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DISPLACEMENT-TRACTION BOUNDARY VALUE PROBLEMS FOR ELASTIC PLATES WITH TRANSVERSE SHEAR DEFORMATION

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ABSTRACT. The existence, uniqueness and continuous dependence on the data are studied in a Sobolev space setting for the solutions of boundary integral equations arising in the interior and exterior mixed boundary value problems for bending of thin elastic plates.

1. Introduction. Applied mathematicians and engineers find closed-form solutions to continuum mechanics problems very convenient, since they facilitate the computation of highly accurate results. The boundary integral equation (BIE) method offers one of the best and most elegant ways of generating such solutions. For the Dirichlet, Neumann and Robin boundary value problems (BVPs) it is possible to construct classical (regular) solutions if the boundary and data are sufficiently smooth. Unfortunately, this cannot be done satisfactorily for mixed BVPs, where the data are usually discontinuous at the points separating the displacement and traction boundary conditions. In this case the net has to be cast wider in order to look for weak solutions to the corresponding BIEs. This technique has two additional advantages: it is also applicable to less smooth boundaries and data, and helps to estimate the convergence rate in boundary element methods associated with the problem, since error bounds are defined quite naturally by means of Sobolev space norms. Weak solution procedures are now familiar to practitioners, who exploit their generality and usefulness extensively.

Mixed BVPs for bending of elastic plates occur very frequently in the modeling of industrial processes, for example, in aerospace engineering, ship and marine technology, and car manufacture. Today's computational power and design sophistication, on a general background of a multitude of environmental issues (such as conservation of natural resources), require more accurate plate bending models than Kirchhoff's classical one. The latter, which reduces to solving a nonhomogeneous biharmonic equation with two boundary conditions, ignores the effects of transverse shear deformation and gives rise to certain mathematical discrepancies. Such drawbacks do not occur in the Mindlin-type model proposed in [**1**], where a system of three second order equations is solved together with three independent boundary conditions to give a fuller picture of the behavior of the plate. This model, used in what follows, is rigorous in the sense that it is based solely on the kinematic assumption that the displacement field $v = (v_1, v_2, v_3)^T$ with respect to a system of rectangular Cartesian coordinates (x_1, x_2, x_3) (here (x_1, x_2) are in the middle plane of the plate) is of the form

$$v_\alpha = x_3 u_\alpha(x_1, x_2), \quad \alpha = 1, 2, \quad v_3 = u_3(x_1, x_2). \tag{1}$$

Boundary value problems for the system of partial differential equations obtained from (1) have already been studied by means of integral equation methods in [**1**–**5, 9, 14**], where the existence of regular, [**1**–**5, 14**], and weak [**9**] solutions has been proved by potential methods for the Dirichlet, Neumann and Robin BVPs. The results in [**15**] illustrate the difficulties arising when regular solutions are sought for the mixed problem.

Below we study the existence, uniqueness and continuous dependence on the data of the weak solutions to the integral equations arising in the interior and exterior displacement-traction case for thin plates when the solutions are represented in terms of single or double layer potentials. We offer a choice of four different integral representations for the solution, of which the last two are of a type that, to our knowledge, has never been used in such problems. The main results are contained in Theorems 2–5.

This analysis is an essential first step in any applied problem and needs to be carried out in order to validate the construction of subsequent numerical approximations.

Applications of the BIE approach in conjunction with weak solutions in three-dimensional elastodynamics are described in [**6**–**8**]. The use of a BIE technique in the numerical solution of a problem for Reissner's plate model is illustrated in [**16**]. An example of boundary element Galerkin method based on the BIE formulation of a BVP with error estimates derived in terms of Sobolev norms can be found in [**13**].

2. Preliminary results. Unless otherwise stated, in what follows Greek and Latin subscripts take the values 1, 2 and 1, 2, 3, respectively, and the convention of summation over repeated indices is understood. For simplicity, we use the same notation for spaces, norms and inner products of vector functions as for scalar ones. Also the generic symbol c, with or without subscripts, denotes various strictly positive constants, $(\dots)_{,i} \equiv \partial(\dots)/\partial x_i$ and a superscript T indicates matrix transposition.

Suppose that a homogeneous and isotropic plate occupies a region $\bar{S} \times [-h_0/2, h_0/2]$, where $S \subset \mathbf{R}^2$ is a domain bounded by a simple, closed, Lipschitz, piecewise C^2-curve ∂S, and $h_0 = \text{const}$ is the plate thickness. We denote by S^+ the finite domain interior to ∂S and set $S^- = \mathbf{R}^2 \backslash (S^+ \cup \partial S)$. The equilibrium equations for bending can be written in the form [**1**]

$$A(\partial_x)u(x) + q(x) = 0, \quad x \in S^+ \text{ or } x \in S^-, \tag{2}$$

where $x = (x_1, x_2)$, $u = (u_1, u_2, u_3)^T$ is the vector characterizing the displacements in accordance with (1), q is a combination of the body forces and moments and of the forces and moments on the faces, the partial differential matrix operator $A(\partial_x) = A(\partial/\partial x_1, \partial/\partial x_2)$ is defined by

$$A(\xi_1, \xi_2) = \begin{pmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 - \mu & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu & -\mu\xi_2 \\ \mu\xi_1 & \mu\xi_2 & \mu\Delta \end{pmatrix},$$

λ and μ are the Lamé constants of the material, $h^2 = h_0^2/12$, and $\Delta = \xi_1^2 + \xi_2^2$. We also consider the boundary operator $T(\partial_x)$ of the normal moments and shear force, given by

$$T(\xi_1, \xi_2) = \begin{pmatrix} h^2(\lambda+2\mu)\nu_1\xi_1 + h^2\mu\nu_2\xi_2 & h^2\mu\nu_2\xi_1 + h^2\lambda\nu_1\xi_2 & 0 \\ h^2\lambda\nu_2\xi_1 + h^2\mu\nu_1\xi_2 & h^2\mu\nu_1\xi_1 + h^2(\lambda+2\mu)\nu_2\xi_2 & 0 \\ \mu\nu_1 & \mu\nu_2 & \mu(\nu_1\xi_1 + \nu_2\xi_2) \end{pmatrix},$$

where $\nu = (\nu_1, \nu_2)^T$ is the unit outward normal to ∂S. In what follows we assume that $\lambda + \mu > 0$ and $\mu > 0$, which ensures that (2) is an elliptic system.

Let $\mathcal{F}$ be the space of rigid displacements, which is spanned by $z^{(1)} = (1, 0, -x_1)^T$, $z^{(2)} = (0, 1, -x_2)^T$ and $z^{(3)} = (0, 0, 1)^T$ [**1**]. Since every $z \in \mathcal{F}$ is infinitely differentiable, to simplify the notation we use the symbols z and $\mathcal{F}$ regardless of whether the rigid displacements are considered as functions in S^+, S^-, on ∂S, or on the whole of $\mathbf{R}^2$.

We denote by $\mathcal{A}$ the space of functions in S^- that, as $r = |x| \to \infty$, admit an asymptotic expansion (in terms of polar coordinates) of the form

$$\begin{aligned}
u_1(r,\theta) &= r^{-1}[a_0 \sin\theta + 2a_1 \cos\theta - a_0 \sin 3\theta + (a_2 - a_1)\cos 3\theta] \\
&\quad + r^{-2}[(2a_3 + a_4)\sin 2\theta + a_5 \cos 2\theta - 3a_3 \sin 4\theta + 2a_6 \cos 4\theta] \\
&\quad + r^{-3}[2a_7 \sin 3\theta + 2a_8 \cos 3\theta + 3(a_9 - a_7)\sin 5\theta \\
&\qquad + 3(a_{10} - a_8)\cos 5\theta] + O(r^{-4}), \\
u_2(r,\theta) &= r^{-1}[2a_2 \sin\theta + a_0 \cos\theta + (a_2 - a_1)\sin 3\theta + a_0 \cos 3\theta] \\
&\quad + r^{-2}[(2a_6 + a_5)\sin 2\theta - a_4 \cos 2\theta + 3a_6 \sin 4\theta + 2a_3 \cos 4\theta] \\
&\quad + r^{-3}[2a_{10} \sin 3\theta - 2a_9 \cos 3\theta + 3(a_{10} - a_8)\sin 5\theta \\
&\qquad + 3(a_7 - a_9)\cos 5\theta] + O(r^{-4}), \\
u_3(r,\theta) &= -(a_1 + a_2)\ln r - [a_1 + a_2 + a_0 \sin 2\theta + (a_1 - a_2)\cos 2\theta] \\
&\quad + r^{-1}[(a_3 + a_4)\sin\theta + (a_5 + a_6)\cos\theta - a_3 \sin 3\theta + a_6 \cos 3\theta] \\
&\quad + r^{-2}[a_{11} \sin 2\theta + a_{12} \cos 2\theta + (a_9 - a_7)\sin 4\theta \\
&\qquad + (a_{10} - a_8)\cos 4\theta] + O(r^{-3}),
\end{aligned}$$

where $a_0, \dots, a_{12}$ are arbitrary constants.

In [**10**] it is shown that boundary value problems for (2) can be reduced to similar ones for the homogeneous equations. Consequently, without loss of generality, from now on we assume that $q = 0$.

Let $\partial S = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, where $\operatorname{mes}\Gamma_\alpha \neq 0$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$. The interior and exterior mixed problems for (2) are formulated as follows:

(M^+) Find u satisfying (2), with $q = 0$, in S^+ and $u|_{\Gamma_1} = f$, $Tu|_{\Gamma_2} = g$.

(M^-) Find $u \in \mathcal{A}$ satisfying (2), with $q = 0$, in S^- and $u|_{\Gamma_1} = f$, $Tu|_{\Gamma_2} = g$.

Here f and g are prescribed functions. The corresponding variational formulations of these problems can be found in [**11**] and [**12**].

Let Ω be a domain in $\mathbf{R}^2$, and let $b_\Omega(u,v) = \int_\Omega 2E(u,v)\,dx$, where

$$\begin{aligned}
2E(u,v) &= h^2 E_0(u,v) + h^2\mu(u_{1,2}+u_{2,1})(v_{1,2}+v_{2,1})\\
&\quad + \mu[(u_1+u_{3,1})(v_1+v_{3,1}) + (u_2+u_{3,2})(v_2+v_{3,2})],\\
E_0(u,v) &= (\lambda+2\mu)(u_{1,1}v_{1,1}+u_{2,2}v_{2,2}) + \lambda(u_{1,1}v_{2,2}+u_{2,2}v_{1,1}).
\end{aligned}$$

$E(u,u)$ is the internal energy density [**1**], which, in view of the conditions on λ and μ, is a positive quadratic form. When $\Omega = S^+$ and $\Omega = S^-$, we write $b_\Omega = b_+$ and $b_\Omega = b_-$.

We denote by $H_s(\Omega)$, $s \in \mathbf{R}$, the well-known Sobolev space with norm $\|\cdot\|_{s;\Omega}$. We also consider the space $\mathring{H}_s(\Omega)$ consisting of all $u \in H_s(\mathbf{R}^2)$ with $\operatorname{supp} u \subset \bar{\Omega}$.

We write $\bar{u} = (u_1, u_2)$ and introduce the space $L^2_\omega(\Omega)$ of all u such that

$$\begin{aligned}
\|u\|^2_{0,\omega;\Omega} = \int_\Omega &\frac{|\bar{u}(x)|^2}{(1+|x|)^2(1+\ln^2(1+|x|))}\,dx\\
&+ \int_\Omega \frac{|u_3(x)|^2}{(1+|x|)^4(1+\ln^2(1+|x|))}\,dx < \infty.
\end{aligned}$$

It is easily verified that $\|\cdot\|_{0,\omega;\Omega}$ is fully compatible with the class $\mathcal{A}$.

Let $H_{1,\omega}(\mathbf{R}^2)$ be the space of three-component distributions u on $\mathbf{R}^2$ with finite norm $\|u\|^2_{1,\omega} = \|u\|^2_{0,\omega;\mathbf{R}^2} + b_{\mathbf{R}^2}(u,u)$, and let $H_{1,\omega}(\Omega)$ be the space of the restrictions to Ω of all $u \in H_{1,\omega}(\mathbf{R}^2)$. The norm in this space can be defined in two equivalent ways, namely,

$$\|u\|^2_{1,\omega;\Omega} = \|u\|^2_{0,\omega;\Omega} + b_\Omega(u,u)$$

or

$$\|u\|_{1,\omega;\Omega} = \inf_{v\in H_{1,\omega}(\mathbf{R}^2):\, v|_\Omega = u} \|v\|_{1,\omega},$$

but in what follows we make use of the former. Clearly, if Ω is bounded, then the norm in $L^2_\omega(\Omega)$ is equivalent to that in $L^2(\Omega)$, and the norm in $H_{1,\omega}(\Omega)$ is equivalent to that in $H_1(\Omega)$. Finally, $\mathring{H}_{1,\omega}(\Omega)$ is the subspace of all $u \in H_{1,\omega}(\mathbf{R}^2)$ such that $\operatorname{supp} u \subset \bar{\Omega}$.

If Ω has a compact boundary $\partial\Omega$, we denote by γ^Ω the trace operator defined first on $C_0^\infty(\bar{\Omega})$ and then extended by continuity to a surjection

$\gamma^\Omega : H_{1,\omega}(\Omega) \to H_{1/2}(\partial\Omega)$. (This is possible because of the local equivalence of $H_{1,\omega}(\Omega)$ and $H_1(\Omega)$.) When $\Omega = S^+$ and $\Omega = S^-$, we denote the corresponding trace operators by γ^+ and γ^-. We also consider a continuous extension operator $l^\Omega : H_{1/2}(\partial\Omega) \to H_1(\Omega)$, which, since the norm in $H_1(\Omega)$ is stronger than that in $H_{1,\omega}(\Omega)$, may be regarded as a continuous operator from $H_{1/2}(\partial\Omega)$ to $H_{1,\omega}(\Omega)$. The symbols l^+ and l^- have the obvious meaning.

Let $\mathring{H}_{-1,\omega}(\Omega)$ (with norm $\|\cdot\|_{-1,\omega}$) and $H_{-1,\omega}(\Omega)$ (with norm $\|\cdot\|_{-1,\omega;\Omega}$) be the duals of $H_{1,\omega}(\Omega)$ and $\mathring{H}_{1,\omega}(\Omega)$. It can be shown that if $u \in \mathring{H}_{-1}(\Omega)$ and has compact support in Ω, or if

$$\int_\Omega |\tilde{u}(x)|^2(1+|x|)^2(1+\ln^2(1+|x|))\,dx + \int_\Omega |u_3(x)|^2(1+|x|)^4(1+\ln^2(1+|x|))\,dx < \infty,$$

then $u \in \mathring{H}_{-1,\omega}(\Omega)$.

We denote by $\langle\cdot\,,\cdot\rangle_{0;\Omega}$ the inner product in $L^2(\Omega)$, and consider the spaces

$$\begin{aligned}
\mathring{H}_1(S^+,\Gamma_\alpha) &= \{u \in H_1(S^+) : \gamma^+ u \in \mathring{H}_{1/2}(\Gamma_\alpha)\},\\
\mathring{H}_{1,\omega}(S^-,\Gamma_\alpha) &= \{u \in H_{1,\omega}(S^-) : \gamma^- u \in \mathring{H}_{1/2}(\Gamma_\alpha)\},\\
\hat{H}_{1/2}(\partial S) &= \{f \in H_{1/2}(\partial S) : \langle f, z^{(i)}\rangle_{0;\partial S} = 0\},\\
\mathcal{H}_{-1/2}(\partial S) &= \{f \in H_{-1/2}(\partial S) : \langle f, z^{(i)}\rangle_{0;\partial S} = 0\}.
\end{aligned}$$

3. Boundary operators. Let $f \in H_{1/2}(\partial S)$, and let $u \in H_1(S^+)$ be the (unique) solution [**11**] of the variational problem

$$b_+(u,v) = 0 \quad \forall\, v \in \mathring{H}_1(S^+), \qquad \gamma^+ u = f.$$

We consider an arbitrary $\alpha \in H_{1/2}(\partial S)$ and write $w = l^+\alpha$. Using the Riesz representation theorem, we can define an operator $\mathcal{T}^+$ on $H_{1/2}(\partial S)$ by

$$\langle \mathcal{T}^+ f, \alpha\rangle_{0;\partial S} = b_+(u,w). \tag{3}$$

This definition is consistent, for if $\tilde{w} \in H_1(S^+)$ is another extension of α, then $w - \tilde{w} \in \overset{\circ}{H}_1(S^+)$ and $b_+(u, w - \tilde{w}) = 0$.

Now let $u \in H_{1,\omega}(S^-)$ be the (unique) solution [**12**] of the variational problem

$$b_-(u,v) = 0 \quad \forall\, v \in \overset{\circ}{H}_{1,\omega}(S^-), \qquad \gamma^- u = f,$$

and let $w = l^-\alpha$. Similarly, we define an operator $\mathcal{T}^-$ by $\langle \mathcal{T}^- f, \alpha\rangle_{0;\partial S} = -b_-(u,w)$.

$\mathcal{T}^\pm$ are known as the Poincare-Steklov operators corresponding to (2). Some important properties of $\mathcal{T}^\pm$, which are used in what follows, can be found in [**9**].

We introduce boundary operators $\pi^\pm_{\alpha\beta} : H_{1/2}(\partial S) \to H_{1/2}(\Gamma_\alpha) \times H_{-1/2}(\Gamma_\beta)$, $\alpha \neq \beta$, defined by $\pi^\pm_{\alpha\beta} f = \{\pi_\alpha f, \pi_\beta \mathcal{T}^\pm f\}$, where π_α are the operators of restriction from ∂S to Γ_α.

Theorem 1. *The operators $\pi^\pm_{\alpha\beta}$ are homeomorphisms.*

Proof. It is obvious that the $\pi^\pm_{\alpha\beta}$ are continuous. To prove the existence and continuity of their inverses, we first consider functions $f_1 \in \overset{\circ}{H}_{1/2}(\Gamma_\beta)$, in other words, such that $\pi_\alpha f_1 = 0$. Let $g_1 = \pi_\beta \mathcal{T}^+ f_1$, and let $u_1 \in H_1(S^+)$ be the (unique) solution of the interior Dirichlet problem with boundary value $\gamma^+ u_1 = f_1$. Then, by formula (7) in [**11**], which is valid for $u_1 \in \overset{\circ}{H}_1(S^+, \Gamma_\beta)$,

$$\begin{aligned} \|f_1\|^2_{1/2;\partial S} &\le c\|u_1\|^2_{1;S^+} \le c b_+(u,u) \\ &= c\langle \mathcal{T}^+ f_1, f_1\rangle_{0;\partial S} \le c\|g_1\|_{-1/2;\Gamma_\beta}\|f_1\|_{1/2;\partial S}; \end{aligned}$$

therefore,

$$\|f_1\|_{1/2;\partial S} \le c\|g_1\|_{-1/2;\Gamma_\beta} = c\|\pi^+_{\alpha\beta} f_1\|_{1/2;\Gamma_\alpha,-1/2;\Gamma_\beta},$$

where

$$\|\{f,g\}\|_{1/2;\Gamma_\alpha,-1/2;\Gamma_\beta} = \|f\|_{1/2;\Gamma_\alpha} + \|g\|_{-1/2;\Gamma_\beta}. \tag{4}$$

The estimate

$$\|f_1\|_{1/2;\partial S} \le c\|\pi^-_{\alpha\beta} f_1\|_{1/2;\Gamma_\alpha,-1/2;\Gamma_\beta} \tag{5}$$

is derived analogously.

We now consider functions $f_2 \in H_{1/2}(\partial S)$ such that $\mathcal{T}^+ f_2 \in \mathring{H}_{-1/2}(\Gamma_\alpha)$, that is, for which $\pi_\beta \mathcal{T}^+ f_2 = 0$. If $u_2 \in H_1(S^+)$ is the solution of the interior Dirichlet problem with boundary value $\gamma^+ u_2 = f_2$, then $\|u_2\|^2_{1;S^+} \le c[b_+(u_2,u_2) + |\int_{\Gamma_\alpha} u_2\,ds|^2]$. (This inequality follows from Theorem 7 in [**11**] and the fact that $|\int_{\Gamma_\alpha} z\,ds| = 0$ implies that $z = 0$.) Consequently,

$$\begin{aligned}\|f_2\|^2_{1/2;\partial S} &\le c\|u_2\|^2_{1;S^+} \le c\left[b_+(u_2,u_2) + \left|\int_{\Gamma_\alpha} u_2\,ds\right|^2\right]\\ &\le c(\langle \mathcal{T}^+ f_2, f_2\rangle_{0;\partial S} + \|\pi_\alpha f_2\|^2_{1/2;\Gamma_\alpha})\\ &\le c(\|\mathcal{T}^+ f_2\|_{-1/2;\partial S}\|\pi_\alpha f_2\|_{1/2;\Gamma_\alpha} + \|\pi_\alpha f_2\|^2_{1/2;\partial S})\\ &\le c\|f_2\|_{1/2;\partial S}\|\pi_\alpha f_2\|_{1/2;\Gamma_\alpha},\end{aligned}$$

which reduces to

$$\|f_2\|_{1/2;\partial S} \le c\|\pi_\alpha f_2\|_{1/2;\Gamma_\alpha} = c\|\pi^+_{\alpha\beta} f_2\|_{1/2;\Gamma_\alpha,-1/2;\Gamma_\beta}. \tag{6}$$

The estimate

$$\|f_2\|_{1/2;\partial S} \le c\|\pi^-_{\alpha\beta} f_2\|_{1/2;\Gamma_{\alpha,-1/2;\Gamma_\beta}} \tag{7}$$

is derived similarly.

Combining (4)–(7), we now find that

$$\begin{aligned}\|f_1 + f_2\|_{1/2;\partial S} &\le c(\|\pi_\alpha f_2\|_{1/2;\Gamma_\alpha} + \|\pi_\beta \mathcal{T}^\pm f_1\|_{-1/2;\Gamma_\beta})\\ &= c(\|\pi_\alpha(f_1+f_2)\|_{1/2;\Gamma_\alpha} + \|\pi_\beta \mathcal{T}^\pm (f_1+f_2)\|_{-1/2;\Gamma_\beta})\\ &= c\|\pi^\pm_{\alpha\beta}(f_1+f_2)\|_{1/2;\Gamma_\alpha,-1/2;\Gamma_\beta}.\end{aligned}$$

We claim that the sets $\{\pi^\pm_{\alpha\beta}(f_1+f_2) : f_1 \in \mathring{H}_{1/2}(\Gamma_\beta),\ \mathcal{T}^\pm f_2 \in \mathring{H}_{-1/2}(\Gamma_\alpha)\}$ are dense in $H_{1/2}(\Gamma_\alpha) \times H_{1/2}(\Gamma_\beta)$. Assuming the opposite, we can find a nonzero $\{\sigma,\tau\} \in \mathring{H}_{-1/2}(\Gamma_\alpha) \times \mathring{H}_{1/2}(\Gamma_\beta)$ (the dual of $H_{1/2}(\Gamma_\alpha) \times H_{-1/2}(\Gamma_\beta)$) such that

$$\langle \pi_\alpha f_2, \sigma\rangle_{0;\partial S} + \langle \tau, \pi_\beta \mathcal{T}^\pm f_1\rangle_{0;\partial S} = 0. \tag{8}$$

If we take $f_2 = 0$, $f_1 = \tau$, then $\langle \tau, \pi_\beta \mathcal{T}^\pm \tau \rangle_{0;\partial S} = \langle \tau, \mathcal{T}^\pm \tau \rangle_{0;\partial S} = 0$. Hence, $\tau \in \mathring{H}_{1/2}(\Gamma_\beta)$ is a rigid displacement that vanishes on Γ_α, so $\tau = 0$, and (8) becomes

$$\langle \pi_\alpha f_2, \sigma \rangle_{0;\partial S} = 0. \tag{9}$$

Setting $f_2 = z^{(i)}$ in (9), we find that $\sigma \in \mathcal{H}_{-1/2}(\partial S)$, which means that the equation $\mathcal{T}^\pm f_2 = \sigma$ is solvable. Any solution f_2 of this equation satisfies $\langle \mathcal{T}^\pm f_2, f_2 \rangle_{0;\partial S} = 0$. This yields $f_2 \in \mathcal{F}$ and $\mathcal{T}^\pm f_2 = \sigma = 0$, which contradicts our assumption and thus completes the proof.

In what follows we make extensive use of the single and double layer plate potentials $V\varphi$ and $W\psi$. Their properties can be found in [**1**] and [**9**], together with the properties of the boundary operators $\hat{V}_0$ and $\hat{W}^\pm$ defined by

$$\begin{aligned} \hat{V}_0\varphi &= V_0\varphi - \langle V_0\varphi, \tilde{z}^{(i)} \rangle_{0;\partial S} \tilde{z}^{(i)}, \quad \varphi \in \mathcal{H}_{-1/2}(\partial S), \\ \hat{W}^\pm\psi &= W^\pm\psi - \langle W^\pm\psi, \tilde{z}^{(i)} \rangle_{0;\partial S} \tilde{z}^{(i)}, \quad \psi \in \hat{H}_{1/2}(\partial S), \end{aligned}$$

where $\{\tilde{z}^{(i)}\}$ is the set obtained from $\{z^{(i)}\}$ by orthonormalization in $L^2(\partial S)$ and V_0 and $W^\pm$ are the boundary operators defined by $V_0\varphi = \gamma^+ V\varphi = \gamma^- V\varphi$ and $W^\pm\psi = \gamma^\pm W\psi$.

4. First representation of the solution.

For simplicity, we refer in the singular to the equations corresponding to the interior and exterior problems written simultaneously by means of the symbol $\pm$.

We seek solutions of ($\mathrm{M}^\pm$) of the form

$$u = \hat{V}\varphi + z \quad \text{in } S^\pm, \tag{10}$$

where the density $\varphi \in \mathcal{H}_{-1/2}(\partial S)$ and $z \in \mathcal{F}$ are unknown. In view of the properties of the single layer potential [**1**], ($\mathrm{M}^\pm$) reduce to the pair of boundary integral equations

$$\pi_1(\hat{V}_0\varphi + z) = f, \qquad \pi_2\mathcal{T}^\pm\hat{V}_0\varphi = g. \tag{11}$$

Theorem 2. (i) *System* (11) *has a unique solution for every* $f \in H_{1/2}(\Gamma_1)$ *and* $g \in H_{-1/2}(\Gamma_2)$.

(ii) *If* $\{\varphi, z\} \in \mathcal{H}_{-1/2}(\partial S) \times \mathcal{F}$ *is the solution of* (11)*, then* (10) *is the solution of* ($\mathrm{M}^{\pm}$)*.*

Proof. (i) Let $\alpha = (\pi_{12}^{\pm})^{-1}\{f, g\} \in H_{1/2}(\partial S)$, and let z_0 be such that $\alpha - z_0 \in \hat{H}_{1/2}(\partial S)$. By Theorem 5 in [**11**], the equation $\hat{V}_0\varphi = \alpha - z_0 = 0$ has a unique solution $\varphi_0 \in \mathcal{H}_{-1/2}(\partial S)$. We claim that $\{\varphi_0, z_0\}$ is a solution of (11). Indeed, since $\hat{V}_0\varphi_0 + z_0 = \alpha$, it follows that $\pi_{12}^{\pm}(\hat{V}_0\varphi_0 + z_0) = \pi_{12}^{\pm}\alpha = \{f, g\}$, so $\pi_1(\hat{V}_0\varphi_0 + z_0) = f$ and $\pi_2\mathcal{T}^{\pm}\hat{V}_0\varphi_0 = g$.

The difference (φ, z) of any two such solutions satisfies

$$\pi_1(\hat{V}_0\varphi + z) = 0, \qquad \pi_2\mathcal{T}^{\pm}\hat{V}_0\varphi = \pi_2\mathcal{T}^{\pm}(\hat{V}_0\varphi + z) = 0.$$

Then $\hat{V}_0\varphi + z = (\pi_{12}^{\pm})^{-1}\{0, 0\} = 0$. Since $\hat{V}_0\varphi \in \hat{H}_{1/2}(\partial S)$ and $z \in \mathcal{F}$, we conclude that $z = 0$ and $\hat{V}_0\varphi = 0$; therefore, $\varphi = 0$, which proves the uniqueness of the solution.

(ii) If $\varphi \in \mathcal{H}_{-1/2}(\partial S)$, then $\hat{V}\varphi$ belongs to both $H_1(S^+)$ and $H_{1,\omega}(S^-)$ [**9**], and so, too, does $\hat{V}\varphi + z$.

5. Second representation of the solution. We now seek solutions of ($\mathrm{M}^{\pm}$) of the form

$$u = \hat{W}\psi + z \quad \text{in } S^{\pm}, \tag{12}$$

where $\psi \in \hat{H}_{1/2}(\partial S)$ and $z \in \mathcal{F}$ are unknown. The corresponding system of boundary integral equations in this case is

$$\pi_1(\hat{W}^{\pm}\psi + z) = f, \qquad \pi_2\mathcal{T}^{\pm}\hat{W}^{\pm}\psi = g. \tag{13}$$

Theorem 3. (i) *System* (13) *has a unique solution for every* $f \in H_{1/2}(\Gamma_1)$ *and* $g \in H_{-1/2}(\Gamma_2)$*.*

(ii) *If* $\{\psi, z\} \in \hat{H}_{1/2}(\partial S) \times \mathcal{F}$ *is the solution of* (13)*, then* (12) *is the solution of* ($\mathrm{M}^{\pm}$)*.*

Proof. (i) Let α and z_0 be the same as in the proof of Theorem 2, and let $\psi_0 \in \hat{H}_{1/2}(\partial S)$ be the solution of the equation $\hat{W}^{\pm}\psi_0 =$

$\alpha - z_0$, whose existence is guaranteed by Theorem 6 in [**9**]. Since $\hat{W}^{\pm}\psi_0 + z_0 = \alpha$, it follows that $\pi_{12}^{\pm}(\hat{W}^{\pm}\psi_0 + z_0) = \pi_{12}^{\pm}\alpha = \{f, g\}$, so $\pi_1(\hat{W}^{\pm}\psi_0 + z_0) = f$ and $\pi_2\mathcal{T}^{\pm}\hat{W}^{\pm}\psi_0 = g$. Consequently, $\{\psi_0, z_0\}$ is a solution of (13). The difference $\{\psi, z\}$ of two such solutions satisfies

$$\pi_1(\hat{W}^{\pm}\psi + z) = 0, \qquad \pi_2\mathcal{T}^{\pm}\hat{W}^{\pm}\psi = \pi_2\mathcal{T}^{\pm}(\hat{W}^{\pm}\psi + z) = 0.$$

Hence, $\hat{W}^{\pm}\psi + z = (\pi_{12}^{\pm})^{-1}\{0, 0\} = 0$. Since $\hat{W}^{\pm}\psi \in \hat{H}_{1/2}(\partial S)$ and $z \in \mathcal{F}$, we conclude that $z = 0$ and $\hat{W}^{\pm}\psi = 0$, which yields $\psi = 0$. This means that the solution of (13) is unique.

(ii) If $\psi \in \hat{H}_{1/2}(\partial S)$, then $\hat{W}\psi$ belongs to both $H_1(S^+)$ and $H_{1,\omega}(S^-)$ [**9**]; hence, so does $\hat{W}\psi + z$.

6. Third representation of the solution. This time we seek solutions of $(M^{\pm})$ of the form

$$u = \hat{V}\varphi_1 + W\psi_2 + z \quad \text{in } S^{\pm}, \tag{14}$$

where $\varphi_1 \in \mathring{H}_{-1/2}(\Gamma_1) \cap \mathcal{H}_{-1/2}(\partial S)$ and $\psi_2 \in \mathring{H}_{1/2}(\Gamma_2)$ are unknown densities and z is an unknown rigid displacement. Such a representation leads to the pair of boundary integral equations

$$\pi_1(\hat{V}_0\varphi_1 + W^{\pm}\psi_2 + z) = f, \qquad \pi_2(\mathcal{T}^{\pm}\hat{V}_0\varphi_1 + \mathcal{T}^{\pm}W^{\pm}\psi_2) = g. \tag{15}$$

Theorem 4. (i) *System* (15) *has a unique solution for every* $f \in H_{1/2}(\Gamma_1)$ *and* $g \in H_{-1/2}(\Gamma_2)$.

(ii) *If* $\{\varphi_1, \psi_2, z\} \in (\mathring{H}_{-1/2}(\Gamma_1) \cap \mathcal{H}_{-1/2}(\partial S)) \times \mathring{H}_{1/2}(\Gamma_2) \times \mathcal{F}$ *is the solution of* (15), *then* (14) *is the solution of* $(\mathrm{M}^{\pm})$.

Proof. We consider (M^+); (M^-) is treated similarly.

(i) Let $u^+ \in H_1(S^+)$ and $u^- \in H_{1,\omega}(S^-)$ be, respectively, the unique solutions of the variational problems

$$b_+(u^+, v) = \langle g, \gamma^+ v\rangle_{0;\partial S} \quad \forall\, v \in \mathring{H}(S^+, \Gamma_2), \qquad \pi_1\gamma^+ u^+ = f,$$

$$b_-(u^-, v) = -\langle g, \gamma^- v\rangle_{0;\partial S} \quad \forall\, v \in \mathring{H}_{1,\omega}(S^-, \Gamma_2), \qquad \pi_1\gamma^- u^- = f.$$

We write

$$U(x) = \begin{cases} u^+(x) & x \in S^+, \\ u^-(x) & x \in S^-, \end{cases}$$

and $\psi_{20} = \gamma^- u^- - \gamma^+ u^+$. Clearly, $\psi_{20} \in \overset{\circ}{H}_{1/2}(\Gamma_2)$ and $\Phi = U - W\psi_{20}$ satisfies

$$\gamma^+\Phi = \gamma^+ u^+ - W^+\psi_{20}, \qquad \gamma^-\Phi = \gamma^- u^- - W^-\psi_{20},$$
$$\gamma^+\Phi - \gamma^-\Phi = \gamma^+ u^+ - \gamma^- u^- - (W^+ - W^-)\psi_{20} = -\psi_{20} + \psi_{20} = 0.$$

We choose z_0 so that $\gamma^+\Phi - z_0 = \gamma^-\Phi - z_0 \in \hat{H}_{1/2}(\partial S)$ and set $\varphi_{10} = \hat{V}_0^{-1}(\gamma^+\Phi - z_0) = \hat{V}_0^{-1}(\gamma^-\Phi - z_0)$.

We now verify that $\{\varphi_{10}, \psi_{20}, z_0\}$ is a solution of (15). We already know that $\psi_{20} \in \overset{\circ}{H}_{1/2}(\Gamma_2)$ and $\varphi_{10} \in \mathcal{H}_{-1/2}(\partial S)$. It remains to show that $\varphi_{10} \in \overset{\circ}{H}_{-1/2}(\Gamma_1)$. Since $\hat{V}\varphi_{10}$ is a solution of both the interior and exterior Dirichlet problems with boundary data $\gamma^+\Phi - z_0 = \gamma^-\Phi - z_0$, it follows that

$$\mathcal{T}^+\hat{V}_0\varphi_{10} = \mathcal{T}^+(\gamma^+\Phi - z_0) = \mathcal{T}^+\gamma^+\Phi = \mathcal{T}^+\gamma^+ u^+ - \mathcal{T}^+ W^+\psi_{20},$$
$$\mathcal{T}^-\hat{V}_0\varphi_{10} = \mathcal{T}^-(\gamma^-\Phi - z_0) = \mathcal{T}^-\gamma^-\Phi = \mathcal{T}^-\gamma^- u^- - \mathcal{T}^- W^-\psi_{20},$$

so $\varphi_{10} = (\mathcal{T}^+\hat{V}_0 - \mathcal{T}^-\hat{V}_0)\varphi_{10} = \mathcal{T}^+\gamma^+ u^+ - \mathcal{T}^-\gamma^- u^-$. Hence, $\pi_2\varphi_{10} = \pi_2(\mathcal{T}^+\gamma^+ u^+ - \mathcal{T}^-\gamma^- u^-) = g - g = 0$, and we deduce that $\varphi_{10} \in \overset{\circ}{H}_{-1/2}(\Gamma_1)$.

Next, from the definition of φ_{10}, we see that

$$\hat{V}_0\varphi_{10} + z_0 - \gamma^+\Phi = \hat{V}_0\varphi_{10} + z_0 + W^+\psi_{20} - \gamma^+ u^+ = 0;$$

consequently, $\pi_1(\hat{V}_0\varphi_{10} + W^+\psi_{20} + z_0) = \pi_1\gamma^+ u^+ = f$. Finally, since $\hat{V}_0\varphi_{10} + W^+\psi_{20} + z_0 = \gamma^+ u^+$, we find that

$$\mathcal{T}^+(\hat{V}_0\varphi_{10} + W^+\psi_{20}) = \mathcal{T}^+\gamma^+ u^+,$$
$$\pi_2(\mathcal{T}^+\hat{V}_0\varphi_{10} + \mathcal{T}^+ W^+\psi_{20}) = \pi_2\mathcal{T}^+\gamma^+ u^+ = g,$$

which means that $\{\varphi_{10}, \psi_{20}, z_0\}$ is a solution of (15).

The difference $\{\varphi_1, \psi_2, z\} \in (\mathring{H}_{-1/2}(\Gamma_1) \cap \mathcal{H}_{-1/2}(\partial S)) \times \mathring{H}_{1/2}(\Gamma_2) \times \mathcal{F}$ of two solutions of (15) satisfies

$$\pi_1(\hat{V}_0\varphi_1 + W^+\psi_2 + z) = 0, \qquad \pi_2(\mathcal{T}^+\hat{V}_0\varphi_1 + \mathcal{T}^+W^+\psi_2) = 0,$$

so $\hat{V}_0\varphi_1 + W^+\psi_2 + z = (\pi_{12}^+)^{-1}\{0,0\} = 0$. Since in this case $u^+ = 0$ in S^+ and $u^- = 0$ in S^-, we conclude that $\psi_2 = \gamma^- u^- - \gamma^+ u^+ = 0$. The equality $\hat{V}_0\varphi_1 + z = 0$ implies that $z = 0$ and $\hat{V}_0\varphi_1 = 0$, so $\varphi_1 = 0$. Hence, the solution of (15) is unique.

(ii) Since $\varphi_1 \in \mathcal{H}_{-1/2}(\partial S)$, we have $\hat{V}\varphi_1 \in H_1(S^+)$. Now $W\psi_2$ differs from $\hat{W}\psi_2$ by a rigid displacement; therefore, since $\psi_2 \in H_{1/2}(\partial S)$, it follows that $W\psi_2 \in H_1(S^+)$. Consequently, (14) is a solution of (M$^+$).

7. Fourth representation of the solution. We seek solutions of (M$^\pm$) of the form

$$u = W\psi_1 + \hat{V}\varphi_2 + z \quad \text{in } S^\pm, \tag{16}$$

where $\psi_1 \in \mathring{H}_{1/2}(\Gamma_1)$ and $\varphi_2 \in \mathring{H}_{-1/2}(\Gamma_2) \cap \mathcal{H}_{-1/2}(\partial S)$ are unknown densities and z is an unknown rigid displacement. This representation yields the boundary integral equations

$$\pi_1(W^\pm\psi_1 + \hat{V}_0\varphi_2 + z) = f, \qquad \pi_1(\mathcal{T}^\pm W^\pm\psi_1 + \mathcal{T}^\pm\hat{V}_0\varphi_2) = g. \tag{17}$$

Theorem 5. (i) *System* (17) *has a unique solution for every $f \in H_{1/2}(\Gamma_1)$ and $g \in H_{-1/2}(\Gamma_2)$.*

(ii) *If $\{\psi_1, \varphi_2, z\} \in \mathring{H}_{1/2}(\Gamma_1) \times (\mathring{H}_{-1/2}(\Gamma_2) \cap \mathcal{H}_{-1/2}(\partial S)) \times \mathcal{F}$ is the solution of* (17)*, then* (16) *is the solution of* (M$^\pm$).

Proof. As above, we consider only (M$^+$).

(i) Let $u^+ \in H_1(S^+)$ and $u^- \in H_{1,\omega}(S^-)$ be, respectively, the (unique) solutions of the variational problems

$$b_+(u^+, v) = \langle g, \gamma^+ v\rangle_{0;\partial S} \quad \forall\, v \in \mathring{H}_1(S^+, \Gamma_2), \qquad \pi_1\gamma^+u^+ = f,$$
$$b_-(u^-, v) = -\langle \mathcal{T}^+\gamma^+u^+, \gamma^- v\rangle_{0;\partial S} \quad \forall\, v \in \mathring{H}_1(S^-, \Gamma_1),$$
$$\pi_2\gamma^-u^- = \pi_2\gamma^+u^+.$$

We write

$$U(x) = \begin{cases} u^+(x) & x \in S^+, \\ u^-(x) & x \in S^-, \end{cases}$$

and $\psi_{10} = \gamma^- u^- - \gamma^+ u^+$. It is obvious that $\psi_{10} \in \mathring{H}_{1/2}(\Gamma_1)$ and that $\Psi = U - W\psi_{10}$ satisfies

$$\gamma^+\Psi = \gamma^+ u^+ - W^+\psi_{10}, \qquad \gamma^-\Psi = \gamma^- u^- - W^-\psi_{10},$$
$$\gamma^+\Psi - \gamma^-\Psi = \gamma^+ u^+ - \gamma^- u^- - (W^+ - W^-)\psi_{10} = -\psi_{10} + \psi_{10} = 0.$$

We choose z_0 so that $\gamma^+\Psi - z_0 = \gamma^-\Psi - z_0 \in \hat{H}_{1/2}(\partial S)$ and set $\varphi_{20} = \hat{V}_0^{-1}(\gamma^+\Psi - z_0) \in \mathcal{H}_{-1/2}(\partial S)$.

We claim that $\varphi_{20} \in \mathring{H}_{-1/2}(\Gamma_2)$. Indeed, since $\hat{V}\varphi_{20}$ is the solution of the interior and exterior Dirichlet problems with boundary data $\gamma^+\Psi - z_0 = \gamma^-\Psi - z_0$, we can write

$$\mathcal{T}^+\hat{V}_0\varphi_{20} = \mathcal{T}^+(\gamma^+\Psi - z_0) = \mathcal{T}^+\gamma^+\Psi = \mathcal{T}^+\gamma^+ u^+ - \mathcal{T}^+W^+\psi_{10},$$
$$\mathcal{T}^-\hat{V}_0\varphi_{20} = \mathcal{T}^-(\gamma^-\Psi - z_0) = \mathcal{T}^-\gamma^-\Psi = \mathcal{T}^-\gamma^- u^- - \mathcal{T}^-W^-\psi_{10},$$
$$\varphi_{20} = (\mathcal{T}^+\hat{V}_0 - \mathcal{T}^-\hat{V}_0)\varphi_{20} = \mathcal{T}^+\gamma^+ u^+ - \mathcal{T}^-\gamma^- u^-.$$

Consequently, $\pi_1\varphi_{20} = \pi_1(\mathcal{T}^+\gamma^+ u^+ - \mathcal{T}^-\gamma^- u^-) = 0$, and we conclude that $\varphi_{20} \in \mathring{H}_{-1/2}(\Gamma_2)$.

The definition of φ_{20} implies that $\hat{V}_0\varphi_{20} + z_0 - \gamma^+\Psi = \hat{V}_0\varphi_{20} + W^+\psi_{10} + z_0 - \gamma^+ u^+ = 0$, or $W^+\psi_{10} + \hat{V}_0\varphi_{20} + z_0 = \gamma^+ u^+$. Hence,

$$\pi_1(W^+\psi_{10} + \hat{V}_0\varphi_{20} + z_0) = \pi_1\gamma^+ u^+ = f,$$
$$\pi_2(\mathcal{T}^+W^+\psi_{10} + \mathcal{T}^+\hat{V}_0\varphi_{20} + z_0) = \pi_2\mathcal{T}^+\gamma^+ u^+ = g,$$

which shows that $\{\psi_{10}, \varphi_{20}, z_0\}$ is a solution of (17).

The difference $\{\psi_1, \varphi_2, z\} \in \mathring{H}_{1/2}(\Gamma_1) \times (\mathring{H}_{-1/2}(\Gamma_2) \cap \mathcal{H}_{-1/2}(\partial S)) \times \mathcal{F}$ of two solutions satisfies

$$\pi_1(W^+\psi_1 + \hat{V}_0\varphi_2 + z) = 0, \qquad \pi_2(\mathcal{T}^+W^+\psi_2 + \mathcal{T}^+\hat{V}_0\varphi_2) = 0,$$

so $W^+\psi_1 + \hat{V}_0\varphi_2 + z = (\pi_{12}^+)^{-1}\{0, 0\} = 0$. Since in this case $u^+ = 0$ in S^+ and $u^- = 0$ in S^-, it follows that $\psi_1 = \gamma^- u^- - \gamma^+ u^+ = 0$. The equality $\hat{V}_0\varphi_2 + z = 0$ now implies that $z = 0$ and $\hat{V}_0\varphi_2 = 0$, which means that $\varphi_2 = 0$. This proves the uniqueness of the solution.

(ii) We use the same procedure as in the proof of the second part of the preceding theorem to show that $u \in H_1(S^+)$ and that u is a solution of (M^+).

The case of (M^-) is treated similarly.

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