right)^{-1} f(x)\right\|_{\nu,p} \\
\le{}& \|f-\varphi\|_{\nu,p} \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{(\cos\varepsilon + t\sin\varepsilon)^{1-\nu}\, dt}{t^2+1} \\
&+ \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1} \left\|\varphi(x) - \left(1-\frac{\varepsilon}{\pi}\right)^{-1} f(x)\right\|_{\nu,p} \\
&+ (\|\varphi\|_{\nu,p} + \|\varphi'\|_{1+\nu,p}) \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1} \left|\int_1^{\cos\varepsilon + t\sin\varepsilon} y^{-\nu}\, dy\right| \\
\le{}& \left[1 + \frac{4-\varepsilon}{\pi} + \frac{4}{\pi\nu}\right] \|f-\varphi\|_{\nu,p} + \frac{\varepsilon}{\pi}\|f\|_{\nu,p} + \frac{\|\varphi\|_{\nu,p} + \|\varphi'\|_{1+\nu,p}}{\pi(1-\nu)} \\
&\cdot \int_{-\cot\varepsilon}^{\infty} \frac{|1-(\cos\varepsilon + t\sin\varepsilon)^{1-\nu}|}{t^2+1}\, dt.
\end{aligned}
$$

The latter integral in (2.19) we treat by making the substitution

$e^v = \cos\varepsilon + t\sin\varepsilon$. Then it takes the form

$$\begin{aligned}
&\int_{-\cot\varepsilon}^{\infty} \frac{|1-(\cos\varepsilon + t\sin\varepsilon)^{1-\nu}|}{t^2+1}\,dt \\
&\qquad = \sin\varepsilon \int_0^{\infty} \frac{\sinh((1-\nu)v)}{\cosh v - \cos\varepsilon}\,dv \\
&\qquad = \sin\varepsilon\left(\int_0^1 + \int_1^{\infty}\right)\frac{\sinh((1-\nu)v)}{\cosh v - \cos\varepsilon}\,dv \\
&\qquad \le \sin\varepsilon\left(\log(\cosh v - \cos\varepsilon)|_0^1 + \int_1^{\infty}\frac{\sin(v(1-\nu))}{\cosh v - 1}\,dv\right) \\
&\qquad \le \sin\varepsilon \log\left(2^{-1}\sin^{-2}\frac{\varepsilon}{2}\right) + A_\nu \sin\varepsilon,
\end{aligned} \tag{2.21}$$

where

$$A_\nu = 1 + \int_1^{\infty} \frac{\sinh(v(1-\nu))}{\cosh v - 1}\,dv, \quad 0<\nu<1.$$

Thus, combining with (2.19), we see that $\lim_{\varepsilon\to 0} I_1(\varepsilon) = 0$. Therefore, by virtue of the above estimates $\lim_{\varepsilon\to 0}\|KL_\varepsilon f - f\|_{\nu,p} = 0$ and relation (2.16) is proved.

In order to verify the pointwise convergence we use the fact that any sequence of function $\{\varphi_n\} \in C_0^1(\mathbf{R}_+)$ which converges to f in $L_{\nu,p}(\mathbf{R}_+)$-norm (1.4) contains a subsequence $\{\varphi_{n_k}\}$ convergent almost everywhere, i.e., $\lim_{k\to\infty}\varphi_{n_k}(x) = f(x)$ for almost all $x>0$. Then via (2.19), (2.20) and (2.21), we obtain

$$\begin{aligned}
&|(KL_\varepsilon f)(x) - f(x)| \\
&\quad \le \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1} \\
&\qquad \cdot \left| f(x(\cos\varepsilon + t\sin\varepsilon))(\cos\varepsilon + t\sin\varepsilon)R(x,t,\varepsilon) - \left(1-\frac{\varepsilon}{\pi}\right)^{-1} f(x)\right| \\
&\quad \le \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{(\cos\varepsilon + t\sin\varepsilon)\,dt}{t^2+1} \\
&\qquad \cdot \left| f(x(\cos\varepsilon + t\sin\varepsilon) - \varphi_{n_k}(x(\cos\varepsilon + t\sin\varepsilon))\right|
\end{aligned} \tag{2.22}$$

$$+\frac{1}{\pi(1-\nu)}\sup_{t\geq 0}|t^{\nu}\varphi_{n_k}(xt)+xt^{1+\nu}\varphi'_{n_k}(xt)|$$
$$\cdot\sin\varepsilon\Big[\log\Big(2^{-1}\sin^{-2}\frac{\varepsilon}{2}\Big)+A_{\nu}\Big]+\Big(1-\frac{\varepsilon}{\pi}\Big)|\varphi_{n_k}(x)-f(x)|+\frac{\varepsilon}{\pi}\,|f(x)|.$$

Meantime, by taking $1\leq p<\infty$, $q=p/(p-1)$, we have

(2.23)

$$\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty}\frac{(\cos\varepsilon+t\sin\varepsilon)\,dt}{t^2+1}\,|f(x(\cos\varepsilon+t\sin\varepsilon)-\varphi_{n_k}(x(\cos\varepsilon+t\sin\varepsilon))|$$
$$\leq\frac{x^{-\nu}\,\|f-\varphi_{n_k}\|_{\nu,p}}{\pi\sin^{\frac{1}{p}}\varepsilon}\bigg(\int_{-\cot\varepsilon}^{\infty}\frac{(\cos\varepsilon+t\sin\varepsilon)^{q(1-\nu+(\frac{1}{p}))}\,dt}{(t^2+1)^q}\bigg)^{1/q}$$
$$=\frac{x^{-\nu}}{\pi}\,\|f-\varphi_{n_k}\|_{\nu,p}\sin\varepsilon\bigg(\int_0^{\infty}\frac{v^{q(2-\nu)-1}\,dv}{(v^2-2\cos\varepsilon v+1)^q}\bigg)^{1/q}.$$

However, appealing to the relation (2.2.9.7) from [**4**, Volume I] we get the value of the latter integral in terms of the Gauss function

$$\int_0^{\infty}\frac{v^{q(2-\nu)-1}\,dv}{(v^2-2\cos\varepsilon v+1)^q}=\frac{\Gamma(q(2-\nu))\Gamma(q\nu)}{\Gamma(2q)}$$
$$\cdot{}_2F_1\Big(q-\frac{q\nu}{2},\frac{q\nu}{2};\,q+\frac{1}{2};\sin^2\varepsilon\Big)$$
$$\to\frac{\Gamma(q(2-\nu))\Gamma(q\nu)}{\Gamma(2q)},\quad \varepsilon\to 0.$$

Therefore we see that the righthand side of inequality (2.22) tends to zero for almost all $x>0$ when $\varepsilon\to 0$, $k\geq k_0$.

Finally let us consider the pointwise convergence when $f\in L_{\nu,1}(\mathbf{R}_+)$. In this case for any $k\geq k_0$, we get $\|f-\varphi_{n_k}\|_{\nu,1}\leq\delta$. Making substitution $v=\cos\varepsilon+t\sin\varepsilon$ the first integral in (2.23) becomes

$$\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty}\frac{(\cos\varepsilon+t\sin\varepsilon)\,dt}{t^2+1}\,|f(x(\cos\varepsilon+t\sin\varepsilon)-\varphi_{n_k}(x(\cos\varepsilon+t\sin\varepsilon))|$$
$$=\frac{\sin\varepsilon}{\pi}\int_0^{\infty}\frac{v^{2-\nu}}{(v-\cos\varepsilon)^2+\sin^2\varepsilon}\,v^{\nu-1}\,|f(xv)-\varphi_{n_k}(xv)|\,dv$$
$$\leq\frac{x^{-\nu}\sin\varepsilon}{\pi}\|f-\varphi_{n_k}\|_{\nu,1}\sup_{v\geq 0}\frac{v^{2-\nu}}{(v-\cos\varepsilon)^2+\sin^2\varepsilon}\,.$$

However, by using the methods of calculus it is not difficult to show that

$$\sup_{v\ge 0} \frac{v^{2-\nu}}{(v-\cos\varepsilon)^2+\sin^2\varepsilon} = \frac{v_0^{2-\nu}}{(v_0-\cos\varepsilon)^2+\sin^2\varepsilon} \le \frac{2^{2-\nu}}{\sin^2\varepsilon},$$

where $v_0 = 1-((\cos\varepsilon)/\nu)+\sqrt{((\cos^2\varepsilon)/\nu^2)+(2(1-\cos\varepsilon)/\nu)}\,.$

Thus,

$$\frac{\sin\varepsilon}{\pi}\int_0^\infty \frac{v^{2-\nu}}{(v-\cos\varepsilon)^2+\sin^2\varepsilon}\, v^{\nu-1}\, |f(xv)-\varphi_{n_k}(xv)|\, dv \le \frac{2^{2-\nu}x^{-\nu}\delta}{\pi\sin\varepsilon},$$

which tends to 0 by choosing first ε and then δ. The theorem is proved.

3. Fredholm equation. In this final section we consider an example of Fredholm integral equation in the space $L_2(\mathbf{R}_+)$, which is associated with the Kontorovich-Lebedev operator (1.1). Precisely, let us consider the following integral equation of the second kind

$$f(x) - \sqrt{\frac{2}{\pi}}\int_0^\infty K_{ix}(t)f(t)\,dt = g(x), \tag{3.1}$$

where $x>0$, $g\in L_2(\mathbf{R}_+)$ and the function $f\in L_2(\mathbf{R}_+)$ is to be determined. According to Corollary 1 and norm inequality (2.8) the Kontorovich-Lebedev transformation is a bounded operator in $L_2(\mathbf{R}_+)$ and we have that $\|K_{ix}\| \le \sqrt{\pi}/2 < 1$. Consequently, by virtue of the Banach theorem integral operator $I-K_{ix}$ has a bounded inverse operator $[I-K_{ix}]^{-1}$ with the norm

$$\|[I-K_{ix}]^{-1}\| \le \frac{1}{1-\sqrt{\pi}/2}\,.$$

Furthermore, the unique L_2-solution of the equation (3.1) is represented through the absolutely and uniformly convergent Neumann series

$$f(x) = \sum_{n=0}^{\infty} K_{ix}^n[g], \tag{3.2}$$

where the iterates $K_{ix}^n[g]$ are defined by the recurrence relation

$$K_{ix}^0[g] = g, \; K_{ix}^n[g] = K_{ix}[K_{ix}^{n-1}[g]] = \int_0^\infty \mathcal{K}_n(x,t)g(t)\,dt, \quad n \geq 1. \tag{3.3}$$

We show that the kernel $\mathcal{K}_n(x,t)$ of the compositions (3.3) can be expressed in terms of the Fourier transform (2.1) and Fourier convolution operator

$$(\varphi * \psi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \varphi(x-t)\psi(t)\,dt. \tag{3.4}$$

To do this, we use the fact from [**6**, Theorem 65] that $(\varphi*\psi)(x) \in L_2(\mathbf{R})$ and $\mathcal{F}(\varphi*\psi)(x) = (\mathcal{F}\varphi)(x)(\mathcal{F}\psi)(x)$ in $L_2(\mathbf{R})$ if $\varphi \in L_2(\mathbf{R})$ and $\psi \in L_1(\mathbf{R})$. Then from (1.6) we easily find that the kernel of $K_{ix}^1[g]$ is $\mathcal{K}_1(x,t) = (\mathcal{F}e^{-t\cosh u})(x)$.

Further, for $n = 2$, we obtain the composition

$$\begin{aligned} K_{ix}^2[g] &= \frac{2}{\pi}\int_0^\infty K_{ix}(y)\,dy \int_0^\infty K_{iy}(t)g(t)\,dt \\ &= \int_0^\infty \mathcal{K}_2(x,t)g(t)\,dt, \end{aligned} \tag{3.5}$$

where

$$\mathcal{K}_2(x,t) = \frac{2}{\pi}\int_0^\infty K_{ix}(y)K_{iy}(t)\,dy. \tag{3.6}$$

Note that the interchange of the order of integration in (3.5) is plainly verified via Fubini's theorem by means of the inequality (1.7) and condition $g \in L_2(\mathbf{R}_+)$. Hence, invoking (1.5) and the value of integral (2.16.14.1) from [**4**, Volume 2] after using the Parseval equality for the Fourier cosine transform (cf. in [**6**]), we derive

(3.7)

$$\begin{aligned} \mathcal{K}_2(x,t) &= \frac{1}{\cosh(\pi x/2)}\int_0^\infty e^{-t\cosh u}\cos\left(\tau\log(u+\sqrt{u^2+1})\right)\frac{du}{\sqrt{u^2+1}} \\ &= \frac{1}{\cosh(\pi x/2)}\int_0^\infty e^{-t\cosh\sinh u}\cos(xu)\,du. \end{aligned}$$

Meantime with the integral (2.5.46.5) in [**4**, Volume 1] and the factorization property for Fourier convolution (3.4), the latter integral in (3.7) implies

$$\begin{aligned}\mathcal{K}_2(x,t) &= \frac{1}{\cosh(\pi x/2)}\int_0^\infty e^{-t\cosh\sinh u}\cos(xu)\,du\\ &= (\mathcal{F}e^{-t\cosh\sinh u})(x)\left(\mathcal{F}\frac{1}{\cosh u}\right)(x)\\ &= \mathcal{F}\left(\frac{1}{\cosh u} * e^{-t\cosh\sinh u}\right)(x).\end{aligned}$$

Thus for the nth iterate it is not difficult to deduce the corresponding formula for the kernel $\mathcal{K}_n(x,t)$. It gives

$$\begin{aligned}\mathcal{K}_n(x,t) &= \mathcal{F}\Big(\frac{1}{\cosh u} * \mathcal{K}_{n-1}(\sinh u, t)\Big)(x)\\ &= \sqrt{\frac{\pi}{2}}\,\frac{1}{\cosh(\pi x/2)}\,\mathcal{F}(\mathcal{K}_{n-1}(\sinh u,t))(x), \quad n \geq 2.\end{aligned}$$

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